The Spectrum Problem for Two Multigraphs with Four Vertices and Seven Edges


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Abstract
Let $G$ be one of the two multigraphs obtained from $K_4-e$ by replacing two edges with a double-edge while maintaining a minimum degree of 2. We find necessary and sufficient conditions on $n$ and $\lambda$ for the existence of a $G$-decomposition of $\lambda K_n$.

1 Introduction
Throughout this paper, we may refer to a multigraph as a graph; however, our graphs contain no loops. If we wish to emphasize that a given graph does not contain parallel edges, then we refer to it as a simple graph. For a graph $G$, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set (or multiset) of $G$, respectively. For a simple graph $G$ and a positive integer $\lambda$, we use $\lambda G$ to denote the graph obtained from $G$ by replacing each edge in $E(G)$ with $\lambda$ parallel edges. Alternatively, we let $\lambda G$ denote the graph consisting of $\lambda$ vertex-disjoint copies of $G$. For edge-disjoint graphs $G$ and $H$, we use $G \cup H$ to represent the graph with edge set (or multiset) $E(G) \cup E(H)$ and vertex set $V(G) \cup V(H)$. We define the join of two vertex-disjoint graphs $G$ and $H$, denoted $G \vee H$, as the graph with vertex

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set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \cup \{a, b : a \in V(G), b \in V(H)\} \).

We use \( K_{s \times t} \) to denote the complete multipartite simple graph with \( s \) parts of size \( t \), and we use \( K_{r, s \times t} \) to denote the complete multipartite simple graph with one part of size \( r \) and \( s \) parts of size \( t \). If \( G \) is a subgraph of \( H \), we use \( H \setminus G \) to denote the graph obtained from \( H \) by deleting the edges in \( E(G) \).

1.1 The Spectrum Problem

Let \( K \) and \( G \) be graphs with \( G \) a subgraph of \( K \). A \( G \)-decomposition of \( K \) is a set (or multiset) \( \Delta = \{G_1, G_2, \ldots, G_t\} \) of subgraphs of \( K \) such that each \( G_i \in \Delta \) is isomorphic to \( G \) and such that each edge of \( K \) appears in exactly one such \( G_i \). Similarly, if \( G \) and \( H \) are each subgraphs of \( K \), then a \( \{G,H\} \)-decomposition of \( K \) is defined to be a set \( \{H_1, H_2, \ldots, H_t\} \) of subgraphs of \( K \) such that each \( H_i \in \Delta \) is isomorphic to either \( G \) or \( H \) and such that each edge of \( K \) appears in exactly one such \( H_i \). A \( G \)-decomposition of \( K \) is also known as a \( (K,G) \)-design or, if \( K \) is the complete graph on \( n \) vertices, a \( G \)-design of order \( n \).

A classic problem in the study of graph designs is to find necessary and sufficient conditions for the existence of a \( G \)-decomposition of \( \lambda K_n \). This is known as the spectrum problem for \( G \) because the set of all such \( n \) is called the spectrum for \( G \)-designs of index \( \lambda \). The spectrum for \( G \)-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and simple graphs with at most 5 vertices (see [3]).

In recent years, there have been some investigations of \( G \)-designs of index \( \lambda \) where \( G \) is a multigraph that is not simple. For example, in [8] Carter determined the spectra for \( G \)-designs of index \( \lambda \) for all connected cubic multigraphs \( G \) of order at most 6. The spectra for \( G \)-designs of index \( \lambda \) have been investigated for various multigraphs \( G \) of small order, e.g., certain subgraphs of \( 2K_4 \) with 5 edges (see [12, 6, 13]), 6 edges (see [2, 7]), 7 edges (see [4]), or 8 edges (see [5]). In this paper we consider two multigraphs, \( G_1 \) and \( G_2 \), each with 7 edges and minimum degree 2 obtained by replacing two edges of \( K_4 - e \) with a double edge (see Figure 1). We settle the spectrum problem for these multigraphs.

1.2 Some Basic Results

The necessary conditions for the existence of a \( G \)-decomposition of \( \lambda K_n \) include the following:

- \( |V(G)| \leq n \),
- \( |E(G)| \) divides \( |E(\lambda K_n)| = \lambda n(n-1) \), and
- \( \gcd\{\deg(v) : v \in V(G)\} \) divides \( \lambda(n-1) \).
Applying these necessary conditions to the two multigraphs under consideration, we obtain the following necessary conditions on their spectra.

**Lemma 1.1.** Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a $G_1$-decomposition of $\lambda K_n$ only if the following hold:

- if $\gcd(\lambda, 7) = 1$, then $n \equiv 0$ or 1 (mod 7);
- if $\gcd(\lambda, 7) = 7$, then $n \geq 4$.

**Proof.** Let $\lambda \geq 2$ and $n \geq 4$ and suppose there exists a $G_1$-decomposition of $\lambda K_n$. Since $|E(G_1)|$ must divide $|E(\lambda K_n)|$ for such a $G_1$-decomposition to exist, we must have that $7 \mid \lambda n(n-1)/2$, and thus $14 \mid \lambda n(n-1)$. First, if $\gcd(\lambda, 7) = 1$, then $14 \mid n(n-1)$, and thus $n \equiv 0$ or 1 (mod 7). Finally, if $\gcd(\lambda, 7) = 7$, then $2 \mid n(n-1)$, which is true for any $n \geq 4$. ■

**Lemma 1.2.** Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a $G_2$-decomposition of $\lambda K_n$ only if the following hold:

- if $\gcd(\lambda, 14) = 1$, then $n \equiv 1$ or 7 (mod 14);
- if $\gcd(\lambda, 14) = 2$, then $n \equiv 0$ or 1 (mod 7);
- if $\gcd(\lambda, 14) = 7$, then $n \equiv 1$ (mod 2);
- if $\gcd(\lambda, 14) = 14$, then $n \geq 4$.

