

Poisson process approximation for dependent superposition of point processes

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Although the study of weak convergence of superpositions of point processes to the Poisson process dates back to the work of Grigelionis in 1963, it was only recently that Schuhmacher [*Stochastic Process. Appl.* **115** (2005) 1819–1837] obtained error bounds for the weak convergence. Schuhmacher considered dependent superposition, truncated the individual point processes to 0–1 point processes and then applied Stein’s method to the latter. In this paper, we adopt a different approach to the problem by using Palm theory and Stein’s method, thereby expressing the error bounds in terms of the mean measures of the individual point processes, which is not possible with Schuhmacher’s approach. We consider locally dependent superposition as a generalization of the locally dependent point process introduced in Chen and Xia [*Ann. Probab.* **32** (2004) 2545–2569] and apply the main theorem to the superposition of thinned point processes and of renewal processes.

Keywords: dependent superposition of point processes; Poisson process approximation; renewal processes; sparse point processes; Stein’s method; thinned point processes

1. Introduction

The study of weak convergence of superpositions of point processes dates back to Grigelionis [22] who proved that the superposition of independent sparse point processes converges weakly to a Poisson process on the carrier space \mathbb{R}_+ . His result was subsequently extended to more general carrier spaces by Goldman [19] and Jagers [23]; see [15] and [9] for further discussion. It was further extended to superpositions of dependent sparse point processes by Banys [1,3], Kallenberg [24], Brown [10] and Banys [2]. For a systematic account of these developments, see [25].

Surprisingly, it was only recently that error bounds for such convergence of point processes were studied. Using Stein’s method for Poisson process approximation, as developed by Barbour [4] and Barbour and Brown [5], Schuhmacher [29] obtained an error bound on the d_2 Wasserstein distance between a sum of weakly dependent sparse point processes $\{\xi_{ni}, 1 \leq i \leq k_n\}_{n \in \mathbb{N}}$ and an approximating Poisson process. As he truncated the sparse point processes to 0–1 point processes, as in the proof of Grigelionis’ theorem, his error bound contains the term $\sum_{i=1}^{k_n} \mathbb{P}[\xi_{ni}(B) \geq 2]$, whose convergence to 0 for every bounded Borel subset B of the carrier space is a condition for Grigelionis’ theorem to hold. A consequence of such truncation is that the mean measure of the approximating Poisson process is not equal to the sum of the mean measures of the individual point processes.

In this paper, we adopt a different approach to Poisson process approximation in which we do not use the truncation, but apply Palm theory and express the error bounds in terms of the mean measures of the individual sparse point processes. Such an approach also ensures that the mean measure of the approximating Poisson process is equal to the sum of the mean measures of the sparse point processes.

As in [29], we study the dependent superposition of sparse point processes. But we consider only locally dependent superposition, which is a natural extension of the point processes $\sum I_i \delta_{U_i}$ studied in [14], Section 4, where δ_x is the point mass at x , the U_i 's are \mathcal{S} -valued independent random elements with \mathcal{S} a locally compact metric space, the indicators I_i 's are locally dependent and the U_i 's are independent of the I_i 's.

In our main theorem (Theorem 2.1), with the help of Brown, Weinberg and Xia [11], Lemma 3.1, it is possible to recover a factor of order $1/\lambda$ from the term $1/(|\Xi^{(i)}| + 1)$. Hence, the error bound on the d_2 Wasserstein distance yields the so-called Stein factor $1/\lambda$, by which approximation remains good for large λ , a feature always sought after for Poisson-type approximations. In the error bound obtained by Schuhmacher [29], a leading term does not have the Stein factor; see Remark 4.4 for further details.

Our main theorem and some corollaries are presented in Section 2. Applications to thinned point processes and renewal processes are given in Sections 3 and 4, respectively.

2. The main theorem

Throughout this paper, we assume that Γ is a locally compact metric space with metric d_0 bounded by 1. In estimating the error of Poisson process approximation to the superposition of dependent point processes $\{\Xi_i, i \in \mathcal{I}\}$ on the carrier space Γ with \mathcal{I} a finite or countably infinite index set, one natural approach is to partition the index set \mathcal{I} into $\{\{i\}, \mathcal{I}_i^s, \mathcal{I}_i^w\}$, where \mathcal{I}_i^s is the set of indices of the point processes which are strongly dependent on Ξ_i and \mathcal{I}_i^w the set of the indices of the point processes which are weakly dependent on Ξ_i ; see [29]. Another approach is to divide the index set according to various levels of local dependence, a successful structure for studying normal approximation; see [13]. The latter approach has been generalized by Barbour and Xia [8] to randomly indexed sums with a particular interest in random variables resulting from integrating a random field with respect to a point process.

Parallel to the local dependence structures defined in [13], we introduce the following:

[LD1] for each $i \in \mathcal{I}$, there exists a neighborhood A_i such that $i \in A_i$ and Ξ_i is independent of $\{\Xi_j, j \in A_i^c\}$;

[LD2] condition [LD1] holds and for each $i \in \mathcal{I}$, there exists a neighborhood B_i such that $A_i \subset B_i$ and $\{\Xi_j, j \in A_i\}$ is independent of $\{\Xi_j, i \in B_i^c\}$.

The index set \mathcal{I} in [LD1] and [LD2] will be assumed to be finite or countably infinite in this paper, although it may be as general as that considered in [8]. The superposition of $\{\Xi_i : i \in \mathcal{I}\}$ which satisfies the condition [LD1] is more general than point processes of the form $\sum I_i \delta_{U_i}$, where the I_i 's are locally dependent indicators with one level of dependent neighborhoods in \mathcal{I} (i.e., the I_i 's satisfy [LD1] in [13], page 1986). Such a point process is a typical example of locally dependent point processes defined in [13], page 2548. Likewise, the superposition of

$\{\Xi_i : i \in \mathcal{I}\}$ which satisfies the condition [LD2] is more general than point processes of the form $\sum I_i \delta_{U_i}$, where the I_i 's are locally dependent indicators with two levels of dependent neighborhoods in \mathcal{I} (i.e., the I_i 's satisfy [LD2] in [13], page 1986).

