

## CONSERVATION RELATIONS FOR LOCAL THETA CORRESPONDENCE

BINYONG SUN AND CHEN-BO ZHU

*In memory of Stephen Rallis*

### 1. INTRODUCTION

The main goal of this article is to prove “conservation relations” for local theta correspondence, which was first conjectured by Kudla [Ku3] and Rallis [KR3] in the mid 1990s.

**1.1. Dual pairs of type I.** Fix a triple  $(F, D, \epsilon)$  where  $\epsilon = \pm 1$ ;  $F$  is a local field of characteristic zero; and  $D$  is either  $F$ , or a quadratic field extension of  $F$ , or a central division quaternion algebra over  $F$ . Denote by  $\iota$  the involutive anti-automorphism of  $D$  which is respectively the identity map, the non-trivial Galois element, or the main involution.

Let  $U$  be an  $\epsilon$ -Hermitian right  $D$ -vector space; namely,  $U$  is a finite dimensional right  $D$ -vector space, equipped with a non-degenerate  $F$ -bilinear map

$$\langle \cdot, \cdot \rangle_U : U \times U \rightarrow D$$

satisfying

$$\langle u, u'a \rangle_U = \langle u, u' \rangle_U a, \quad u, u' \in U, a \in D,$$

and

$$\langle u, u' \rangle_U = \epsilon \langle u', u \rangle_U^\iota, \quad u, u' \in U.$$

Denote by  $G(U)$  the isometry group of  $U$ , namely, the group of all  $D$ -linear automorphisms of  $U$  preserving the form  $\langle \cdot, \cdot \rangle_U$ . It is naturally a locally compact topological group and is a so-called classical group as summarized in Table 1.

Let  $V$  be a  $-\epsilon$ -Hermitian left  $D$ -vector space, defined in an analogous way. Denote by  $G(V)$  the isometry group of  $V$ . Following Howe [Ho1], we call  $(G(U), G(V))$  an irreducible dual pair of type I. The tensor product  $U \otimes_D V$  is a symplectic space over  $F$  under the bilinear form

$$(1) \quad \langle u \otimes v, u' \otimes v' \rangle := \frac{\langle u, u' \rangle_U \langle v, v' \rangle_V^\iota + \langle v, v' \rangle_V \langle u, u' \rangle_U^\iota}{2}, \quad u, u' \in U, v, v' \in V.$$

Here  $\langle \cdot, \cdot \rangle_V$  denotes the underlying  $-\epsilon$ -Hermitian form on  $V$  (similar notations will be used without further explanation).

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TABLE 1. The classical group  $G(U)$

D	F	quadratic extension	quaternion algebra
$\epsilon = 1$	orthogonal group	unitary group	quaternionic symplectic group
$\epsilon = -1$	symplectic group	unitary group	quaternionic orthogonal group

**Definition 1.1.** *The Heisenberg group  $H(W)$  attached to a (finite dimensional) symplectic space  $W$  over  $F$  is the topological group which equals  $W \times F$  as a topological space, and whose group multiplication is given by*

$$(w, t)(w', t') := (w + w', t + t' + \langle w, w' \rangle_W), \quad w, w' \in W, t, t' \in F.$$

The group  $G(U) \times G(V)$  acts continuously on the Heisenberg group  $H(U \otimes_D V)$  as group automorphisms by

$$(2) \quad (g, h) \cdot \left( \sum_{i=1}^r u_i \otimes v_i, t \right) := \left( \sum_{i=1}^r g(u_i) \otimes h(v_i), t \right),$$

for all  $g \in G(U)$ ,  $h \in G(V)$ ,  $r \geq 0$ ,  $u_1, u_2, \dots, u_r \in U$ ,  $v_1, v_2, \dots, v_r \in V$  and  $t \in F$ . Using this action, we form the Jacobi group

$$J(U, V) := (G(U) \times G(V)) \ltimes H(U \otimes_D V).$$

For the study of local theta correspondence, it is natural to introduce the following modification of  $G(U)$ :

$$\bar{G}(U) := \begin{cases} \widetilde{\text{Sp}}(U), & \text{if } U \text{ is a symplectic space, namely, } (D, \epsilon) = (F, -1); \\ G(U), & \text{otherwise.} \end{cases}$$

Here  $\widetilde{\text{Sp}}(U)$  denotes the metaplectic group: it is the unique topological central extension of the symplectic group  $\text{Sp}(U)$  by  $\{\pm 1\}$  which does not split unless  $U = 0$  or  $F \cong \mathbb{C}$ . Define  $\bar{G}(V)$  analogously. Using the covering homomorphism  $\bar{G}(U) \times \bar{G}(V) \rightarrow G(U) \times G(V)$  and the action (2), we form the modified Jacobi group

$$\bar{J}(U, V) := (\bar{G}(U) \times \bar{G}(V)) \ltimes H(U \otimes_D V).$$

**1.2. Local theta correspondence.** The center of the Heisenberg group  $H(W)$  of Definition 1.1 is obviously identified with  $F$ . Fix a non-trivial unitary character  $\psi : F \rightarrow \mathbb{C}^\times$  throughout the article. As usual, a superscript “ $\times$ ” over a ring indicates its multiplicative group of invertible elements.

From now on until Section 6, we assume that  $F$  is non-archimedean. The smooth version of the Stone–von Neumann Theorem asserts the following.

**Theorem 1.2** (cf. [MVW, 2.I.2]). *For any symplectic space  $W$  over  $F$ , there exists a unique (up to isomorphism) irreducible smooth representation of  $H(W)$  with central character  $\psi$ .*

For any totally disconnected, locally compact, Hausdorff topological space  $Z$ , denote by  $C^\infty(Z)$  the space of locally constant  $\mathbb{C}$ -valued functions on  $Z$ , and by  $\mathcal{S}(Z)$  the space of all functions in  $C^\infty(Z)$  with compact support. Taking a complete polarization  $W = X \oplus Y$  of a symplectic space  $W$  over  $F$ , let  $H(W)$  act on  $\mathcal{S}(X)$  by

$$(3) \quad ((x_0 + y_0, t) \cdot \phi)(x) := \phi(x + x_0) \psi(t + \langle 2x + x_0, y_0 \rangle_W),$$

for all  $\phi \in \mathcal{S}(X)$ ,  $x, x_0 \in X$ ,  $y_0 \in Y$  and  $t \in F$ . It is easy to check that this defines an irreducible smooth representation of  $H(W)$  with central character  $\psi$ .

**Definition 1.3.** Let  $J$  be a totally disconnected, locally compact, Hausdorff topological group. Assume that  $J$  contains the Heisenberg group  $H(W)$  as a closed normal subgroup, where  $W$  is a symplectic space over  $F$ . A smooth representation of  $J$  is called a smooth oscillator representation (associated to  $\psi$ , unless otherwise specified) if its restriction to  $H(W)$  is irreducible and has central character  $\psi$ .

Dixmier’s version of Schur’s Lemma [Wa1, 0.5.2] implies that smooth oscillator representations are unique up to twisting by characters.