**Proof.** Let $\lambda \geq 2$ and $n \geq 4$ and suppose there exists a $G_2$-decomposition of $\lambda K_n$. Since $|E(G_2)|$ must divide $|E(\lambda K_n)|$ for such a $G_2$-decomposition to exist, we must have that $7 \mid \lambda n(n-1)/2$, and thus $14 \mid \lambda n(n-1)$. Also, since all the vertices of $G_2$ have even degree, each vertex of $\lambda K_n$ must similarly have even degree; thus, $2 \mid \lambda(n-1)$. First, if $\gcd(\lambda, 14) = 1$, then $14 \mid n(n-1)$ and $2 \mid (n-1)$, and thus $n \equiv 1$ or 7 (mod 14). Second, if $\gcd(\lambda, 14) = 2$, then $\lambda$ is even and $7 \mid n(n-1)$, and thus $n \equiv 0$ or 1 (mod 7). Third, if $\gcd(\lambda, 14) = 7$, then $2 \mid n(n-1)$ but also $2 \mid (n-1)$, and thus $n \equiv 1$ (mod 2). Finally, if $\gcd(\lambda, 14) = 14$, then $14 \mid \lambda$, and thus there are no further restrictions on $n$. ■
The following theorems on decompositions of complete graphs and complete multipartite graphs are used extensively in proving our main results. All of these results can be found in the Handbook of Combinatorial Designs [9] (see [1] and [10]).

**Theorem 1.3** ([1]). If \( n \) is an odd positive integer, then there exists a \( \{K_3, K_5\} \)-decomposition of \( K_n \).

**Theorem 1.4** ([10]). The necessary and sufficient conditions for the existence of a \( K_3 \)-decomposition of \( K_{t \times m} \) are (i) \( t \geq 3 \), (ii) \( (t - 1)m \equiv 0 \pmod{2} \), and (iii) \( t(t-1)m^2 \equiv 0 \pmod{6} \).

**Theorem 1.5** ([10]). If \( t \geq 3 \) and \( t \equiv 0 \pmod{3} \), then there exists a \( K_3 \)-decomposition of \( K_{4,t \times 2} \).

Combining the previous two results, we have the following corollary that is more directly applicable in our general constructions.

**Corollary 1.6.** Let \( t \geq 3 \). There exists a \( K_3 \)-decomposition of \( K_{t \times 2} \) if \( t \equiv 0 \) or 1 \pmod{3} \) and of \( K_{4,(t-2)\times 2} \) if \( t \equiv 2 \) \pmod{3}.

The following is a well-known result that is a special case of Wilson’s Fundamental Construction (see [11]).

**Theorem 1.7.** Let \( m, n, r, s, \) and \( t \) be positive integers. If there exists a \( (K_{t \times m}, K_n) \)-design, then there exists a \( (K_{t \times ms}, K_{nm}) \)-design. Similarly, if there exists a \( (K_{r,t \times m}, K_n) \)-design, then there exists a \( (K_{rs,t \times ms}, K_{nxs}) \)-design.

## 2 Some Small Examples

In this section we present \( G_1 \)- and \( G_2 \)-decompositions of various graphs that are needed for the constructions presented in Section 3. Let \( G \in \{G_1, G_2\} \) and let \( G[a, b, c, d] \) denote the graph with vertex set \( \{a, b, c, d\} \) and edge set as represented in Figure 1. For example, \( G_2[0, 1, 2, 3] \) denotes the graph with vertex set \( \{0, 1, 2, 3\} \) and edge multiset \( \{\{0, 1\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 2\}, \{2, 3\}\} \). Given the graphs represented by the notation \( G[a, b, c, d] \) and some \( i \in \mathbb{Z}_n \), we define \( G[a, b, c, d] + i = G[a+i, b+i, c+i, d+i] \) where all addition is performed in \( \mathbb{Z}_n \). By convention, define \( \infty + 1 = \infty \).

### 2.1 Small Designs of Index 2

**Example 2.1.** Let \( V(2K_7) = \mathbb{Z}_6 \cup \{\infty\} \) and let \( \Delta_1 = \{G_1[1, 0, 3, \infty] + i : i \in \mathbb{Z}_6\} \) and \( \Delta_2 = \{G_2[0, 1, 3, \infty] + i : i \in \mathbb{Z}_6\} \). Then \( \Delta_1 \) and \( \Delta_2 \) are respectively \( G_1 \)- and \( G_2 \)-decompositions of \( 2K_7 \).
Example 2.2. Let $V(2K_8) = Z_8$ and let $\Delta_1 = \{G_1[0,2,3,7] + i : i \in Z_8\}$ and $\Delta_2 = \{G_2[0,3,5] + i : i \in Z_8\}$. Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $2K_8$.

Example 2.3. Let $V(2K_{14}) = Z_{13} \cup \{\infty\}$ and let $\Delta_1 = \{G_1[5,3,0,\infty] + i : i \in Z_{13}\} \cup \{G_1[0,4,1,7] + i : i \in Z_{13}\}$ and $\Delta_2 = \{G_2[0,3,1,\infty] + i : i \in Z_{13}\} \cup \{G_2[0,5,1,7] + i : i \in Z_{13}\}$. Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $2K_{14}$.

Example 2.4. Let $V(2K_{28}) = Z_{27} \cup \{\infty\}$ and let
$$
\Delta_1 = \{G_1[17,0,11,\infty] + i : i \in Z_{27}\} \cup \{G_1[1,0,15,23] + i : i \in Z_{27}\}
\cup \{G_1[0,0,12,17] + i : i \in Z_{27}\} \cup \{G_1[4,11,0,2] + i : i \in Z_{27}\},
$$
$$
\Delta_2 = \{G_2[0,3,1,\infty] + i : i \in Z_{27}\} \cup \{G_2[0,8,1,14] + i : i \in Z_{27}\}
\cup \{G_2[0,11,5,9] + i : i \in Z_{27}\} \cup \{G_2[10,0,15,6] + i : i \in Z_{27}\}.
$$
Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $2K_{28}$.