Three metrics will be used to describe the accuracy of Poisson process approximation: the total variation metric for Poisson random variable approximation d_{tv} ; the total variation metric for Poisson process approximation d_{TV} ; and a Wasserstein metric d_2 (see [7] or [32]).

To briefly define these metrics, let \mathcal{H} be the space of all finite point process configurations on Γ , that is, each $\xi \in \mathcal{H}$ is a non-negative integer-valued finite measure on Γ . Let \mathcal{K} stand for the set of d_0 -Lipschitz functions $k : \Gamma \rightarrow [-1, 1]$ such that $|k(\alpha) - k(\beta)| \leq d_0(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$. The first Wasserstein metric d_1 on \mathcal{H} is defined by

$$d_1(\xi_1, \xi_2) = \begin{cases} 0, & \text{if } |\xi_1| = |\xi_2| = 0, \\ 1, & \text{if } |\xi_1| \neq |\xi_2|, \\ |\xi_1|^{-1} \sup_{k \in \mathcal{K}} \left| \int k \, d\xi_1 - \int k \, d\xi_2 \right|, & \text{if } |\xi_1| = |\xi_2| > 0, \end{cases}$$

where $|\xi_i|$ is the total mass of ξ_i . A metric d'_1 equivalent to d_1 can be defined as follows (see [12]): for two configurations $\xi_1 = \sum_{i=1}^n \delta_{y_i}$ and $\xi_2 = \sum_{i=1}^m \delta_{z_i}$ with $m \geq n$,

$$d'_1(\xi_1, \xi_2) = \min_{\pi} \sum_{i=1}^n d_0(y_i, z_{\pi(i)}) + (m - n),$$

where π ranges over all permutations of $(1, \dots, m)$. Both d_1 and d'_1 generate the weak topology on \mathcal{H} (see [32], Proposition 4.2) and we use $\mathcal{B}(\mathcal{H})$ to stand for the Borel σ -algebra generated by the weak topology. Define three subsets of real-valued functions on \mathcal{H} : $\mathcal{F}_{\text{tv}} = \{1_A(|\xi|) : A \subset \mathbf{Z}_+\}$, $\mathcal{F}_{d_1} = \{f : |f(\xi_1) - f(\xi_2)| \leq d_1(\xi_1, \xi_2) \text{ for all } \xi_1, \xi_2 \in \mathcal{H}\}$ and $\mathcal{F}_{\text{TV}} = \{1_A(\xi) : A \in \mathcal{B}(\mathcal{H})\}$. The pseudo-metric d_{tv} and the metrics d_2 and d_{TV} are then defined on probability measures on \mathcal{H} by

$$\begin{aligned} d_{\text{tv}}(\mathbf{Q}_1, \mathbf{Q}_2) &= \inf_{(X_1, X_2)} \mathbb{P}(|X_1| \neq |X_2|) = \sup_{f \in \mathcal{F}_{\text{tv}}} \left| \int f \, d\mathbf{Q}_1 - \int f \, d\mathbf{Q}_2 \right|, \\ d_2(\mathbf{Q}_1, \mathbf{Q}_2) &= \inf_{(X_1, X_2)} \mathbb{E}[d_1(X_1, X_2)] \\ &= \sup_{f \in \mathcal{F}_{d_1}} \left| \int f \, d\mathbf{Q}_1 - \int f \, d\mathbf{Q}_2 \right|, \\ d_{\text{TV}}(\mathbf{Q}_1, \mathbf{Q}_2) &= \inf_{(X_1, X_2)} \mathbb{P}(X_1 \neq X_2) \\ &= \sup_{f \in \mathcal{F}_{\text{TV}}} \left| \int f \, d\mathbf{Q}_1 - \int f \, d\mathbf{Q}_2 \right|, \end{aligned}$$

where the infima are taken over all *couplings* of (X_1, X_2) such that $\mathcal{L}(X_i) = \mathbf{Q}_i$, $i = 1, 2$, and the second equations are due to the duality theorem; see [27], page 168.

To bound the error of Poisson process approximation, we need the Palm distributions \mathcal{Q}_α of a point process X_2 with respect to a point process X_1 with finite mean measure ν at α . When X_1

is a *simple* point process, that is, it has at most one point at each location, the Palm distribution Q_α may be intuitively interpreted as the conditional distribution of X_2 given that X_1 has a point at α . More precisely, let $\mathcal{B}(\Gamma)$ denote the Borel σ -algebra in Γ generated by the metric d_0 and define the Campbell measure C of (X_1, X_2) on $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathcal{H})$:

$$C(B \times M) = \mathbb{E}[X_1(B)1_{X_2 \in M}], \quad B \in \mathcal{B}(\Gamma), M \in \mathcal{B}(\mathcal{H}).$$

Since the mean measure ν of X_1 is finite, by Kallenberg [25], 15.3.3, there exist probability measures Q_α on $\mathcal{B}(\mathcal{H})$ such that

$$\mathbb{E}[X_1(B)1_{X_2 \in M}] = \int_B Q_\alpha(M)\nu(d\alpha) \quad \forall B \in \mathcal{B}(\Gamma), M \in \mathcal{B}(\mathcal{H}), \quad (2.1)$$

which is equivalent to

$$Q_\alpha(M) = \frac{\mathbb{E}\{1_{[X_2 \in M]}X_1(d\alpha)\}}{\nu(d\alpha)} \quad \forall M \in \mathcal{B}(\mathcal{H}), \alpha \in \Gamma \text{ } \nu\text{-a.s.};$$

see [25], Section 10.1. It is possible to realize a family of point processes Y_α on some probability space such that $Y_\alpha \sim Q_\alpha$ and we say that Y_α is a *Palm process* of X_2 with respect to X_1 at α . Moreover, when $X_1 = X_2$, we call the point process $Y_\alpha - \delta_\alpha$ the *reduced Palm process* of X_2 at α ; see [25], Lemma 10.2.