**Lemma 1.4.** For any smooth oscillator representations  $\omega$  and  $\omega'$  of  $J$  as in Definition 1.3, there exists a unique character  $\chi$  on  $J$  which is trivial on  $H(W)$  such that  $\omega' \cong \chi \otimes \omega$ .

For the dual pair of a non-trivial symplectic group and an odd orthogonal group, smooth oscillator representations of  $J(U, V)$  do not exist. Nevertheless, in all cases, there exists a smooth oscillator representation  $\omega_{U,V}$  of  $\bar{J}(U, V)$  which is genuine in the following sense:

- when  $U$  is a symplectic space,  $\varepsilon_U$  acts through the scalar multiplication by  $(-1)^{\dim V}$  in  $\omega_{U,V}$ ; and likewise for  $V$  when  $V$  is a symplectic space.

Here and henceforth, for a symplectic space  $U$ ,  $\varepsilon_U$  denotes the non-trivial element of the kernel of the covering homomorphism  $\bar{G}(U) \rightarrow G(U)$ . See Lemma 3.4.

Denote by  $\text{Irr}(\bar{G}(U))$  the set of all isomorphism classes of irreducible admissible smooth representations of  $\bar{G}(U)$ . Define  $\text{Irr}(\bar{G}(V))$  similarly. For a genuine smooth oscillator representation  $\omega_{U,V}$  of  $\bar{J}(U, V)$ , put

$$\begin{aligned} \mathcal{R}_{\omega_{U,V}}(U) &:= \{ \pi \in \text{Irr}(\bar{G}(U)) \mid \text{Hom}_{\bar{G}(U)}(\omega_{U,V}, \pi) \neq 0 \}, \\ \mathcal{R}_{\omega_{U,V}}(V) &:= \{ \pi' \in \text{Irr}(\bar{G}(V)) \mid \text{Hom}_{\bar{G}(V)}(\omega_{U,V}, \pi') \neq 0 \}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_{\omega_{U,V}}(U, V) &:= \{ (\pi, \pi') \in \text{Irr}(\bar{G}(U)) \times \text{Irr}(\bar{G}(V)) \mid \text{Hom}_{\bar{G}(U) \times \bar{G}(V)}(\omega_{U,V}, \pi \otimes \pi') \neq 0 \}. \end{aligned}$$

The theory of local theta correspondence begins with the following Howe Duality Conjecture.

**Conjecture 1.5.** The set  $\mathcal{R}_{\omega_{U,V}}(U, V)$  is the graph of a bijection between  $\mathcal{R}_{\omega_{U,V}}(U)$  and  $\mathcal{R}_{\omega_{U,V}}(V)$ .

Waldspurger proves the above conjecture when  $F$  has an odd residue characteristic [Wald]. (We will not assume the Howe duality conjecture, and thus there will not be any assumption on the residue characteristic of  $F$ .) For the theory of local theta correspondence, a basic question is occurrence, which is to determine the sets  $\mathcal{R}_{\omega_{U,V}}(U)$  and  $\mathcal{R}_{\omega_{U,V}}(V)$ . By the symmetric role of  $U$  and  $V$ , without loss of generality we shall focus on the set  $\mathcal{R}_{\omega_{U,V}}(U)$ .

**1.3. A prologue: conservation relations in the Witt group.** Before proceeding to the conservation relations for local theta correspondence, it is instructive to explain certain relations in the Witt group, which is actually the conservation relations for local theta correspondence in the case  $U = 0$  (the zero space).

Denote by  $\widehat{\mathcal{W}}_0^+$  the commutative monoid (under orthogonal direct sum) of all isometry classes of  $-\epsilon$ -Hermitian left  $D$ -vector spaces. When no confusion is possible, we do not distinguish an element of  $\widehat{\mathcal{W}}_0^+$  with a space which represents it.

Recall that the split rank of a space  $V \in \widehat{\mathcal{W}}_0^+$  is defined to be

$$\text{rank } V := \max \{ \dim Y \mid Y \text{ is a totally isotropic D-subspace of } V \}.$$

Denote by  $\mathbb{H}$  the hyperbolic plane in  $\widehat{\mathcal{W}}_0^+$ , namely, the element of  $\widehat{\mathcal{W}}_0^+$  with dimension 2 and split rank 1. A subset of  $\widehat{\mathcal{W}}_0^+$  of the form

$$\mathbf{t} := \{V_0, V_0 + \mathbb{H}, V_0 + 2\mathbb{H}, \dots\}$$

is called a Witt tower in  $\widehat{\mathcal{W}}_0^+$ , where  $V_0$  is an anisotropic element in  $\widehat{\mathcal{W}}_0^+$ , namely, an element of split rank 0. Define the anisotropic degree of  $\mathbf{t}$  by

$$\text{deg } \mathbf{t} := \dim V_0.$$

Denote by  $\mathcal{W}_0$  the set of all Witt towers in  $\widehat{\mathcal{W}}_0^+$ . This is a quotient set of  $\widehat{\mathcal{W}}_0^+$ , and the monoid structure on  $\widehat{\mathcal{W}}_0^+$  descends to a monoid structure on  $\mathcal{W}_0$ . The resulting monoid  $\mathcal{W}_0$  is in fact a finite abelian group (its order is a power of 2), which is called the Witt group of  $-\epsilon$ -Hermitian left D-vector spaces. See Section 3.1 for details.

Write  $d_{D,\epsilon}$  for the maximal dimension of an anisotropic element in  $\widehat{\mathcal{W}}_0^+$ :

$$d_{D,\epsilon} := \max\{\dim V_0 \mid V_0 \text{ is an anisotropic element of } \widehat{\mathcal{W}}_0^+\}.$$

Recall the following well-known result.

**Proposition 1.6.** (cf. [Sch, Chapters 5 and 10]). *One has that*

$$(4) \quad d_{D,\epsilon} = \begin{cases} 0, & \text{if } V \text{ is a symplectic space;} \\ 1, & \text{if } V \text{ is a quaternionic Hermitian space;} \\ 2, & \text{if } V \text{ is a Hermitian space or a skew-Hermitian space;} \\ 3, & \text{if } V \text{ is a quaternionic skew-Hermitian space;} \\ 4, & \text{if } V \text{ is a symmetric bilinear space.} \end{cases}$$

Moreover, there exists a unique element  $V^\circ \in \widehat{\mathcal{W}}_0^+$  which is anisotropic and has dimension  $d_{D,\epsilon}$ ; and every anisotropic element of  $\widehat{\mathcal{W}}_0^+$  is isometrically isomorphic to a subspace of  $V^\circ$ .

Here and henceforth, we refer to the various cases by the types of the spaces under consideration. For example, “ $V$  is a symplectic space” means that  $(D, \epsilon) = (F, 1)$ .