Example 2.5. Let $V(2K_{29}) = Z_{29}$ and let
$$
\Delta_1 = \{G_1[0,2,8,5] + i : i \in Z_{29}\} \cup \{G_1[0,14,11,4] + i : i \in Z_{29}\}
\cup \{G_1[0,12,13,6] + i : i \in Z_{29}\} \cup \{G_1[0,10,9,5] + i : i \in Z_{29}\},
$$
$$
\Delta_2 = \{G_2[14,2,0,3] + i : i \in Z_{29}\} \cup \{G_2[13,4,0,5] + i : i \in Z_{29}\}
\cup \{G_2[5,6,0,13] + i : i \in Z_{29}\} \cup \{G_2[3,10,0,14] + i : i \in Z_{29}\}.
$$
Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $2K_{29}$.

Example 2.6. Let $V(2K_{35}) = Z_{31}$ with partition $\{\{i \in Z_{21} : i \equiv j \pmod{3}\} : j \in Z_3\}$ and let
$$
\Delta_1 = \{G_1[1,5,0,8] + i : i \in Z_{21}\} \cup \{G_1[1,0,10,2,7] + i : i \in Z_{21}\},
$$
$$
\Delta_2 = \{G_2[0,8,7,2] + i : i \in Z_{21}\} \cup \{G_2[0,11,7,2] + i : i \in Z_{21}\}.
$$
Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $2K_{35}$.

Example 2.7. Let $V(2K_{35}) = Z_{35}$ with partition $\{\{i \in Z_{35} : i \equiv j \pmod{5}\} : j \in Z_5\}$ and let
$$
\Delta_1 = \{G_1[9,12,0,1] + i : i \in Z_{35}\} \cup \{G_1[0,21,13,1] + i : i \in Z_{35}\}
\cup \{G_1[6,13,2,0] + i : i \in Z_{35}\} \cup \{G_1[0,19,17,11] + i : i \in Z_{35}\},
$$
$$
\Delta_2 = \{G_2[0,11,4,16] + i : i \in Z_{35}\} \cup \{G_2[0,6,4,13] + i : i \in Z_{35}\}
\cup \{G_2[0,17,9,12] + i : i \in Z_{35}\} \cup \{G_2[0,14,13,16] + i : i \in Z_{35}\}.
$$
Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $^2K_{5\times 7}$.

### 2.2 Small Designs of Index 3

**Example 2.9.** Let $V(3K_7) = \mathbb{Z}_2 \times \mathbb{Z}_3 \cup \{\infty\}$ and let

$$
\Delta_1 = \{G_1[(0, 0 + i), (1, 0 + i), (1, 2 + i), (0, 1 + i) : i \in \mathbb{Z}_3] \}
\cup \{G_1[(1, 2 + i), \infty, (1, 1 + i), (0, 1 + i)] : i \in \mathbb{Z}_3\} 
\cup \{G_1[(0, 2 + i), \infty, (0, 1 + i), (1, 0 + i)] : i \in \mathbb{Z}_3\},
$$

$$
\Delta_2 = \{G_2[\infty, (0, 1 + i), (0, 0 + i), (1, 1 + i)] : i \in \mathbb{Z}_3\}
\cup \{G_2[(1, 1 + i), (0, 1 + i), (1, 0 + i), \infty] : i \in \mathbb{Z}_3\}
\cup \{G_2[(0, 0 + i), (1, 1 + i), (1, 2 + i), (0, 2 + i)] : i \in \mathbb{Z}_3\}.
$$

Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $^3K_7$.

**Example 2.10.** Let $V(3K_8) = \mathbb{Z}_8$ and let

$$
\Delta_1 = \{G_1[4, 1, 3, 6], G_1[2, 6, 0, 3], G_1[2, 3, 7, 0], G_1[0, 1, 3, 4], 
G_1[0, 4, 5, 1], G_1[0, 6, 7, 5], G_1[1, 6, 5, 2], G_1[2, 4, 1, 7], 
G_1[3, 5, 7, 1], G_1[4, 7, 5, 2], G_1[6, 3, 5, 2], G_1[6, 4, 7, 1]\}.
$$

Then $\Delta_1$ is a $G_1$-decomposition of $^3K_8$.

**Example 2.11.** Let $V(3K_{14}) = \mathbb{Z}_{13} \cup \{\infty\}$ and let $\Delta_1 = \{G_1[0, \infty, 10, 4] + i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0, 2, 5, 1] + i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0, 1, 6, 2] + i : i \in \mathbb{Z}_{13}\}$. Then $\Delta_1$ is a $G_1$-decomposition of $^3K_{14}$.

**Example 2.12.** Let $V(3K_{15}) = \mathbb{Z}_{15}$ and let

$$
\Delta_1 = \{G_1[0, 2, 5, 6] + i : i \in \mathbb{Z}_{15}\} \cup \{G_1[0, 4, 7, 5] + i : i \in \mathbb{Z}_{15}\} 
\cup \{G_1[1, 7, 0, 4] + i : i \in \mathbb{Z}_{15}\},
$$

$$
\Delta_2 = \{G_2[0, 1, 7, 2] + i : i \in \mathbb{Z}_{15}\} \cup \{G_2[0, 2, 7, 3] + i : i \in \mathbb{Z}_{15}\} 
\cup \{G_2[0, 3, 7, 1] + i : i \in \mathbb{Z}_{15}\}.
$$

Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $^3K_{15}$.