As noted in [21], when Γ is reduced to one point only, the Palm distribution of X_2 (with respect to itself) is the same as the size-biased distribution; general guidelines for the construction of size-biased variables are investigated in [20].

Theorem 2.1. *Let $\{\Xi_i, i \in \mathcal{I}\}$ be a collection of point processes on Γ with respective mean measures $\lambda_i, i \in \mathcal{I}$. Set $\Xi = \sum_{i \in \mathcal{I}} \Xi_i$ with mean measure denoted by λ and assume that $\lambda := \lambda(\Gamma) < \infty$. If [LD1] holds, then*

$$d_{\text{TV}}(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \frac{1 - e^{-\lambda}}{\lambda} \mathbb{E} \sum_{i \in \mathcal{I}} \int_{\Gamma} \{ ||V_i| - |V_{i,\alpha}| | + ||\Xi_i| - |\Xi_{i,(\alpha)}| | \} \lambda_i(d\alpha), \quad (2.2)$$

$$\begin{aligned} d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) &\leq \mathbb{E} \sum_{i \in \mathcal{I}} \left(\frac{3.5}{\lambda} + \frac{2.5}{|\Xi^{(i)}| + 1} \right) \int_{\Gamma} d'_1(V_i, V_{i,\alpha}) \lambda_i(d\alpha) \\ &\quad + \sum_{i \in \mathcal{I}} \left(\frac{3.5}{\lambda} + \mathbb{E} \frac{2.5}{|\Xi^{(i)}| + 1} \right) \mathbb{E} \int_{\Gamma} d'_1(\Xi_i, \Xi_{i,(\alpha)}) \lambda_i(d\alpha), \end{aligned} \quad (2.3)$$

$$d_{\text{TV}}(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \mathbb{E} \sum_{i \in \mathcal{I}} \int_{\Gamma} \{ \|V_i - V_{i,\alpha}\| + \|\Xi_i - \Xi_{i,(\alpha)}\| \} \lambda_i(d\alpha), \quad (2.4)$$

where $\Xi^{(i)} = \sum_{j \in A_i^c} \Xi_j$, $V_i = \sum_{j \in A_i \setminus \{i\}} \Xi_j$, $\Xi_{i,(\alpha)}$ is the reduced Palm process of Ξ_i at α , $V_{i,\alpha}$ is the Palm process of V_i with respect to Ξ_i at α such that $\Xi^{(i)} + V_{i,\alpha} + \Xi_{i,(\alpha)} + \delta_\alpha$ is the Palm

process of Ξ with respect to Ξ_i at α and $\|\cdot\|$ denotes the variation norm of signed measure. Under the condition [LD2], (2.2) and (2.4) remain the same, but (2.3) can be further reduced to

$$d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \sum_{i \in \mathcal{I}} \left(\frac{3.5}{\lambda} + \mathbb{E} \frac{2.5}{\sum_{j \in B_i^c} |\Xi_j| + 1} \right) \mathbb{E} \int_{\Gamma} d'_1(V_i, V_{i,\alpha}) \lambda_i(d\alpha) + \sum_{i \in \mathcal{I}} \left(\frac{3.5}{\lambda} + \mathbb{E} \frac{2.5}{|\Xi^{(i)}| + 1} \right) \mathbb{E} \int_{\Gamma} d'_1(\Xi_i, \Xi_{i,(\alpha)}) \lambda_i(d\alpha) \tag{2.5}$$

$$\leq \sum_{i \in \mathcal{I}} \left(\frac{3.5}{\lambda} + 2.5 \cdot \frac{\sqrt{\kappa_i(1 + \kappa_i/4)} + 1 + \kappa_i/2}{\sum_{j \in B_i^c} \lambda_j + 1} \right) \times \{ \lambda_i \mathbb{E}|V_i| + \mathbb{E}(|V_i| \cdot |\Xi_i|) + \lambda_i^2 + \mathbb{E}(|\Xi_i|^2) - \lambda_i \}, \tag{2.6}$$

where $\lambda_i = \lambda_i(\Gamma)$ and

$$\kappa_i = \frac{\sum_{j_1 \in B_i^c} \sum_{j_2 \in B_i^c \cap A_{j_1}} \text{cov}(|\Xi_{j_1}|, |\Xi_{j_2}|)}{\sum_{j \in B_i^c} \lambda_j + 1}.$$

Proof. We employ Stein’s method for Poisson process approximation, established in [4] and [5], to prove the theorem. To this end, for a suitable measurable function h on \mathcal{H} , let

$$\mathcal{A}h(\xi) = \int_{\Gamma} [h(\xi + \delta_{\alpha}) - h(\xi)] \lambda(d\alpha) + \int_{\Gamma} [h(\xi - \delta_x) - h(\xi)] \xi(dx).$$

Then \mathcal{A} defines a generator of the spatial immigration–death process with immigration intensity λ and unit per capita death rate, and the equilibrium distribution of the spatial immigration–death process is $\text{Po}(\lambda)$; see [32], Section 3.2, for more details. The Stein equation based on \mathcal{A} is

$$\mathcal{A}h(\xi) = f(\xi) - \text{Po}(\lambda)(f) \tag{2.7}$$

with solution

$$h_f(\xi) = - \int_0^{\infty} [\mathbb{E}f(\mathbf{Z}_{\xi}(t)) - \text{Po}(\lambda)(f)] dt,$$

where $\{\mathbf{Z}_{\xi}(t), t \geq 0\}$ is the spatial immigration–death process with generator \mathcal{A} and initial configuration $\mathbf{Z}_{\xi}(0) = \xi$. To obtain bounds on the errors in the approximation, we need to define

$$\begin{aligned} \Delta h_f(\xi; x) &:= h_f(\xi + \delta_x) - h_f(\xi), \\ \Delta^2 h_f(\xi; x, y) &:= \Delta h_f(\xi + \delta_x; y) - \Delta h_f(\xi; y), \\ \Delta^2 h_f(\xi, \eta; x) &:= \Delta h_f(\xi; x) - \Delta h_f(\eta; x), \end{aligned}$$