By Proposition 1.6, the Witt group  $\mathcal{W}_0$  has a unique element of anisotropic degree  $d_{D,\epsilon}$ . Denote it by  $\mathbf{t}_0^\circ$  and call it the anti-split Witt tower in  $\widehat{\mathcal{W}}_0^+$ . Note that  $\mathcal{W}_0$  is trivial when  $V$  is a symplectic space. Except for this case,  $\mathbf{t}_0^\circ$  has order 2 in  $\mathcal{W}_0$ . By considering the negative of the orthogonal complement of an anisotropic element of  $\widehat{\mathcal{W}}_0^+$  in  $V^\circ$ , Proposition 1.6 implies the following relations in the Witt group.

**Proposition 1.7.** *We have*

$$\text{deg } \mathbf{t}_1 + \text{deg } \mathbf{t}_2 = d_{D,\epsilon}$$

for all  $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{W}_0$  with difference  $\mathbf{t}_0^\circ$ .

1.4. **The generalized Witt group.** Note that the subset  $\mathcal{R}_{\omega_{U,V}}(U)$  of  $\text{Irr}(\bar{G}(U))$  depends only on the restriction of  $\omega_{U,V}$  to the subgroup

$$\bar{J}_U(V) := \bar{G}(U) \times \mathbb{H}(U \otimes_{\mathbb{D}} V) \subset \bar{J}(U, V).$$

For a fixed  $\pi \in \text{Irr}(\bar{G}(U))$ , we shall consider the occurrence of  $\pi$  in  $\omega_{U,V}$ , or the membership of  $\pi$  in  $\mathcal{R}_{\omega_{U,V}}(U)$ , as  $V$  vary. We introduce the following definition.

**Definition 1.8.** *An enhanced oscillator representation of  $\bar{G}(U)$  is a pair  $(V, \omega)$ , where  $V$  is a  $-\epsilon$ -Hermitian left  $\mathbb{D}$ -vector space, and  $\omega$  is a smooth oscillator representation of  $\bar{J}_U(V)$  which is genuine in the following sense: if  $U$  is a symplectic space, then  $\varepsilon_U$  acts through the scalar multiplication by  $(-1)^{\dim V}$  in  $\omega$ . Two enhanced oscillator representations  $(V_1, \omega_1)$  and  $(V_2, \omega_2)$  of  $\bar{G}(U)$  are said to be isomorphic if there is an isometric isomorphism  $V_1 \cong V_2$  such that  $\omega_1$  is isomorphic to  $\omega_2$  with respect to the induced isomorphism  $\bar{J}_U(V_1) \cong \bar{J}_U(V_2)$ .*

Denote by  $\widehat{\mathcal{W}}_U^+$  the set of all isomorphism classes of enhanced oscillator representations of  $\bar{G}(U)$ . The set  $\widehat{\mathcal{W}}_U^+$  has a natural additive structure which makes it a commutative monoid:

$$(V_1, \omega_1) + (V_2, \omega_2) := (V_1 \oplus V_2, \omega_1 \otimes \omega_2).$$

Here  $V_1 \oplus V_2$  denotes the orthogonal direct sum, and the tensor product  $\omega_1 \otimes \omega_2$  carries the representation of the group  $\bar{J}_U(V_1 \oplus V_2)$ , as follows:

$$(g, (w_1 + w_2, t_1 + t_2)) \cdot (\phi_1 \otimes \phi_2) := ((g, (w_1, t_1)) \cdot \phi_1) \otimes ((g, (w_2, t_2)) \cdot \phi_2),$$

where  $g \in \bar{G}(U)$ ,  $w_i \in U \otimes_{\mathbb{D}} V_i$ ,  $t_i \in \mathbb{F}$  and  $\phi_i \in \omega_i$  ( $i = 1, 2$ ). As before, we shall not distinguish an element of  $\widehat{\mathcal{W}}_U^+$  with an enhanced oscillator representation which represents it.

Recall the hyperbolic plane  $\mathbb{H} \in \widehat{\mathcal{W}}_0^+$ . We shall define the hyperbolic plane  $\mathbb{H}_U$  in  $\widehat{\mathcal{W}}_U^+$  as follows. The symplectic space  $U \otimes_{\mathbb{D}} \mathbb{H}$  has a complete polarization

$$(5) \quad U \otimes_{\mathbb{D}} \mathbb{H} = U \otimes e_1 \oplus U \otimes e_2,$$

where  $e_1, e_2$  is a basis of  $\mathbb{H}$  of isotropic vectors. Define the hyperbolic plane  $\mathbb{H}_U$  in  $\widehat{\mathcal{W}}_U^+$  to be the enhanced oscillator representation

$$\mathbb{H}_U := (\mathbb{H}, \omega_U) \in \widehat{\mathcal{W}}_U^+,$$

where  $\omega_U$  is the representation of  $\bar{J}_U(\mathbb{H})$  on the space  $\mathcal{S}(U \otimes e_1)$  so that the Heisenberg group  $\mathbb{H}(U \otimes_{\mathbb{D}} \mathbb{H})$  acts as in (3) (for the complete polarization (5)), and  $\bar{G}(U)$  acts by

$$(\bar{g} \cdot \phi)(x \otimes e_1) := \phi((g^{-1}(x)) \otimes e_1), \quad \bar{g} \in \bar{G}(U), \phi \in \mathcal{S}(U \otimes e_1), x \in U.$$

Here  $g$  denotes the image of  $\bar{g}$  under the covering homomorphism  $\bar{G}(U) \rightarrow G(U)$ .

**Definition 1.9.** *Two elements  $\sigma_1, \sigma_2 \in \widehat{\mathcal{W}}_U^+$  are said to be Witt equivalent if there are non-negative integers  $m_1$  and  $m_2$  such that the equality*

$$\sigma_1 + m_1 \mathbb{H}_U = \sigma_2 + m_2 \mathbb{H}_U$$

*holds in the commutative monoid  $\widehat{\mathcal{W}}_U^+$ . This defines an equivalence relation on  $\widehat{\mathcal{W}}_U^+$  whose equivalence classes are called Witt towers in  $\widehat{\mathcal{W}}_U^+$ .*

For  $\sigma = (V, \omega) \in \widehat{\mathcal{W}}_U^+$ , we shall refer to the dimension and the split rank of  $V$  as the dimension and the split rank of  $\sigma$ :

$$\dim \sigma := \dim V \quad \text{and} \quad \text{rank } \sigma := \text{rank } V.$$

Each Witt tower  $\mathfrak{t} \subset \widehat{\mathcal{W}}_U^+$  has a unique anisotropic representative, namely, an element of split rank 0 (this is a consequence of Lemma 1.4). Write  $\sigma_{\mathfrak{t}}$  for this anisotropic element. Then

$$\mathfrak{t} = \{\sigma_{\mathfrak{t}}, \sigma_{\mathfrak{t}} + \mathbb{H}_U, \sigma_{\mathfrak{t}} + 2\mathbb{H}_U, \dots\}.$$