**Example 2.13.** Let $V(3K_{28}) = \mathbb{Z}_{27} \cup \{\infty\}$ and let

$$
\Delta_1 = \{G_1[0, \infty, 13, 26] + i : i \in \mathbb{Z}_{27}\} \cup \{G_1[0, 12, 16, 5] + i : i \in \mathbb{Z}_{27}\} 
\cup \{G_1[0, 10, 6, 12] + i : i \in \mathbb{Z}_{27}\},
$$

$$
\Delta_2 = \{G_1[0, 9, 5, 8] + i : i \in \mathbb{Z}_{27}\} \cup \{G_1[0, 7, 8, 10] + i : i \in \mathbb{Z}_{27}\} 
\cup \{G_1[0, 3, 2, 9] + i : i \in \mathbb{Z}_{27}\}.
$$

Then $\Delta_1$ is a $G_1$-decomposition of $^3K_{28}$. 
Example 2.14. Let \( V(3K_{29}) = \mathbb{Z}_{29} \) and let

\[
\Delta_1 = \{ G_1[0, 15, 13, 10] + i : i \in \mathbb{Z}_{29} \} \cup \{ G_1[0, 12, 10, 14] + i : i \in \mathbb{Z}_{29} \} \\
\cup \{ G_1[0, 11, 9, 18] + i : i \in \mathbb{Z}_{29} \} \cup \{ G_1[0, 8, 7, 13] + i : i \in \mathbb{Z}_{29} \} \\
\cup \{ G_1[0, 6, 5, 12] + i : i \in \mathbb{Z}_{29} \} \cup \{ G_1[0, 4, 3, 8] + i : i \in \mathbb{Z}_{29} \},
\]

\[
\Delta_2 = \{ G_2[0, 13, 27, 14] + i : i \in \mathbb{Z}_{29} \} \cup \{ G_2[0, 12, 2, 19] + i : i \in \mathbb{Z}_{29} \} \\
\cup \{ G_2[0, 11, 2, 20] + i : i \in \mathbb{Z}_{29} \} \cup \{ G_2[0, 8, 1, 22] + i : i \in \mathbb{Z}_{29} \} \\
\cup \{ G_2[0, 6, 1, 24] + i : i \in \mathbb{Z}_{29} \} \cup \{ G_2[0, 4, 1, 26] + i : i \in \mathbb{Z}_{29} \}.
\]

Then \( \Delta_1 \) and \( \Delta_2 \) are respectively \( G_1 \)- and \( G_2 \)-decompositions of \( 3K_{29} \).

Example 2.15. Let \( V(3K_{3 \times 7}) = \mathbb{Z}_{21} \) with partition \( \{ i \in \mathbb{Z}_{21} : i \equiv j \pmod{3} \} : j \in \mathbb{Z}_3 \) and let

\[
\Delta_1 = \{ G_1[0, 5, 10, 14] + i : i \in \mathbb{Z}_{21} \} \cup \{ G_1[0, 1, 14, 13] + i : i \in \mathbb{Z}_{21} \} \\
\cup \{ G_1[0, 4, 2, 10] + i : i \in \mathbb{Z}_{21} \},
\]

\[
\Delta_2 = \{ G_2[0, 8, 7, 11] + i : i \in \mathbb{Z}_{21} \} \cup \{ G_2[0, 2, 7, 8] + i : i \in \mathbb{Z}_{21} \} \\
\cup \{ G_2[0, 11, 7, 2] + i : i \in \mathbb{Z}_{21} \}.
\]

Then \( \Delta_1 \) and \( \Delta_2 \) are respectively \( G_1 \)- and \( G_2 \)-decompositions of \( 3K_{3 \times 7} \).

Example 2.16. Let \( V(3K_{5 \times 7}) = \mathbb{Z}_{35} \) with partition \( \{ i \in \mathbb{Z}_{35} : i \equiv j \pmod{5} \} : j \in \mathbb{Z}_5 \) and let

\[
\Delta_1 = \{ G_1[0, 17, 16, 14] + i : i \in \mathbb{Z}_{35} \} \cup \{ G_1[0, 13, 14, 1] + i : i \in \mathbb{Z}_{35} \} \\
\cup \{ G_1[0, 12, 9, 16] + i : i \in \mathbb{Z}_{35} \} \cup \{ G_1[0, 8, 11, 17] + i : i \in \mathbb{Z}_{35} \} \\
\cup \{ G_1[0, 7, 4, 12] + i : i \in \mathbb{Z}_{35} \} \cup \{ G_1[0, 6, 2, 11] + i : i \in \mathbb{Z}_{35} \},
\]

\[
\Delta_2 = \{ G_2[0, 4, 1, 7] + i : i \in \mathbb{Z}_{35} \} \cup \{ G_2[0, 9, 1, 4] + i : i \in \mathbb{Z}_{35} \} \\
\cup \{ G_2[0, 7, 1, 9] + i : i \in \mathbb{Z}_{35} \} \cup \{ G_2[0, 13, 2, 14] + i : i \in \mathbb{Z}_{35} \} \\
\cup \{ G_2[0, 18, 2, 13] + i : i \in \mathbb{Z}_{35} \} \cup \{ G_2[0, 14, 2, 18] + i : i \in \mathbb{Z}_{35} \}.
\]

Then \( \Delta_1 \) and \( \Delta_2 \) are respectively \( G_1 \)- and \( G_2 \)-decompositions of \( 3K_{5 \times 7} \).

2.3 Small Designs of Index 7

Example 2.17. Let \( V(7K_4) = \mathbb{Z}_4 \cup \{ \infty \} \) and let \( \Delta_1 = \{ G_1[\infty, 0, 1, 2] + i : i \in \mathbb{Z}_3 \} \cup \{ G_1[0, 1, 2, \infty] + i : i \in \mathbb{Z}_3 \} \). Then \( \Delta_1 \) is a \( G_1 \)-decomposition of \( 7K_4 \).

Example 2.18. Let \( V(7K_5) = \mathbb{Z}_5 \) and let \( \Delta_1 = \{ G_1[0, 4, 3, 1] + i : i \in \mathbb{Z}_5 \} \cup \{ G_1[0, 3, 1, 2] + i : i \in \mathbb{Z}_5 \} \) and \( \Delta_2 = \{ G_2[0, 3, 2, 1] + i : i \in \mathbb{Z}_5 \} \cup \{ G_2[0, 2, 3, 1] + i : i \in \mathbb{Z}_5 \} \). Then \( \Delta_1 \) and \( \Delta_2 \) are respectively \( G_1 \)- and \( G_2 \)-decompositions of \( 7K_5 \).
Example 2.19. Let $V(K_6) = Z_5 \cup \{\infty\}$ and let $\Delta_1 = \{G_1[0, 1, 2, \infty] + i : i \in Z_5\} \cup \{G_1[5, 0, 1, 3] + i : i \in Z_5\} \cup \{G_1[0, 2, 3, 4] + i : i \in Z_5\}$. Then $\Delta_1$ is a $G_1$-decomposition of $K_6$.