for corresponding test functions f . Xia [32], Propositions 5.6 and 5.12 (see [5,6]) and Lemma 5.26, state that, for all $x, y \in \Gamma$,

$$|\Delta^2 h_f(\xi; x, y)| \leq \frac{1 - e^{-\lambda}}{\lambda} \quad \forall f \in \mathcal{F}_{\text{TV}}, \quad (2.8)$$

$$|\Delta^2 h_f(\xi; x, y)| \leq 1 \quad \forall f \in \mathcal{F}_{\text{TV}}, \quad (2.9)$$

$$|\Delta^2 h_f(\xi, \eta; x)| \leq \left(\frac{3.5}{\lambda} + \frac{2.5}{|\eta| \wedge |\xi| + 1} \right) d'_1(\xi, \eta) \quad \forall f \in \mathcal{F}_{d_1}. \quad (2.10)$$

Now, since $\Xi^{(i)} + V_{i,\alpha} + \Xi_{i,(\alpha)} + \delta_\alpha$ is the Palm process of Ξ with respect to Ξ_i at α , it follows from (2.1) that

$$\begin{aligned} \mathbb{E} \int_{\Gamma} [h(\Xi) - h(\Xi - \delta_\alpha)] \Xi(d\alpha) &= \sum_{i \in \mathcal{I}} \mathbb{E} \int_{\Gamma} [h(\Xi) - h(\Xi - \delta_\alpha)] \Xi_i(d\alpha) \\ &= \sum_{i \in \mathcal{I}} \mathbb{E} \int_{\Gamma} \Delta h(\Xi^{(i)} + V_{i,\alpha} + \Xi_{i,(\alpha)}; \alpha) \lambda_i(d\alpha). \end{aligned}$$

On the other hand, by the Stein equation (2.7), we have

$$\begin{aligned} &|\mathbb{E} f(\Xi) - \text{Po}(\lambda)(f)| \\ &= \left| \mathbb{E} \int_{\Gamma} [h_f(\Xi + \delta_\alpha) - h_f(\Xi)] \lambda(d\alpha) + \mathbb{E} \int_{\Gamma} [h_f(\Xi - \delta_x) - h_f(\Xi)] \Xi(dx) \right| \\ &= \left| \sum_{i \in \mathcal{I}} \mathbb{E} \int_{\Gamma} \{ \Delta h_f(\Xi; \alpha) - \Delta h_f(\Xi^{(i)} + V_{i,\alpha} + \Xi_{i,(\alpha)}; \alpha) \} \lambda_i(d\alpha) \right| \end{aligned} \quad (2.11)$$

$$\begin{aligned} &\leq \sum_{i \in \mathcal{I}} \mathbb{E} \int_{\Gamma} \{ |\Delta h_f(\Xi^{(i)} + V_i + \Xi_i; \alpha) - \Delta h_f(\Xi^{(i)} + V_{i,\alpha} + \Xi_i; \alpha)| \\ &\quad + |\Delta h_f(\Xi^{(i)} + V_{i,\alpha} + \Xi_i; \alpha) - \Delta h_f(\Xi^{(i)} + V_{i,\alpha} + \Xi_{i,(\alpha)}; \alpha)| \} \lambda_i(d\alpha). \end{aligned} \quad (2.12)$$

To prove (2.2), we note that the test functions $f \in \mathcal{F}_{\text{TV}}$ satisfy $f(\xi) = f(|\xi| \delta_z)$ for a fixed point $z \in \Gamma$ and so we have $h_f(\xi) = h_f(|\xi| \delta_z)$. Hence, for all $\eta, \xi_1, \xi_2 \in \mathcal{H}$,

$$\begin{aligned} &|\Delta h_f(\eta + \xi_1; \alpha) - \Delta h_f(\eta + \xi_2; \alpha)| \\ &= |\Delta h_f(\eta + (|\xi_1| \vee |\xi_2|) \delta_z; \alpha) - \Delta h_f(\eta + (|\xi_1| \wedge |\xi_2|) \delta_z; \alpha)| \\ &\leq \sum_{j=1}^{||\xi_1| - |\xi_2||} |\Delta^2 h_f(\eta + (|\xi_1| \wedge |\xi_2| + j - 1) \delta_z; z, \alpha)| \leq ||\xi_1| - |\xi_2|| \frac{1 - e^{-\lambda}}{\lambda}, \end{aligned} \quad (2.13)$$

where the last inequality is due to (2.8). Combining (2.13) with (2.12) yields (2.2).

Next, (2.10) and (2.11) imply that for $f \in \mathcal{F}_{d_1}$,

$$|\mathbb{E}f(\Xi) - \text{Po}(\lambda)(f)| \leq \sum_{i \in \mathcal{I}} \mathbb{E} \int_{\Gamma} \left(\frac{3.5}{\lambda} + \frac{2.5}{|\Xi^{(i)}| + 1} \right) d'_1(V_i + \Xi_i, V_{i,\alpha} + \Xi_{i,(\alpha)}) \lambda_i(d\alpha).$$

Because $d'_1(V_i + \Xi_i, V_{i,\alpha} + \Xi_{i,(\alpha)}) \leq d'_1(V_i, V_{i,\alpha}) + d'_1(\Xi_i, \Xi_{i,(\alpha)})$ and, for each $i \in \mathcal{I}$, Ξ_i is independent of $\Xi^{(i)}$, (2.3) follows. On the other hand, due to the independence between $\{V_i, \Xi_i\}$ and $\{\Xi_j, j \in B_i^c\}$ implied by [LD2], (2.5) is immediate. To prove (2.6), one can verify that

$$\text{Var}\left(\sum_{j \in B_i^c} |\Xi_j|\right) = \sum_{j_1 \in B_i^c} \sum_{j_2 \in B_i^c \cap A_{j_1}} \text{cov}(|\Xi_{j_1}|, |\Xi_{j_2}|)$$

and that $\mathbb{E} \sum_{j \in B_i^c} |\Xi_j| = \sum_{j \in B_i^c} \lambda_j$. Hence, (2.6) follows from Lemma 3.1 in [11] and the facts that $d'_1(V_i, V_{i,\alpha}) \leq |V_i| + |V_{i,\alpha}|$ and $d'_1(\Xi_i, \Xi_{i,(\alpha)}) \leq |\Xi_i| + |\Xi_{i,(\alpha)}|$.