We define the anisotropic degree of  $\mathfrak{t}$  to be

$$(6) \quad \text{deg } \mathfrak{t} := \dim \sigma_{\mathfrak{t}}.$$

Write  $\mathcal{W}_U$  for the set of all Witt towers in  $\widehat{\mathcal{W}}_U^+$ . Similar to the Witt group  $\mathcal{W}_0$ , the additive structure on  $\widehat{\mathcal{W}}_U^+$  descends to an additive structure on  $\mathcal{W}_U$  which makes  $\mathcal{W}_U$  an abelian group (see Proposition 3.11, part (c)). There is a short exact sequence (see Proposition 3.11, part (d))

$$1 \rightarrow (G(U))^* \rightarrow \mathcal{W}_U \rightarrow \mathcal{W}_0 \rightarrow 1.$$

Here and henceforth, for all topological group  $G$ ,  $G^* := \text{Hom}(G, \mathbb{C}^\times)$  denotes the group of all characters on  $G$ . When  $G(U)$  is a perfect group, we have identifications

$$\widehat{\mathcal{W}}_U^+ = \widehat{\mathcal{W}}_0^+ \quad \text{and} \quad \mathcal{W}_U = \mathcal{W}_0.$$

(This includes the case when  $U = 0$ , with clearly consistent notation.)

**1.5. First occurrence index and conservation relations.** With the above notion, we may rephrase the question of occurrence as follows: for a given  $\sigma = (V, \omega) \in \widehat{\mathcal{W}}_U^+$ , one seeks to determine the set

$$\mathcal{R}_\sigma := \{\pi \in \text{Irr}(\bar{G}(U)) \mid \text{Hom}_{\bar{G}(U)}(\omega, \pi) \neq 0\}.$$

A clear necessary condition for  $\pi \in \mathcal{R}_\sigma$  is that  $\pi$  is genuine with respect to  $\sigma$  in the following sense:

- if  $U$  is a symplectic space, then  $\varepsilon_U$  acts through the scalar multiplication by  $(-1)^{\dim \sigma}$  in  $\pi$ .

Let  $\pi \in \text{Irr}(\bar{G}(U))$  and let  $\mathfrak{t} \in \mathcal{W}_U$ . Assume that  $\pi$  is genuine with respect to  $\mathfrak{t}$ ; namely,  $\pi$  is genuine with respect to some (and hence all) elements of  $\mathfrak{t}$ . There are two basic properties concerning occurrence:

- Occurrence in the so-called stable range (see [Li1] and (63)):

$$(7) \quad \text{for all } \sigma \in \mathfrak{t}, \text{ if } \text{rank } \sigma \geq \dim U, \text{ then } \pi \in \mathcal{R}_\sigma.$$

- Kudla’s persistence principle ([Ku1]):

$$\text{for all } \sigma_1, \sigma_2 \in \mathfrak{t}, \text{ if } \dim \sigma_1 \leq \dim \sigma_2, \text{ then } \mathcal{R}_{\sigma_1} \subset \mathcal{R}_{\sigma_2}.$$

Write  $1_U$  for the trivial representation of  $\bar{G}(U)$ . We note that in our formulation, Kudla’s persistence principle follows clearly from the fact that  $\text{Hom}_{\bar{G}(U)}(\omega_U, 1_U) \neq 0$ . (Recall that  $\omega_U$  denotes the underlying representation of the hyperbolic plane  $\mathbb{H}_U \in \widehat{\mathcal{W}}_U^+$ .) Define the first occurrence index

$$(8) \quad \mathfrak{n}_{\mathfrak{t}}(\pi) := \min\{\dim \sigma \mid \sigma \in \mathfrak{t}, \pi \in \mathcal{R}_\sigma\}.$$

In view of the aforementioned two properties, the first occurrence index is finite and is of clear interest.

Generalizing the anti-split Witt tower  $\mathfrak{t}_0^\circ \in \mathcal{W}_0$ , we will define in Section 4 the anti-split Witt tower  $\mathfrak{t}_U^\circ \in \mathcal{W}_U$ . When  $U$  is a symmetric bilinear space,  $\mathfrak{t}_U^\circ \in \mathcal{W}_U = (\mathrm{O}(U))^*$  is the sign character; when  $U$  is a symplectic space or a quaternionic Hermitian space (in which case  $\mathrm{G}(U)$  is a perfect group),  $\mathfrak{t}_U^\circ \in \mathcal{W}_U = \mathcal{W}_0$  is identical to  $\mathfrak{t}_0^\circ$ . In all cases,  $\mathfrak{t}_U^\circ$  is characterized by the equality

$$(9) \quad n_{\mathfrak{t}_U^\circ}(1_U) = 2 \dim U + d_{D,\epsilon}.$$

The element  $\mathfrak{t}_U^\circ \in \mathcal{W}_U$  has anisotropic degree  $d_{D,\epsilon}$  and has order 2 unless  $U$  is the zero symmetric bilinear space (in this exceptional case the group  $\mathcal{W}_U$  is trivial).

In the non-archimedean case, the conservation relations assert the following.

**Theorem 1.10.** *Let  $\mathfrak{t}_1$  and  $\mathfrak{t}_2$  be two Witt towers in  $\mathcal{W}_U$  with difference  $\mathfrak{t}_U^\circ$ . Then for any  $\pi \in \mathrm{Irr}(\bar{\mathrm{G}}(U))$  which is genuine with respect to  $\mathfrak{t}_1$  (and hence genuine with respect to  $\mathfrak{t}_2$ ), one has that*

$$n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) = 2 \dim U + d_{D,\epsilon}.$$

**Remarks:** (a) For orthogonal-symplectic and unitary-unitary dual pairs, the conservation relations were conjectured by Kudla and Rallis in the mid 1990s [Ku3, KR3]. For quaternionic dual pairs, the conjectured statements first appeared in Gan-Tantono [GT, Section 4].

(b) For orthogonal-symplectic dual pairs and  $\pi$  supercuspidal, the conservation relations were due to Kudla and Rallis [KR3]. This was later extended to all irreducible dual pairs of type I by Minguez [Mi], again for  $\pi$  supercuspidal.

(c) The inequality  $n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) \geq 2 \dim U + d_{D,\epsilon}$  is due to Rallis [Ra1, Ra2] and Kudla-Rallis [KR3, Theorem 3.8] (for orthogonal-symplectic dual pairs), and Gong-Grenié [GG, Theorem 1.8] (for unitary-unitary dual pairs, following an earlier work of Harris-Kudla-Sweet [HKS]). See also Gan-Ichino [GI, Theorem 5.4].