Example 2.20. Let $V(K_9) = Z_9$ and let

$$\Delta_1 = \{G_1[0, 1, 3, 4] + i : i \in Z_9\} \cup \{G_1[0, 2, 4, 3] + i : i \in Z_9\} \cup \{G_1[0, 5, 6, 8] + i : i \in Z_9\} \cup \{G_1[0, 6, 5, 7] + i : i \in Z_9\}$$

$$\Delta_2 = \{G_2[2, 0, 4, 8] + i : i \in Z_9\} \cup \{G_2[1, 0, 2, 3] + i : i \in Z_9\} \cup \{G_2[3, 0, 5, 8] + i : i \in Z_9\}.$$ 

Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $K_9$.

Example 2.21. Let $V(K_{10}) = Z_{10} \cup \{\infty\}$ and let

$$\Delta_1 = \{G_1[0, \infty, 3, 4] + i : i \in Z_{10}\} \cup \{G_1[2, 0, \infty, 1] + i : i \in Z_{10}\} \cup \{G_1[0, 1, 4, 3] + i : i \in Z_{10}\} \cup \{G_1[0, 4, 3, 1] + i : i \in Z_{10}\} \cup \{G_1[4, 6, 0, 2] + i : i \in Z_{10}\}.$$ 

Then $\Delta_1$ is a $G_1$-decomposition of $K_{10}$.

Example 2.22. Let $V(K_{11}) = Z_{11}$ and let

$$\Delta_1 = \{G_1[0, 1, 8, 9] + i : i \in Z_{11}\} \cup \{G_1[0, 4, 3, 9] + i : i \in Z_{11}\} \cup \{G_1[0, 3, 5, 1] + i : i \in Z_{11}\} \cup \{G_1[0, 5, 4, 2] + i : i \in Z_{11}\} \cup \{G_1[0, 2, 5, 4] + i : i \in Z_{11}\}.$$ 

$$\Delta_2 = \{G_2[1, 0, 2, 6] + i : i \in Z_{11}\} \cup \{G_2[3, 0, 4, 9] + i : i \in Z_{11}\} \cup \{G_2[4, 0, 5, 7] + i : i \in Z_{11}\} \cup 2\{G_2[2, 0, 3, 7] + i : i \in Z_{11}\}.$$ 

Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $K_{11}$.

Example 2.23. Let $V(K_3 \times 2) = Z_6$ with partition $\{\{0, 3\}, \{1, 4\}, \{2, 5\}\}$ and let

$$\Delta_1 = \{G_1[0, 5, 1, 2] + i : i \in Z_6\} \cup \{G_1[0, 4, 2, 1] + i : i \in Z_6\},$$

$$\Delta_2 = \{G_2[0, 1, 2, 4], G_2[0, 1, 5, 4], G_2[0, 2, 1, 5], G_2[0, 5, 4, 2], G_2[3, 4, 2, 1], G_2[3, 4, 5, 1], G_2[3, 2, 4, 5], G_2[3, 5, 1, 2], G_2[1, 3, 2, 0], G_2[1, 3, 5, 0], G_2[4, 0, 2, 3], G_2[4, 0, 5, 3]\}.$$ 

Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $K_3 \times 2$. 
Example 2.24. Let $V(\overline{7K_6} \setminus \overline{7K_2}) = Z_6$ with $V(\overline{7K_2}) = \{0, 1\}$ and let

$$\Delta_1 = \{G_1[5, 3, 2, 4], G_1[3, 2, 4, 5], G_1[2, 1, 3, 5], G_1[0, 2, 3, 4], G_1[1, 3, 5, 2], G_1[4, 1, 5, 0], G_1[4, 2, 1, 3], G_1[5, 0, 2, 1], G_1[5, 0, 4, 1]\}.$$ 

Then $\Delta_1$ is a $G_1$-decomposition of $\overline{7K_6} \setminus \overline{7K_2}$.

Example 2.25. Let $V(\overline{7K_7} \setminus \overline{7K_3}) = Z_7$ with $V(\overline{7K_3}) = \{0, 1, 2\}$ and let

$$\Delta_1 = \{G_1[0, 6, 4, 3], G_1[6, 3, 2, 5], G_1[0, 6, 3, 4], G_1[0, 3, 4, 5], G_1[0, 3, 4, 5], G_1[1, 3, 4, 5], G_1[1, 3, 4, 5], G_1[2, 3, 4, 5], G_1[2, 3, 5, 4], G_1[2, 4, 6, 3], G_1[2, 4, 6, 3], G_1[5, 0, 6, 1], G_1[5, 0, 6, 3], G_1[5, 2, 3, 1], G_1[5, 3, 2, 4], G_1[6, 1, 4, 2], G_1[6, 1, 4, 2], G_1[6, 1, 5, 0]\},$$

$$\Delta_2 = \{G_2[2, 5, 4, 6], G_2[2, 3, 4, 5], G_2[2, 3, 4, 5], G_2[0, 3, 4, 5], G_2[0, 3, 4, 5], G_2[0, 4, 5, 6], G_2[0, 4, 5, 6], G_2[0, 6, 5, 3], G_2[0, 6, 5, 3], G_2[2, 3, 6, 4], G_2[2, 3, 6, 4], G_2[3, 1, 5, 2], G_2[3, 1, 5, 2], G_2[3, 6, 5, 1], G_2[4, 0, 6, 1], G_2[4, 1, 6, 2], G_2[4, 1, 6, 2], G_2[5, 0, 6, 2]\}.$$ 

Then $\Delta_1$ and $\Delta_2$ are respectively $G_1$- and $G_2$-decompositions of $\overline{7K_7} \setminus \overline{7K_3}$.