Finally, we show (2.4). For $\xi_1, \xi_2 \in \mathcal{H}$, we define

$$\xi_1 \wedge \xi_2 = \sum_{j=1}^k (a_{1j} \wedge a_{2j}) \delta_{x_j},$$

where $\{x_1, \dots, x_k\}$ is the support of the point measure $\xi_1 + \xi_2$, so that $\xi_i = \sum_{j=1}^k a_{ij} \delta_{x_j}$ for $i = 1, 2$ with the a_{ij} 's being non-negative integers. Then, for all $f \in \mathcal{F}_{\text{TV}}, \eta, \xi_1, \xi_2 \in \mathcal{H}$,

$$\begin{aligned} & |\Delta h_f(\eta + \xi_1; \alpha) - \Delta h_f(\eta + \xi_2; \alpha)| \\ & \leq |\Delta h_f(\eta + \xi_1; \alpha) - \Delta h_f(\eta + \xi_1 \wedge \xi_2; \alpha)| \\ & \quad + |\Delta h_f(\eta + \xi_2; \alpha) - \Delta h_f(\eta + \xi_1 \wedge \xi_2; \alpha)| \\ & \leq (|\xi_1| - |\xi_1 \wedge \xi_2|) + (|\xi_2| - |\xi_1 \wedge \xi_2|) \\ & = \|\xi_1 - \xi_2\|, \end{aligned} \tag{2.14}$$

where the last inequality is due to (2.9). Applying (2.14) in (2.12), we obtain (2.4). □

Corollary 2.2. *With the notation of Theorem 2.1, if $\{\Xi_i, i \in \mathcal{I}\}$ are all independent, then*

$$d_{\text{TV}}(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \frac{1 - e^{-\lambda}}{\lambda} \mathbb{E} \sum_{i \in \mathcal{I}} \int_{\Gamma} \left| |\Xi_i| - |\Xi_{i,(\alpha)}| \right| \lambda_i(d\alpha), \tag{2.15}$$

$$d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \sum_{i \in \mathcal{I}} \left(\frac{3.5}{\lambda} + \mathbb{E} \frac{2.5}{\sum_{j \neq i} |\Xi_j| + 1} \right) \mathbb{E} \int_{\Gamma} d'_1(\Xi_i, \Xi_{i,(\alpha)}) \lambda_i(d\alpha) \tag{2.16}$$

$$\leq \left(\frac{3.5}{\lambda} + 2.5 \cdot \frac{\sqrt{\kappa(1 + \kappa/4)} + 1 + \kappa/2}{\lambda - \max_{j \in \mathcal{I}} \lambda_j + 1} \right) \sum_{i \in \mathcal{I}} \{\lambda_i^2 + \mathbb{E}(|\Xi_i|^2) - \lambda_i\}, \tag{2.17}$$

$$d_{TV}(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \mathbb{E} \sum_{i \in \mathcal{I}} \int_{\Gamma} \|\Xi_i - \Xi_{i,(\alpha)}\| \lambda_i(d\alpha), \tag{2.18}$$

where $\kappa = \frac{\sum_{i \in \mathcal{I}} \text{Var}(|\Xi_i|)}{\lambda - \max_{j \in \mathcal{I}} \lambda_j + 1}$.

Proof. Let $A_i = B_i = \{i\}$, then (2.15)–(2.18) follow from (2.2), (2.5), (2.6) and (2.4), respectively. \square

Corollary 2.3 (cf. [14], Theorem 4.1). Let $\{I_i, i \in \mathcal{I}\}$ be dependent indicators with \mathcal{I} a finite or countably infinite index set and let $\{\mathcal{U}_i, i \in \mathcal{I}\}$ be Γ -valued independent random elements independent of $\{I_i, i \in \mathcal{I}\}$. Define $\Xi = \sum_{i \in \mathcal{I}} I_i \delta_{\mathcal{U}_i}$ with mean measure λ , let $\mathbb{E}I_i = p_i$ and assume that $\lambda = \sum_{i \in \mathcal{I}} p_i < \infty$. For each $i \in \mathcal{I}$, let A_i be the set of indices of those I_j 's which are dependent on I_i , that is, I_i is independent of $\{I_j : j \in A_i^c\}$. Then,

$$d_{TV}(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i \in \mathcal{I}} \left\{ \sum_{j \in A_i \setminus \{i\}} \mathbb{E}I_i I_j + \sum_{j \in A_i} p_i p_j \right\}, \tag{2.19}$$

$$d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \mathbb{E} \sum_{i \in \mathcal{I}} \sum_{j \in A_i \setminus \{i\}} \left(\frac{3.5}{\lambda} + \frac{2.5}{S_i + 1} \right) I_i I_j + \sum_{i \in \mathcal{I}} \sum_{j \in A_i} \left(\frac{3.5}{\lambda} + \mathbb{E} \left[\frac{2.5}{S_i + 1} \mid I_j = 1 \right] \right) p_i p_j, \tag{2.20}$$

$$d_{TV}(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \sum_{i \in \mathcal{I}} \left\{ \sum_{j \in A_i \setminus \{i\}} \mathbb{E}I_i I_j + \sum_{j \in A_i} p_i p_j \right\}, \tag{2.21}$$

where $S_i = \sum_{j \notin A_i} I_j$. For each $i \in \mathcal{I}$, let B_i be the set of indices of those I_l 's which are dependent on $\{I_j, j \in A_i\}$ so that $\{I_j : j \in A_i\}$ is independent of $\{I_l : l \in B_i^c\}$. Then, (2.19) and (2.21) remain the same, but (2.20) can be further reduced to