We now comment on the organization of this article. In Section 2, we explain the idea of the proof of Theorem 1.10, with a special focus on the upper bound, namely the inequality  $n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) \leq 2 \dim U + d_{D,\epsilon}$  (a key contribution of the current article). We give its main argument for the case when  $U$  is a quaternionic Hermitian space. In Section 3, we introduce the Grothendieck group  $\widehat{\mathcal{W}}_U$  of the commutative monoid  $\widehat{\mathcal{W}}_U^+$ , to be called the generalized Witt-Grothendieck group, and we give their basic properties. In Section 4, we introduce the notion of Kudla characters and the Kudla homomorphism, which are then used to explicitly determine the generalized Witt-Grothendieck groups. A main purpose for introducing these new notions is to formulate the conservation relations for all irreducible dual pairs of type I in a uniform and conceptually simple manner. This approach is justified, and in our view appealing, due to the commonality of the underlying principles, in both the occurrence and the non-occurrence aspects. In Section 5, we review the doubling method and some results on the structure of degenerate principal series of  $\bar{\mathrm{G}}(U)$  for the  $U$  split, and prove the upper bound in the conservation relations. Section 6 is devoted to the phenomenon of non-occurrence of the trivial representation before stable range, which is responsible for the lower bound in the conservation relations. We follow the method of Rallis [Ra1, Ra2] (which treat the case of orthogonal groups). It is worth mentioning that for the base case ( $\dim U = 1$ , and  $V$  anisotropic; see Lemma 6.6), proving non-occurrence of the trivial representation again requires the use of the doubling method. In the final Section 7, we discuss the conservation relations in the archimedean case. Then three different phenomena

occur: the same conservation relations as in the non-archimedean case hold when  $U$  is a real or complex symmetric bilinear space; no conservation relations hold when  $U$  is a complex symplectic space or a real quaternionic Hermitian space; when  $U$  is a real symplectic space, a complex Hermitian or skew-Hermitian space, or a real quaternionic skew-Hermitian space, a more involved version of the conservation relations hold.

2. ABOUT THE PROOF OF THEOREM 1.10

**2.1. The strategy of the proof.** Let  $\mathbf{t}_1, \mathbf{t}_2$  and  $\pi$  be as in Theorem 1.10. As usual, we use a superscript “ $\vee$ ” to indicate the contragredient representation of an admissible smooth representation. There are two equally important aspects of the conservation relations, which respectively assert the non-occurrence (Proposition 6.2):

$$(10) \quad n_{\mathbf{t}_1}(\pi) + n_{-\mathbf{t}_2}(\pi^\vee) \geq 2 \dim U + d_{D,\epsilon},$$

and the occurrence (Corollary 5.7):

$$(11) \quad n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) \leq 2 \dim U + d_{D,\epsilon}.$$

Assuming both (10) and (11), we then have

$$(12) \quad \begin{cases} n_{\mathbf{t}_1}(\pi) + n_{-\mathbf{t}_2}(\pi^\vee) \geq 2 \dim U + d_{D,\epsilon}; \\ n_{\mathbf{t}_2}(\pi) + n_{-\mathbf{t}_1}(\pi^\vee) \geq 2 \dim U + d_{D,\epsilon}; \\ n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) \leq 2 \dim U + d_{D,\epsilon}; \\ n_{-\mathbf{t}_1}(\pi^\vee) + n_{-\mathbf{t}_2}(\pi^\vee) \leq 2 \dim U + d_{D,\epsilon}. \end{cases}$$

This forces all the above inequalities to be equalities! In particular we arrive at Theorem 1.10.

For the non-occurrence, the key phenomenon is the late occurrence of the trivial representation  $1_U$  in the anti-split Witt tower  $\mathbf{t}_U^\circ$  (Proposition 6.1):

$$(13) \quad n_{\mathbf{t}_U^\circ}(1_U) \geq 2 \dim U + d_{D,\epsilon}.$$

This asserts the vanishing of certain vector valued  $\bar{G}(U)$ -invariant distributions. We show (13) following the method of Rallis [Ra1,Ra2]. The proof consists of reduction to the null cone, and within the null cone, proving vanishing on small orbits and homogeneity for the main orbits, and finally employing the Fourier transform. This is long and involved, and the details are given in Section 6.

As is well-known, (13) implies (10), as follows. Let  $\sigma_1 = (V_1, \omega_1)$  be the element of  $\mathbf{t}_1$  so that  $n_{\mathbf{t}_1}(\pi) = \dim \sigma_1$ . Likewise, let  $\sigma_2 = (V_2, \omega_2)$  be the element of  $-\mathbf{t}_2$  so that  $n_{-\mathbf{t}_2}(\pi^\vee) = \dim \sigma_2$ . Then

$$\text{Hom}_{\bar{G}(U)}(\omega_1, \pi) \neq 0 \quad \text{and} \quad \text{Hom}_{\bar{G}(U)}(\omega_2, \pi^\vee) \neq 0.$$

Consequently,

$$\text{Hom}_{\bar{G}(U)}(\omega_1 \otimes \omega_2, 1_U) \neq 0,$$

which implies that

$$1_U \in \mathcal{R}_{\sigma_1 + \sigma_2} \quad \text{and so} \quad \dim(\sigma_1 + \sigma_2) \geq n_{\mathbf{t}_1 - \mathbf{t}_2}(1_U) = n_{\mathbf{t}_U^\circ}(1_U).$$

Therefore (13) implies that

$$n_{\mathbf{t}_1}(\pi) + n_{-\mathbf{t}_2}(\pi^\vee) = \dim \sigma_1 + \dim \sigma_2 = \dim(\sigma_1 + \sigma_2) \geq 2 \dim U + d_{D,\epsilon}.$$



**2.2. The methods of Kudla and Rallis.** For the occurrence, we use the doubling method, theory of local zeta integrals, and critically the known structure of degenerate principle series, which are in fact all part of the foundational work of Kudla and Rallis [KR3]. To illustrate how the methods of Kudla and Rallis will lead to (11), we give the main argument for one special case, namely, theta lifting from a quaternionic symplectic group to a quaternionic orthogonal group.

We thus assume that  $U$  is a quaternionic Hermitian space. Then  $G(U)$  is a perfect group, and  $\widehat{\mathcal{W}}_U^+ = \widehat{\mathcal{W}}_0^+$ . For each space  $V \in \widehat{\mathcal{W}}_0^+$ , define its discriminant to be

$$\text{disc } V := \left( \prod_{i=1}^m \langle e_i, e_i \rangle_V \langle e_i, e_i \rangle_V^c \right) (\mathbb{F}^\times)^2 \in \mathbb{F}^\times / (\mathbb{F}^\times)^2,$$

where  $e_1, e_2, \dots, e_m$  is an orthogonal basis of  $V$ . This is independent of the choice of orthogonal basis.

For each  $V \in \widehat{\mathcal{W}}_0^+$ , denote by  $\omega_V$  the unique (up to isomorphism) smooth oscillator representation of

$$J_U(V) := G(U) \ltimes H(U \otimes_{\mathbb{D}} V) \subset J(U, V).$$

Likewise, denote by  $\omega_V^\square$  the unique (up to isomorphism) smooth oscillator representation of

$$J_U^\square(V) := G(U^\square) \ltimes H(U^\square \otimes_{\mathbb{D}} V) \subset J(U^\square, V).$$

Here

$$(14) \quad U^\square := U \oplus U^-,$$

and  $U^-$  denotes the space  $U$  equipped with the form scaled by  $-1$ . Put

$$U^\Delta := \{(u, u) \mid u \in U\}.$$

It is a Lagrangian subspace of  $U^\square$ , namely, it is totally isotropic and  $\dim U^\square = 2 \dim U^\Delta$ . Consequently,  $U^\Delta \otimes_{\mathbb{D}} V$  is a Lagrangian subspace of the symplectic space  $U^\square \otimes_{\mathbb{D}} V$ .