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Example 2.26. Let $V(\overline{14K_4}) = Z_4$ and let $\Delta_2 = \{G_2[0, 1, 2, 3] + i : i \in Z_4\} \cup 2\{G_2[0, 2, 1, 3] + i : i \in Z_4\}$. Then $\Delta_2$ is a $G_2$-decomposition of $\overline{14K_4}$.

Example 2.27. Let $V(\overline{14K_6}) = Z_5 \cup \{\infty\}$ and let $\Delta_2 = 3\{G_2[0, \infty, 2] + i : i \in Z_5\} \cup \{G_2[0, 1, 2, \infty] + i : i \in Z_6\} \cup 2\{G_2[0, 1, 2, 3] + i : i \in Z_5\}$. Then $\Delta_2$ is a $G_2$-decomposition of $\overline{14K_6}$.

Example 2.28. Let $V(\overline{14K_{10}}) = Z_9 \cup \{\infty\}$ and let

$$\Delta_2 = 2\{G_2[0, 4, \infty, 2] + i : i \in Z_9\} \cup \{G_2[0, 2, \infty, 4] + i : i \in Z_9\} \cup \{G_2[0, 2, 4, 7] + i : i \in Z_9\} \cup \{G_2[0, 1, 4, 2] + i : i \in Z_9\} \cup 4\{G_2[0, 1, 4, 3] + i : i \in Z_9\}.$$ 

Then $\Delta_2$ is a $G_2$-decomposition of $\overline{14K_{10}}$.

Example 2.29. Let $V(\overline{14K_6} \setminus \overline{14K_2}) = Z_4 \cup \{\infty_1, \infty_2\}$ with $V(\overline{14K_2}) = \{\infty_1, \infty_2\}$ and let

$$\Delta_2 = 3\{G_2[0 + i, \infty_1, 2 + i, \infty_2] : i \in Z_4\} \cup \{G_2[0 + i, \infty_2, 2 + i, \infty_1] : i \in Z_4\} \cup \{G_2[0 + i, \infty_2, 2 + i, 1 + i] : i \in Z_4\} \cup 2\{G_2[0 + i, 1 + i, 2 + i, 3 + i] : i \in Z_4\}.$$ 

Then $\Delta_2$ is a $G_2$-decomposition of $\overline{14K_6} \setminus \overline{14K_2}$.
3 Main Results

Through judicious use of the examples from the previous section, we show that the necessary conditions on $G_1$- and $G_2$-designs that we established in Section 1.2 are sufficient for any index $\lambda \geq 2$. In the following constructions, we make extensive use of the join of complete graphs. Of special note is our use of the null graph $K_0$, which has an empty vertex set. For example, $K_7 \vee K_0$ is simply $K_7$. Similarly, $K_7 \setminus K_0$ is also $K_7$. On the other hand, $K_7 \vee K_1 = K_8$, but $K_7 \setminus K_1 = K_7$.

First, we settle the spectra for $G_1$- and $G_2$-designs of certain indices.

Lemma 3.1. Let $G \in \{G_1, G_2\}$. There exists a $G$-decomposition of $2K_n$ if $n \equiv 0$ or 1 (mod 7).

Proof. Let $G \in \{G_1, G_2\}$ and let $n = 7r + t$ for some positive integer $r$ and $t \in \{0, 1\}$. If $(r, t)$ is $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(4, 0)$, or $(4, 1)$, then the result follows from Examples 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6, respectively. The remainder of the proof breaks into two cases.

Case 1: $r$ is odd with $r \geq 3$.

By Theorem 1.3 there exists a $\{K_3, K_3\}$-decomposition of $K_r$. Thus by Theorem 1.7 there exists a $\{K_{3 \times 7}, K_{5 \times 7}\}$-decomposition of $K_{r \times 7}$. Since there exist $G$-decompositions of both $2K_{3 \times 7}$ and $2K_{5 \times 7}$ (by Examples 2.7 and 2.8, respectively), a $G$-decomposition of $2K_{r \times 7}$ also exists by transitivity. Finally, we note that $K_{7r+t} = (rK_7 \vee K_t) \cup K_{r \times 7} = K_{r \times 7} \cup \bigcup_{i=1}^{r} K_{7+t_i}$. Thus $2K_{7r+t} = 2K_{r \times 7} \cup \bigcup_{i=1}^{r} 2K_{7+t_i}$, and the result follows from the existence of $G$-decompositions of $2K_{r \times 7}$, $2K_7$, and $2K_3$.

Case 2: $r$ is even with $r \geq 6$.

Let $r = 2s$ for some integer $s \geq 3$; hence, $n = 14s + t$. By Corollary 1.6 there exists a $K_3$-decomposition either of $K_{s \times 2}$ if $s \not\equiv 2$ (mod 3) or of $K_4, (s-2) \times 2$ otherwise. Thus by Theorem 1.7 there exists a $K_{3 \times 7}$-decomposition of either $K_{s \times 14}$ or $K_{28, (s-2) \times 14}$. Since there exists a $G$-decomposition of $2K_{3 \times 7}$ (by Example 2.7), a $G$-decomposition of either $2K_{s \times 14}$ or $2K_{28, (s-2) \times 14}$ also exists by transitivity. Finally, we note that $K_{14s+t}$ can be described as either $(sK_{14} \vee K_t) \cup K_{s \times 14} = K_{s \times 14} \cup \bigcup_{i=1}^{s} K_{14+t_i}$ or $((K_{28} \cup (s-2)K_{14}) \vee K_t) \cup K_{28, (s-2) \times 14} = K_{28, (s-2) \times 14} \cup K_{28+t} \cup \bigcup_{i=1}^{s} K_{14+t_i}$. Thus, we describe $2K_{14s+t}$ as $2K_{s \times 14} \cup \bigcup_{i=1}^{s} 2K_{14+t_i}$ when $s \not\equiv 2$ (mod 3) and as $2K_{28, (s-2) \times 14} \cup 2K_{28+t} \cup \bigcup_{i=1}^{s} 2K_{14+t_i}$ when $s \equiv 2$ (mod 3), and the result follows from the existence of $G$-decompositions of $2K_{s \times 14}$ or $2K_{28, (s-2) \times 14}$, $2K_{14}, 2K_{15}, 2K_{28}$, and $2K_{29}$.

\[\blacksquare\]
Lemma 3.2. There exists a $G_1$-decomposition of $3K_n$ if $n \equiv 0 \text{ or } 1 \pmod{7}$.