$$d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \sum_{i \in \mathcal{I}} \sum_{j \in A_i \setminus \{i\}} \left(\frac{3.5}{\lambda} + \mathbb{E} \frac{2.5}{W_i + 1} \right) \mathbb{E}(I_i I_j) + \sum_{i \in \mathcal{I}} \sum_{j \in A_i} \left(\frac{3.5}{\lambda} + \mathbb{E} \frac{2.5}{W_i + 1} \right) p_i p_j \tag{2.22}$$

$$\leq \sum_{i \in \mathcal{I}} \left(\frac{3.5}{\lambda} + 2.5 \cdot \frac{\sqrt{\kappa_i(1 + \kappa_i/4)} + 1 + \kappa_i/2}{\sum_{j \in B_i^c} p_j + 1} \right) \times \left(\sum_{j \in A_i \setminus \{i\}} \mathbb{E}I_i I_j + \sum_{j \in A_i} p_i p_j \right), \tag{2.23}$$

where $W_i = \sum_{j \notin B_i} I_j$ and

$$\kappa_i = \frac{\sum_{j_1 \in B_i^c} \sum_{j_2 \in B_i^c \cap A_{j_1}} \text{cov}(I_{j_1}, I_{j_2})}{\sum_{j \in B_i^c} p_j + 1}.$$

Proof. If we set $\Xi_i = I_i \delta_{\mathcal{U}_i}$, $i \in \mathcal{I}$, then Ξ_i is independent of $\{\Xi_j : j \notin A_i\}$, so [LD1] holds, $\Xi_{i,(\alpha)} = 0$, and the claims (2.19)–(2.21) follow from (2.2)–(2.4), respectively. On the other hand, $\{\Xi_j : j \in A_i\}$ is independent of $\{\Xi_j : j \notin B_i\}$, so [LD2] holds and (2.22) and (2.23) are direct consequences of (2.5) and (2.6). \square

A typical example of Poisson process approximation is that of the Bernoulli process defined as follows (see [32], Section 6.1, for further discussion). Let I_1, \dots, I_n be independent indicators with

$$\mathbb{P}(I_i = 1) = 1 - \mathbb{P}(I_i = 0) = p_i, \quad i = 1, \dots, n.$$

Let $\Gamma = [0, 1]$, $\Xi = \sum_{i=1}^n I_i \delta_{i/n}$ and $\lambda = \sum_{i=1}^n p_i \delta_{i/n}$ be the mean measure of Ξ . If we set $\Xi_i = I_i \delta_{i/n}$, $i = 1, \dots, n$, then the reduced Palm process of Ξ_i at $\alpha \in \Gamma$ is $\Xi_{i,(\alpha)} = 0$ and the Palm distribution of Ξ_j with respect to point process Ξ_i at α for $j \neq i$ is the same as that of Ξ_j . Hence, Corollary 2.2, together with (2.16) and [14], Proposition 4.5, can be used to obtain immediately the following (known) result.

Example 2.4 ([32], Section 6.1). For the Bernoulli process Ξ on $\Gamma = [0, 1]$ with mean measure λ ,

$$\begin{aligned} d_{\text{tv}}(\mathcal{L}(\Xi), \text{Po}(\lambda)) &\leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^n p_i^2, \\ d_{\text{TV}}(\mathcal{L}(\Xi), \text{Po}(\lambda)) &\leq \sum_{i=1}^n p_i^2, \\ d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) &\leq \frac{6}{\lambda - \max_{1 \leq i \leq n} p_i} \sum_{i=1}^n p_i^2. \end{aligned}$$

Example 2.5. Throw n points uniformly and independently onto the interval $[0, n]$ and let Ξ be the configuration of the points on $[0, T] := \Gamma$ with $n \gg T$ and λ be the mean measure of Ξ . Then,

$$d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \frac{6T}{n - 1}.$$

Proof. Let $I_i = 1$ if the i th point is in Γ and 0 if it is not in Γ . The configuration of the i th point on Γ can then be written as $\Xi_i = I_i \delta_{\mathcal{U}_i}$ and $\Xi = \sum_{i=1}^n \Xi_i$, where the \mathcal{U}_i 's are independent and identically distributed uniform random variables on Γ and are independent of the I_i 's. Noting

that the reduced Palm process $\Xi_{i,(\alpha)} = 0$, we obtain the bound by applying (2.22) with $p_i = \mathbb{P}(I_i = 1) = T/n$ and [14], Proposition 4.5. \square

3. Superposition of thinned dependent point processes

Assume that q is a measurable retention function on Γ and X is a point process on Γ . For a realization $X(\omega)$ of X , we thin its points as follows. For each point of $X(\omega)$ at α , it is retained with probability $q(\alpha)$ and discarded with probability $1 - q(\alpha)$, independently of the other points; see [16], page 554, for dependent thinning and [30] for discussions of more thinning strategies. The thinned configuration is denoted by $X_q(\omega)$. For retention functions q_1, q_2, \dots, q_n , let $\sum_{i=1}^n X'_{q_i}$ be the process arising from the superposition of independent realizations of $X_{q_1}, X_{q_2}, \dots, X_{q_n}$, that is, $X'_{q_1}, X'_{q_2}, \dots, X'_{q_n}$ are independent and $\mathcal{L}(X'_{q_i}) = \mathcal{L}(X_{q_i})$ for $i = 1, \dots, n$. Fichtner [18] showed that a sequence of such superpositions, obtained from the rows of an infinitesimal array of retention functions, converges to a Poisson process under standard conditions; see also [25], Exercise 8.8). Serfozo [28] presented convergence theorems for sums of dependent point processes that are randomly thinned by a two-step procedure which deletes each entire point process with a given probability and for each retained point process, points are deleted or retained according to another thinning strategy. Necessary and sufficient conditions are given for a sum of two-step thinned point processes to converge in distribution and the limit is shown to be a Cox process; see also [17] and [26].