**Lemma 2.1.** ([Ca, Theorem 1] and [Ho1, Theorem 9.1]). *Let  $X$  be a Lagrangian subspace of a symplectic space  $W$  over  $\mathbb{F}$ , to be viewed as an abelian subgroup of the Heisenberg group  $H(W)$ . Let  $\omega$  be an irreducible smooth representation of  $H(W)$  with central character  $\psi$ . Then there exists a non-zero linear functional on  $\omega$  which is invariant under  $X$ , and such a linear functional is unique up to scalar multiples.*

*Proof.* Using the realization of  $\omega$  in (3), the lemma follows by the existence and uniqueness of the Haar measure on  $X$ . □

Using Lemma 2.1, we obtain a non-zero linear functional  $\lambda_V$  on  $\omega_V^\square$  which is invariant under  $U^\Delta \otimes_{\mathbb{D}} V \subset H(U^\square \otimes_{\mathbb{D}} V)$ . Denote by  $P(U^\Delta)$  the parabolic subgroup of  $G(U^\square)$  stabilizing  $U^\Delta$ . The linear functional  $\lambda_V$  has the following transformation property (cf. [Ya, Section 6]):

$$\lambda_V(p \cdot \phi) = \chi_V(p) \lambda_V(\phi) \quad \text{for all } p \in P(U^\Delta), \phi \in \omega_V^\square,$$

where  $\chi_V(p)$  denotes the image of  $p$  under the composition of

$$P(U^\Delta) \xrightarrow{\text{restriction}} \text{GL}(U^\Delta) \xrightarrow{\text{reduced norm}} \mathbb{F}^\times \xrightarrow{a \mapsto ((-1)^{\dim V} \text{disc } V, a)_\mathbb{F} |a|_\mathbb{F}^{\dim V}} \mathbb{C}^\times.$$

Here and as usual,  $(, )_\mathbb{F}$  denotes the quadratic Hilbert symbol for  $\mathbb{F}$ , and  $|\cdot|_\mathbb{F}$  denotes the normalized absolute value for  $\mathbb{F}$ .

For every character  $\chi \in (P(U^\Delta))^*$ , put

$$I(\chi) := \{f \in C^\infty(G(U^\square)) \mid f(pg) = \chi(p)f(g), p \in P(U^\Delta), g \in G(U^\square)\}.$$

It is a smooth representation of  $G(U^\square)$  under the right translations. This is a so-called degenerate principal series representation.

Let  $\pi \in \text{Irr}(G(U))$  be as in the Introduction. The theory of local zeta integrals [PSR, LR] implies that

$$(15) \quad \text{Hom}_{G(U)}(I(\chi), \pi) \neq 0.$$

Here  $G(U)$  is identified with the subgroup of  $G(U^\square)$  pointwise fixing  $U^-$ .

Via matrix coefficients, the linear functional  $\lambda_V$  induces a  $G(U^\square)$ -intertwining linear map

$$(16) \quad \omega_V^\square \rightarrow I(\chi_V), \quad \phi \mapsto (g \mapsto \lambda_V(g \cdot \phi)).$$

Denote by  $Q_V$  the image of (16). It is easy to see that (Lemma 5.4)

$$(17) \quad \text{Hom}_{G(U)}(Q_V, \pi) \neq 0 \quad \text{implies} \quad \text{Hom}_{G(U)}(\omega_V, \pi) \neq 0.$$

Let  $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{W}_0 = \mathcal{W}_U$  so that their difference is the anti-split Witt tower. Let  $V_1$  be a space in  $\mathfrak{t}_1$  and let  $V_2$  be a space in  $\mathfrak{t}_2$ . The following result about the structure of degenerate principal series is critical for the conservation relations.

**Proposition 2.2.** [Ya, Theorem 1.4]. *If  $\dim V_1 + \dim V_2 = 2 \dim U + 1$ , then  $I(\chi_{V_1})/Q_{V_1} \cong Q_{V_2}$  as smooth representations of  $G(U^\square)$ .*

Using Proposition 2.2, we get Proposition 2.3.

**Proposition 2.3.** *If  $\dim V_1 + \dim V_2 = 2 \dim U + 1$ , then*

$$\text{Hom}_{G(U)}(\omega_{V_1}, \pi) \neq 0 \quad \text{or} \quad \text{Hom}_{G(U)}(\omega_{V_2}, \pi) \neq 0.$$

*Proof.* By Proposition 2.2, Equation (15) (for  $\chi = \chi_{V_1}$ ) implies that

$$\text{Hom}_{G(U)}(Q_{V_1}, \pi) \neq 0 \quad \text{or} \quad \text{Hom}_{G(U)}(Q_{V_2}, \pi) \neq 0.$$

The proposition then follows by (17). □

**Lemma 2.4.** *If  $n_{\mathfrak{t}_1}(\pi) = \deg \mathfrak{t}_1$  or  $n_{\mathfrak{t}_2}(\pi) = \deg \mathfrak{t}_2$ , then*

$$n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) \leq 2 \dim U + 3.$$

*Proof.* Without loss of generality, assume that  $n_{\mathfrak{t}_1}(\pi) = \deg \mathfrak{t}_1$ . By (7), we have

$$n_{\mathfrak{t}_2}(\pi) \leq 2 \dim U + \deg \mathfrak{t}_2.$$

Therefore Proposition 1.7 implies that

$$n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) \leq \deg \mathfrak{t}_1 + 2 \dim U + \deg \mathfrak{t}_2 = 2 \dim U + 3.$$

□

We now finish the proof of the inequality (11). In view of Lemma 2.4, we may assume that  $n_{\mathfrak{t}_1}(\pi) > \deg(\mathfrak{t}_1)$  and  $n_{\mathfrak{t}_2}(\pi) > \deg(\mathfrak{t}_2)$ . Then  $U \neq 0$ . Assume that (11) does not hold. Then

$$n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) \geq 2 \dim U + 5.$$

This implies that there is a space  $V_1$  in  $\mathfrak{t}_1$  and a space  $V_2$  in  $\mathfrak{t}_2$  such that

$$(18) \quad \dim V_i < n_{\mathfrak{t}_i}(\pi), \quad i = 1, 2,$$

and

$$(19) \quad \dim V_1 + \dim V_2 = 2 \dim U + 1.$$

We get a contradiction since (18) implies that

$$\text{Hom}_{G(U)}(\omega_{V_1}, \pi) = 0 \quad \text{and} \quad \text{Hom}_{G(U)}(\omega_{V_2}, \pi) = 0,$$

and by Proposition 2.3, Equation (19) implies that

$$\text{Hom}_{G(U)}(\omega_{V_1}, \pi) \neq 0 \quad \text{or} \quad \text{Hom}_{G(U)}(\omega_{V_2}, \pi) \neq 0.$$

This proves that the inequality (11) always holds.