Proof. Let $n = 7r + t$ for some positive integer $r$ and $t \in \{0, 1\}$. If $(r,t)$ is $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(4,0)$, or $(4,1)$, then the result follows from Examples 2.9, 2.10, 2.11, 2.12, 2.13, and 2.14, respectively. The proof then follows as in the proof of Lemma 3.1, where the requisite $G_1$-decompositions of the multipartite graphs $3K_{3 \times 7}$ and $3K_{5 \times 7}$ can be found in Examples 2.15 and 2.16, respectively.

Lemma 3.3. There exists a $G_2$-decomposition of $3K_n$ if $n \equiv 1 \text{ or } 7 \pmod{14}$.

Proof. If $n$ is 7, 15, or 29, then the result follows from Examples 2.9, 2.12, and 2.14, respectively. The remainder of the proof breaks into two cases.

Case 1: $n \equiv 1 \pmod{14}$ with $n \geq 43$.

Let $n = 14r + 1$ for some integer $r \geq 3$; hence, $n = 7(2r) + 1$. By Corollary 1.6 there exists a $K_3$-decomposition either of $K_{r \times 2}$ if $r \not\equiv 2 \pmod{3}$ or of $K_{4_r \times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{7 \times 7}$-decomposition of either $K_{r \times 14}$ or $K_{28 \times (r-2)\times 14}$. Since there exists a $G_2$-decomposition of $3K_{3 \times 7}$ (by Example 2.15), a $G_2$-decomposition of either $3K_{r \times 14}$ or $3K_{28 \times (r-2)\times 14}$ also exists by transitivity. Finally, we note that $K_{14r+1}$ can be described as either $(rK_{14} \cup K_1) \cup K_{r \times 14} = K_{r \times 14} \cup \bigcup_{i=1}^{r-2} K_{15}$ or $((K_{28} \cup (r-2)K_{14}) \cup K_1) \cup K_{28 \times (r-2) \times 14} = K_{28 \times (r-2) \times 14} \cup K_{29} \cup \bigcup_{i=1}^{r-2} K_{15}$. Thus, we describe $3K_{14r+1}$ as $3K_{r \times 14} \cup \bigcup_{i=1}^{r-2} 3K_{15}$ when $r \not\equiv 2 \pmod{3}$ and as $3K_{28 \times (r-2) \times 14} \cup 3K_{29} \cup \bigcup_{i=1}^{r-2} 3K_{15}$ when $r \equiv 2 \pmod{3}$, and the result follows from the existence of $G_2$-decompositions of $3K_{r \times 14}$ or $3K_{28 \times (r-2) \times 14}$, $3K_{15}$, and $3K_{29}$.

Case 2: $n \equiv 7 \pmod{14}$ with $n \geq 21$.

Let $n = 14r + 7$ for some integer $r$; hence, $n = 7(2r+1)$. By Theorem 1.3 there exists a $(K_3, K_5)$-decomposition of $K_{2r+1}$. Thus by Theorem 1.7 there exists a $(K_{3 \times 7}, K_{5 \times 7})$-decomposition of $K_{(2r+1) \times 7}$. Since there exist $G_2$-decompositions of both $3K_{3 \times 7}$ and $3K_{5 \times 7}$ (by Examples 2.15 and 2.16, respectively), a $G_2$-decomposition of $3K_{(2r+1) \times 7}$ also exists by transitivity. Finally, we note that $K_{14r+7} = K_{(2r+1) \times 7} \cup (2r+1)K_7 = K_{(2r+1) \times 7} \cup \bigcup_{i=1}^{2r+1} K_7$. Thus $3K_{14r+7} = 3K_{(2r+1) \times 7} \cup \bigcup_{i=1}^{2r+1} 3K_7$, and the result follows from the existence of $G_2$-decompositions of $3K_{(2r+1) \times 7}$ and $3K_7$.

Lemma 3.4. There exists a $G_1$-decomposition of $7K_n$ if $n \geq 4$.

Proof. Let $n = 4r + t$ for some positive integer $r$ and $t \in \{0,1,2,3\}$. If $(r,t)$ is $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(2,2)$, or $(2,3)$, then the result follows from Examples 2.17, 2.18, 2.19, 2.20, 2.21, and 2.22, respectively. If $(r,t)$
Lemma 3.5. There exists a $G_1$-decomposition of $7K_n$ if $n \geq 5$ and $n$ is odd.

Proof. Let $n = 4r + t$ for some positive integer $r$ and $t \in \{1, 3\}$. If $(r, t)$ is $(1, 1)$, $(2, 1)$, or $(2, 3)$, then the result follows from Examples 2.18, 2.20, and 2.22, respectively. If $(r, t)$ is $(1, 3)$, then $n = 7$, and the result follows from overlaying 2 copies of a $G_2$-decomposition of $7K_7$ (see Lemma 3.1) and 1 copy of a $G_2$-decomposition of $3K_7$ (see Lemma 3.3). For the remainder of the proof, we assume $r \geq 3$, and the proof then follows as in the proof of Lemma 3.4, where the requisite $G_2$-decompositions of $7K_{3\times2}$ and $7K_7 \setminus 7K_3$ can also be found in Examples 2.23 and 2.25, respectively.

Lemma 3.6. There exists a $G_2$-decomposition of $14K_n$ if $n \geq 4$.