For simplicity, we assume that $\{\Xi_i, i \in \mathcal{I}\}$ is a locally dependent collection of point processes (satisfying [LD1]) on a locally compact metric space Γ with metric d_0 bounded by 1. For each point of Ξ_i , we delete the point with probability $1 - p$ and retain it with probability p , independent of the others. The thinned point process is denoted by $\Xi_i^p, i \in \mathcal{I}$, and, in general, for each point process X , we use X^p to denote its thinned process. Let $\Xi^p = \sum_{i \in \mathcal{I}} \Xi_i^p$. As before, we define A_i to be the collection of indices j of the point processes Ξ_j which are dependent on Ξ_i , that is, Ξ_i is independent of $\{\Xi_j, j \in A_i^c\}$.

Theorem 3.1. *Let μ_i be the mean measure of $\Xi_i, \mu_i = \mu_i(\Gamma) = \mathbb{E}(|\Xi_i|), i \in \mathcal{I}$, and assume that $\lambda = \sum_{i \in \mathcal{I}} \mu_i < \infty$. The mean measure of Ξ^p is then $\lambda^p = p \sum_{i \in \mathcal{I}} \mu_i$ and*

$$d_{TV}(\mathcal{L}(\Xi^p), \text{Po}(\lambda^p)) \leq p \left(1 \wedge \frac{1}{\lambda}\right) \mathbb{E} \sum_{i \in \mathcal{I}} \{[|V_i| + |\Xi_i|]\lambda_i + [|V_i| + |\Xi_i| - 1]|\Xi_i|\}, \quad (3.1)$$

$$d_2(\mathcal{L}(\Xi^p), \text{Po}(\lambda^p)) \leq p \mathbb{E} \sum_{i \in \mathcal{I}} \left(\frac{3.5}{\lambda} + \frac{2.5}{|\sum_{j \in A_i^c} \Xi_j| + 1} \right) \times \{[|V_i| + |\Xi_i|]\lambda_i + [|V_i| + |\Xi_i| - 1]|\Xi_i|\}, \quad (3.2)$$

$$d_{TV}(\mathcal{L}(\Xi^p), \text{Po}(\lambda^p)) \leq p \mathbb{E} \sum_{i \in \mathcal{I}} \{[|V_i| + |\Xi_i|]\lambda_i + [|V_i| + |\Xi_i| - 1]|\Xi_i|\}. \quad (3.3)$$

Proof. We prove only (3.2), as the proofs of (3.1) and (3.3) are similar to that of (3.2). By conditioning on the configurations, we have, for each Borel set $B \subset \Gamma$,

$$\mathbb{E}[\Xi_i^p(B)] = \mathbb{E}\{\mathbb{E}[\Xi_i^p(B)|\Xi_i]\} = \mathbb{E}[\Xi_i(B)p],$$

which implies that the mean measure of Ξ_i^p is $\lambda_i^p = p\mu_i$ and hence that $\lambda^p = p\sum_{i \in \mathcal{I}} \mu_i$. By (2.3) and the fact that $d'_1(\xi_1, \xi_2) \leq |\xi_1| + |\xi_2|$, we obtain

$$\begin{aligned} & d_2(\mathcal{L}(\Xi^p), \text{Po}(\lambda^p)) \\ & \leq \mathbb{E} \sum_{i \in \mathcal{I}} \left(\frac{3.5}{p\lambda} + \frac{2.5}{|\sum_{j \in A_i^c} \Xi_j^p| + 1} \right) \int_{\Gamma} [|V_i^p| + |V_{i,\alpha}^p| + |\Xi_i^p| + |\Xi_{i,(\alpha)}^p|] \lambda_i^p(d\alpha) \\ & \leq \mathbb{E} \sum_{i \in \mathcal{I}} \left(\frac{3.5}{p\lambda} + \frac{2.5}{|\sum_{j \in A_i^c} \Xi_j^p| + 1} \right) \int_{\Gamma} \{ [|V_i^p| + |\Xi_i^p|] \lambda_i^p(d\alpha) + [|V_i^p| + |\Xi_i^p| - 1] \Xi_i^p(d\alpha) \} \\ & \leq \mathbb{E} \sum_{i \in \mathcal{I}} \left(\frac{3.5}{p\lambda} + \frac{2.5}{|\sum_{j \in A_i^c} \Xi_j^p| + 1} \right) \{ [|V_i^p| + |\Xi_i^p|] \lambda_i p + [|V_i^p| + |\Xi_i^p| - 1] \Xi_i^p \}. \end{aligned}$$

Since the points are thinned independently, we can condition on the configuration of $\{\Xi_i, i \in \mathcal{I}\}$. Noting that for $Z \sim \text{Binomial}(n, p)$, $\mathbb{E} \frac{1}{Z+1} \leq \frac{1}{(n+1)p}$ and $\mathbb{E}[(X-1)X] = n(n-1)p^2$, we obtain

$$\begin{aligned} & d_2(\mathcal{L}(\Xi^p), \text{Po}(\lambda^p)) \\ & \leq \mathbb{E} \sum_{i \in \mathcal{I}} \left(\frac{3.5}{p\lambda} + \frac{2.5}{p(|\sum_{j \in A_i^c} \Xi_j| + 1)} \right) \{ [|V_i| + |\Xi_i|] \lambda_i + [|V_i| + |\Xi_i| - 1] \Xi_i \} p^2. \end{aligned}$$

This completes the proof of (3.2). \square

Remark 3.2. Serfozo [28], Example 3.6, obtained the rate p for the convergence of a sum of thinned point processes to a Poisson process. Theorem 3.1 shows that the rate p is valid for all of the three metrics used.

4. Superposition of renewal processes

Viswanathan [31], page 290, states that if $\{\Xi_i, 1 \leq i \leq n\}$ are independent renewal processes on $[0, T]$, each representing the process of calls generated by a subscriber, then the total number of calls can be modeled by a Poisson process. In this section, we quantify this statement by giving an error bound for Poisson process approximation to the sum of independent sparse renewal processes. We begin with a technical lemma.