### 3. GENERALIZED WITT-GROTHENDIECK GROUPS

**3.1. The Witt-Grothendieck group.** Recall from the Introduction the commutative monoid  $\widehat{\mathcal{W}}_0^+$  of isometry classes of  $-\epsilon$ -Hermitian left D-vector spaces.

**Lemma 3.1.** [Sch, Chapter 7, Corollary 9.2 (i)] *The monoid  $\widehat{\mathcal{W}}_0^+$  is cancellative; namely, for all  $V_1, V_2, V_3 \in \widehat{\mathcal{W}}_0^+$ , if  $V_1 + V_3 = V_2 + V_3$ , then  $V_1 = V_2$ .*

Denote by  $\widehat{\mathcal{W}}_0$  the Grothendieck group [Sch, Chapter 2, Definition 1.2] of the commutative monoid  $\widehat{\mathcal{W}}_0^+$ . By definition, it is an abelian group together with a monoid homomorphism  $j_0 : \widehat{\mathcal{W}}_0^+ \rightarrow \widehat{\mathcal{W}}_0$  with the following universal property: for each abelian group  $A$  and each monoid homomorphism  $\varphi^+ : \widehat{\mathcal{W}}_0^+ \rightarrow A$ , there is a unique group homomorphism  $\varphi : \widehat{\mathcal{W}}_0 \rightarrow A$  such that  $\varphi \circ j_0 = \varphi^+$ . The group  $\widehat{\mathcal{W}}_0$  is called the Witt-Grothendieck group of  $-\epsilon$ -Hermitian left D-vector spaces. Note that Lemma 3.1 implies that  $j_0$  is injective [Sch, Chapter 2, Lemma 1.3]. Via  $j_0$ , we view  $\widehat{\mathcal{W}}_0^+$  as a submonoid of  $\widehat{\mathcal{W}}_0$ .

Denote by  $E$  the center of  $D$ . Put

$$\Delta := \Delta_{F,D,\epsilon} := \begin{cases} \{1\}, & \epsilon = 1, D = F; \\ E_{\pm}^{\times}/N^{\times}, & \epsilon = 1, D \text{ is a quadratic extension}; \\ F^{\times}/N^{\times}, & \epsilon = 1, D \text{ is a quaternion algebra}; \\ \text{Hil}(F), & \epsilon = -1, D = F; \\ F^{\times}/N^{\times}, & \epsilon = -1, D \text{ is a quadratic extension}; \\ \{1\}, & \epsilon = -1, D \text{ is a quaternion algebra,} \end{cases}$$

where

$$\begin{cases} N^{\times} := \{aa' \mid a \in E^{\times}\} \subset F^{\times}; \\ E_{\pm}^{\times} := \{a \in E^{\times} \mid a' = a \text{ or } -a\}; \\ \text{Hil}(F) := \frac{F^{\times}}{(F^{\times})^2} \times \{\pm 1\}, \end{cases}$$

and  $\text{Hil}(F)$  is viewed as a group with group multiplication

$$(20) \quad (a, t)(a', t') := (aa', tt'(a, a')_F), \quad a, a' \in F^{\times}/(F^{\times})^2, t, t' \in \{\pm 1\}.$$

Here and as usual,  $(, )_F$  denotes the quadratic Hilbert symbol for  $F$ . Then we have a group homomorphism

$$(21) \quad \text{disc} : \widehat{\mathcal{W}}_0 \rightarrow \Delta$$

TABLE 2. The Witt-Grothendieck group  $\widehat{\mathcal{W}}_0$

D	F	quadratic extension	quaternion algebra
$\epsilon = 1$	$2\mathbb{Z}$	$\mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \frac{\mathbb{F}^\times}{\mathbb{N}^\times}$	$\mathbb{Z} \times \frac{\mathbb{F}^\times}{\mathbb{N}^\times}$
$\epsilon = -1$	$\mathbb{Z} \times \text{Hil}(\mathbb{F})$	$\mathbb{Z} \times \frac{\mathbb{F}^\times}{\mathbb{N}^\times}$	$\mathbb{Z}$

such that for each  $V \in \widehat{\mathcal{W}}_0^+$ ,

$$\text{disc } V := \begin{cases} 1, & \epsilon = 1, D = \mathbb{F}; \\ (\prod_{i=1}^m \langle e_i, e_i \rangle_V) \mathbb{N}^\times, & \epsilon = 1, D \text{ is a quadratic extension}; \\ (\prod_{i=1}^m \langle e_i, e_i \rangle_V \langle e_i, e_i \rangle'_V) \mathbb{N}^\times, & \epsilon = 1, D \text{ is a quaternion algebra}; \\ ((\prod_{i=1}^m \langle e_i, e_i \rangle_V) \mathbb{N}^\times, \text{hass } V), & \epsilon = -1, D = \mathbb{F}; \\ (\prod_{i=1}^m \langle e_i, e_i \rangle_V) \mathbb{N}^\times, & \epsilon = -1, D \text{ is a quadratic extension}; \\ 1, & \epsilon = -1, D \text{ is a quaternion algebra,} \end{cases}$$

where  $e_1, e_2, \dots, e_m$  is an orthogonal basis of  $V$ , and  $\text{hass } V$  denotes the Hasse invariant of  $V$  when  $V$  is a symmetric bilinear space,

$$\text{hass } V := \prod_{1 \leq i < j \leq m} (\langle e_i, e_i \rangle_V, \langle e_j, e_j \rangle_V)_F.$$

On the other hand, we also have the dimension homomorphism  $\dim : \widehat{\mathcal{W}}_0 \rightarrow \mathbb{Z}$ . We summarize the well-known results on the classification of  $-\epsilon$ -Hermitian left D-vector spaces as follows in Theorem 3.2.

**Theorem 3.2.** (cf. [Sch, Chapters 5 and 10]). *The homomorphism*

$$(22) \quad \dim \times \text{disc} : \widehat{\mathcal{W}}_0 \rightarrow \mathbb{Z} \times \Delta$$

is injective and its image equals the group of Table 2. If we identify  $\widehat{\mathcal{W}}_0$  with the image of (22), then an element  $(m, \delta)$  of  $\widehat{\mathcal{W}}_0$  belongs to the monoid  $\widehat{\mathcal{W}}_0^+$  if and only if  $m \geq 0$ , and

$$\begin{cases} \delta = 1, & \text{if } m = 0, \\ \delta \neq -1, & \text{if } m = 1, D \text{ is a quaternion algebra and } \epsilon = 1; \\ \delta \in (\mathbb{F}^\times / \mathbb{N}^\times) \times \{1\} \subset \text{Hil}(\mathbb{F}), & \text{if } m = 1, D = \mathbb{F} \text{ and } \epsilon = -1. \end{cases}$$

The fiber product  $\mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \frac{\mathbb{F}^\times}{\mathbb{N}^\times}$  of Table 2 is defined with respect to the quotient homomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ , and the homomorphism  $\frac{\mathbb{F}^\times}{\mathbb{N}^\times} \rightarrow \mathbb{Z}/2\mathbb{Z}$  with kernel  $\frac{\mathbb{F}^\times}{\mathbb{N}^\times}$ .