Proof. If $n$ is odd, then the result follows from overlaying 2 copies of a $G_2$-decomposition of $7K_n$ (see Lemma 3.5). For the remainder of the proof, we assume $n$ is even. Let $n = 4r + t$ for some positive integer $r$ and $t \in \{0, 2\}$. If $(r, t)$ is $(1, 0)$, $(1, 2)$, or $(2, 2)$, then the result follows from Examples 2.26, 2.27, and 2.28, respectively. If $(r, t)$ is $(2, 0)$, then $n = 8$, and the result follows from overlaying 7 copies of a $G_2$-decomposition of $7K_7$ (see Lemma 3.1). For the remainder of the proof, we assume $r \geq 3$. By Corollary 1.6 there exists a $K_3$-decomposition either of $K_{r\times2}$ if $r \not\equiv 2 \pmod{3}$ or of $K_{4,(r-2)\times2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3\times2}$-decomposition of either $K_{r\times4}$ or $K_{8,(r-2)\times4}$. Since there exists a $G_2$-decomposition of $7K_{3\times2}$ (by Example 2.23), a $G_2$-decomposition of either $14K_{r\times4}$ or $14K_{8,(r-2)\times4}$ also exists by transitivity. Finally, we note that
$K_{4r+t}$ can be described as either $(rK_4 \lor K_t) \cup K_{r \times 4} = K_{r \times 4} \cup K_{4+t} \cup \bigcup_{i=1}^{r-1}(K_{4+t} \setminus K_t)$ or $((K_8 \lor (r-2)K_4) \lor K_t) \cup K_{8r},(r-2)\times 4 = K_{8r},(r-2)\times 4 \cup K_{8+t} \cup \bigcup_{i=1}^{r-2}(K_{4+t} \setminus K_t)$. Thus, we describe $14K_{4r+t}$ as $14K_{r \times 4} \cup 14K_{4+t} \cup \bigcup_{i=1}^{r-1}(14K_{4+t} \setminus 14K_t)$ when $r \not\equiv 0 (\mod 3)$ and as $14K_{8r},(r-2)\times 4 \cup 14K_{8+t} \cup \bigcup_{i=1}^{r-2}(14K_{4+t} \setminus K_t)$ when $r \equiv 2 (\mod 3)$, and the result follows from the existence of $G_2$-decompositions of $14K_{r \times 4}$ or $14K_{8r},(r-2)\times 4$, $14K_4$, $14K_6$, $14K_8$, $14K_{10}$, and $14K_6 \setminus 14K_2$, where the latter decompositions is shown to exist in Example 2.29.

Finally, we settle the spectra for $G_1$- and $G_2$-designs of any index $\lambda$ (at least 2).

**Theorem 3.7.** Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a $G_1$-decomposition of $3K_n$ if and only if the following hold:

- if $\gcd(\lambda, 7) = 1$, then $n \equiv 0$ or 1 (mod 7);
- if $\gcd(\lambda, 7) = 7$, then $n \geq 4$.

**Proof.** The necessity of the given conditions is established in Lemma 1.1. We now show sufficiency. Let $n \geq 4$ and let $\lambda = 7r+t$ for some integers $r \geq 0$ and $t \in \{2, 3, \ldots, 8\}$. In the case where $t = 7$, the result follows from overlaying $r+1$ copies of a $G_1$-decomposition of $7K_n$ (see Lemma 3.4). For the remainder of the proof, we assume $n \equiv 0$ or 1 (mod 7). In the case where $t$ is even, the result follows from overlaying $r$ copies of a $G_1$-decomposition of $7K_n$ (see Lemma 3.4) and $t/2$ copies of a $G_1$-decomposition of $2K_n$ (see Lemma 3.1). In the case where $t$ is odd, the result follows from overlaying $r$ copies of a $G_1$-decomposition of $7K_n$ (see Lemma 3.4), 1 copy of a $G_1$-decomposition of $3K_n$ (see Lemma 3.2), and $(t-3)/2$ copies of a $G_1$-decomposition of $2K_n$ (see Lemma 3.1).

**Theorem 3.8.** Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a $G_2$-decomposition of $3K_n$ if and only if the following hold:

- if $\gcd(\lambda, 14) = 1$, then $n \equiv 1$ or 7 (mod 14);
- if $\gcd(\lambda, 14) = 2$, then $n \equiv 0$ or 1 (mod 7);
- if $\gcd(\lambda, 14) = 7$, then $n \equiv 1$ (mod 2);
- if $\gcd(\lambda, 14) = 14$, then $n \geq 4$.

**Proof.** The necessity of the given conditions is established in Lemma 1.2. We now show sufficiency. Let $n \geq 4$ and let $\lambda = 14r+t$ for some integers $r \geq 0$ and $t \in \{2, 3, \ldots, 15\}$. In the case where $t = 14$, the result follows from overlaying $r+1$ copies of a $G_2$-decomposition of $14K_n$ (see Lemma 3.6). In the case where $t = 7$, we assume that $n$ is odd, and the result follows from overlaying $2r+1$ copies of a $G_2$-decomposition of $7K_n$. In the case where $t \not\equiv 0 (\mod 3)$ and as $14K_{8r},(r-2)\times 4 \cup 14K_{8+t} \cup \bigcup_{i=1}^{r-2}(14K_{4+t} \setminus K_t)$ when $r \equiv 2 (\mod 3)$, and the result follows from the existence of $G_2$-decompositions of $14K_{r \times 4}$ or $14K_{8r},(r-2)\times 4$, $14K_4$, $14K_6$, $14K_8$, $14K_{10}$, and $14K_6 \setminus 14K_2$, where the latter decompositions is shown to exist in Example 2.29. □
For the remainder of the proof, we assume $n \equiv 0$ or 1 (mod 7). In the case where $t$ is even, the result follows from overlaying $r$ copies of a $G_2$-decomposition of $14K_n$ (see Lemma 3.6) and $t/2$ copies of a $G_2$-decomposition of $2K_n$ (see Lemma 3.1). In the case where $t$ is odd, we assume that $n$ is odd, and the result follows from overlaying $r$ copies of a $G_2$-decomposition of $14K_n$ (see Lemma 3.6), 1 copy of a $G_2$-decomposition of $3K_n$ (see Lemma 3.3), and $(t-3)/2$ copies of a $G_2$-decomposition of $2K_n$ (see Lemma 3.1).

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