Lemma 4.1. *Let $\eta \sim G$, $\xi_i \sim F, i \geq 1$, be independent non-negative random variables and define*

$$N_t = \max\{n : \eta + \xi_1 + \cdots + \xi_{n-1} \leq t\}, \quad t \geq 0.$$

Then,

$$G(t) \leq \mathbb{E}(N_t) \leq \frac{G(t)}{1 - F(t)}, \quad (4.1)$$

$$\mathbb{E}(N_t^2) - \mathbb{E}(N_t) \leq \frac{2F(t)\mathbb{E}(N_t)}{1 - F(t)} \leq \frac{2F(t)G(t)}{(1 - F(t))^2}. \quad (4.2)$$

Proof. Let $V(t) = \mathbb{E}(N_t)$. The renewal equation gives

$$V(t) = G(t) + \int_0^t V(t - s) dF(s) \leq G(t) + V(t)F(t), \quad (4.3)$$

which implies (4.1). For (4.2), define $V_2(t) = \mathbb{E}[N_t(N_t + 1)]$. Then, using the same arguments as for proving the renewal equation,

$$V_2(t) = 2V(t) + \int_0^t V_2(t - s) dF(s).$$

This implies that $V_2(t) \leq 2V(t) + V_2(t)F(t)$, which, in turn, implies that

$$V_2(t) \leq \frac{2V(t)}{1 - F(t)}.$$

Since

$$\mathbb{E}(N_t^2) - \mathbb{E}(N_t) = V_2(t) - 2V(t) = \int_0^t V_2(t - s) dF(s) \leq V_2(t)F(t) \leq \frac{2F(t)V(t)}{1 - F(t)},$$

(4.2) follows from (4.1). □

Theorem 4.2. Suppose that $\{\Xi_i, 1 \leq i \leq n\}$ are independent renewal processes on $[0, T]$ with the first arrival time of Ξ_i having distribution G_i and its inter-arrival time having distribution F_i . Let $\Xi = \sum_{i=1}^n \Xi_i$ and λ be its mean measure. Then,

$$d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \frac{6 \sum_{i=1}^n [2F_i(T) + G_i(T)]G_i(T)}{(\sum_{i=1}^n G_i(T) - \max_j G_j(T))(1 - F_i(T))^2}. \quad (4.4)$$

Proof. We view a renewal process as a point process whose points occur at the renewal times. For a renewal process X with renewal times $\tau_1 \leq \tau_2 \leq \dots$, we further define $X' = \delta_{\tau_1}$. Since $\lambda_i = \mathbb{E}(|\Xi_i|)$, it follows from (2.16) that

$$\begin{aligned} d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) &\leq \sum_{i=1}^n \left(\frac{3.5}{\lambda} + \mathbb{E} \frac{2.5}{\sum_{j \neq i} |\Xi_j| + 1} \right) [\lambda_i^2 + \mathbb{E}(|\Xi_i|^2) - \lambda_i] \\ &\leq \sum_{i=1}^n \left(\frac{3.5}{\lambda} + \mathbb{E} \frac{2.5}{\sum_{j \neq i} |\Xi'_j| + 1} \right) [\lambda_i^2 + \mathbb{E}(|\Xi_i|^2) - \lambda_i]. \end{aligned} \quad (4.5)$$

However, applying Proposition 4.5 of [13] gives

$$\mathbb{E} \frac{1}{\sum_{j \neq i} |\Xi'_j| + 1} \leq \frac{1}{\sum_{j \neq i} \mathbb{E} |\Xi'_j|} = \frac{1}{\sum_{j \neq i} G_j(T)}$$

and using (4.1), we obtain

$$\lambda \geq \sum_{i=1}^n G_i(T).$$

By combining (4.5), (4.1) and (4.2), we obtain (4.4). □

Remark 4.3. If $\{\Xi_i, 1 \leq i \leq n\}$ are independent and identically distributed stationary renewal processes on $[0, T]$ with the successive inter-arrival time distribution F , then

$$d_2(\mathcal{L}(\Xi), \text{Po}(\lambda)) \leq \frac{6n[2F(T) + G(T)]}{(n-1)(1-F(T))^2},$$

where $G(t) = \int_0^t (1-F(s)) ds / \int_0^\infty (1-F(s)) ds$; see [16], page 71.

Remark 4.4. An application of [29], Theorem 2.1, to the sum of the renewal processes $\{\Xi_i, 1 \leq i \leq n\}$ in Remark 4.3 with the natural partition $\{\{i\}, \emptyset, \{1, \dots, i-1, i+1, \dots, n\}\}$ for each $1 \leq i \leq n$ will give an error bound

$$n[F(T) + G(T)] + \theta G(T)(1 + \ln^+ n),$$

where θ is a constant. The first term of the bound increases linearly in n and the bound is clearly not as sharp as the bound in Remark 4.3.

Since the thinned process X^p of a renewal process X with mean measure μ is still a renewal process (see [16], pages 75–76) with mean measure $\mu^p = p\mu$ (see the proof of Theorem 3.1), a repetition of the proof of Theorem 4.2 yields the following proposition.

Proposition 4.5. Suppose that $\{\Xi_i, 1 \leq i \leq n\}$ are independent renewal processes on $[0, T]$ with the first arrival time of Ξ_i having distribution G_i and its inter-arrival time having distribution F_i . Let Ξ_i^p be the thinned point process obtained from Ξ_i by deleting each point with probability $1-p$ and retaining it with probability p , independently of the other points. Let $\Xi^p = \sum_{i=1}^n \Xi_i^p$ and λ^p be its mean measure. Then,

$$d_2(\mathcal{L}(\Xi^p), \text{Po}(\lambda^p)) \leq \frac{6p \sum_{i=1}^n [2F_i(T) + G_i(T)]G_i(T)}{(\sum_{i=1}^n G_i(T) - \max_j G_j(T))(1-F_i(T))^2}.$$

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