**3.2. Commutator quotient groups of classical groups.** Let  $U$  be an  $\epsilon$ -Hermitian right D-vector space as before. Then the commutator group  $[G(U), G(U)]$  is closed in  $G(U)$  (cf. [Ri]). Put

$$A(U) := G(U) / [G(U), G(U)],$$

which is an abelian topological group. Given a D-linear isometric embedding

$$\varphi : U \rightarrow U'$$

of  $\epsilon$ -Hermitian right D-vector spaces, we have a continuous group homomorphism

$$(23) \quad G(\varphi) : G(U) \rightarrow G(U')$$

such that for all  $g \in G(G)$ ,

$$(G(\varphi)(g))(\varphi(u) + u') = \varphi(g(u)) + u',$$

for all  $u \in U$ , and all  $u' \in U'$  which is perpendicular to  $\varphi(U)$ . The homomorphism (23) descends to a continuous group homomorphism

$$(24) \quad A(\varphi) : A(U) \rightarrow A(U').$$

The assignments

$$U \mapsto A(U), \quad \varphi \mapsto A(\varphi)$$

are a functor from the category of  $\epsilon$ -Hermitian right D-vector spaces (the morphisms in this category are D-linear isometric embeddings) to the category of abelian topological groups. Write  $A_\infty$  for the direct limit of this functor:

$$A_\infty := \varinjlim_U A(U).$$

Recall that by definition of the direct limit,  $A_\infty$  is an abelian topological group together with a continuous homomorphism  $A(U) \rightarrow A_\infty$  for each  $\epsilon$ -Hermitian right D-vector space  $U$ , satisfying certain universal properties.

We summarize the well-known results on commutator quotient groups of classical groups as follows:

**Theorem 3.3.** (cf. [Wall, Section 2]). *The topological group  $A_\infty$  is canonically isomorphic to the topological group of Table 3. Furthermore the natural homomorphism  $A(U) \rightarrow A_\infty$  is a topological isomorphism in the following cases:*

- (i)  $U$  is a symplectic space or a quaternionic Hermitian space;
- (ii)  $U$  is a non-zero Hermitian or skew-Hermitian space;
- (iii)  $U$  is a non-anisotropic symmetric bilinear space or a non-anisotropic quaternionic skew-Hermitian space.

Identify  $A_\infty$  with the group of Table 3. Write

$$(25) \quad \mu_U : G(U) \rightarrow A_\infty$$

for the homomorphism which descends to the natural homomorphism  $A(U) \rightarrow A_\infty$ . We describe the homomorphism  $\mu_U$  case by case in what follows.

**Case 1:**  $U$  is a quaternionic Hermitian space or a symplectic space. In this case,  $A_\infty$  is trivial, and hence  $\mu_U$  is also trivial.

**Case 2:**  $U$  is a Hermitian space or a skew-Hermitian space. Hilbert's Theorem 90 implies that the map

$$(26) \quad E^\times / F^\times \rightarrow U(E), \quad xF^\times \mapsto \frac{x}{x^t}$$

TABLE 3. The commutator quotient group  $A_\infty$

D	F	quadratic extension	quaternion algebra
$\epsilon = 1$	$\frac{F^\times}{N^\times} \times \{\pm 1\}$	$\frac{E^\times}{F^\times}$	$\{1\}$
$\epsilon = -1$	$\{1\}$	$\frac{E^\times}{F^\times}$	$\frac{F^\times}{N^\times}$

is a topological isomorphism, where  $U(E) := \{a \in E^\times \mid aa^t = 1\}$ . The homomorphism  $\mu_U$  is the composition of

$$G(U) \xrightarrow{\det} U(E) \xrightarrow{\text{the inverse of (26)}} E^\times / F^\times.$$

**Case 3:**  $U$  is a symmetric bilinear space. In this case,  $G(U)$  is generated by reflections [Di1, Proposition 8], namely, elements of the form  $s_v$  such that

$$\begin{cases} s_v(v) = -v, \text{ and} \\ s_v(u) = u \text{ for all element } u \in U \text{ which is perpendicular to } v, \end{cases}$$

where  $v$  is a non-isotropic vector in  $U$ . The homomorphism  $\mu_U$  is determined by

$$\mu_U(s_v) = (\langle v, v \rangle_U N^\times, -1)$$

for all reflections  $s_v \in G(U)$  (cf. [OM, Section 55]).

**Case 4:**  $U$  is a quaternionic skew-Hermitian space. In this case,  $G(U)$  is generated by quasi-symmetries [Di2, Theorem 2], namely, elements of the form  $s_{v,a}$  such that

$$\begin{cases} s_{v,a}(v) = va, \text{ and} \\ s_{v,a}(u) = u \text{ for all element } u \in U \text{ which is perpendicular to } v, \end{cases}$$

where  $v$  is a non-isotropic vector in  $U$ , and  $a$  is an element of  $D$  which commutes with  $\langle v, v \rangle_U$  and satisfies that  $aa^t = 1$ . The homomorphism  $\mu_U$  is determined by

$$\mu_U(s_{v,a}) = \begin{cases} (1+a)(1+a^t)N^\times, & \text{if } a \neq -1; \\ \langle v, v \rangle_U \langle v, v \rangle_U^t N^\times, & \text{if } a = -1 \end{cases}$$

for all quasi-symmetries  $s_{v,a} \in G(U)$  (cf. [Ku2, Corollary 1.5 and Proposition 1.6] and [Ya, Proposition 6.5]).

**3.3. The generalized Witt-Grothendieck group.** Let  $U$  be an  $\epsilon$ -Hermitian right  $D$ -vector space, and let  $V$  be a  $-\epsilon$ -Hermitian left  $D$ -vector space.

**Lemma 3.4.** *There exists a smooth oscillator representation of  $\bar{J}(U, V)$ . Every smooth oscillator representation  $\omega_{U, V}$  of  $\bar{J}(U, V)$  is unitarizable and has the following property: when  $U$  is a non-zero symplectic space,  $\varepsilon_U$  acts through the scalar multiplication by  $(-1)^{\dim V}$  in  $\omega_{U, V}$ , and likewise for  $V$  when  $V$  is a non-zero symplectic space.*

*Proof.* Besides the splitting of metaplectic covers [Ku2, Proposition 4.1], the first assertion and the second part of the last assertion are due to the fact that any two elements in a metaplectic group commute with each other if their projections to the symplectic group commute with each other [MVW, Chapter 2, Lemma II.5]. The unitarizability of  $\omega_{U, V}$  is due to the fact that all characters on  $G(U)$  and  $G(V)$  are unitary (see Theorem 3.3). □

**Lemma 3.5.** *Every smooth oscillator representation of  $\bar{J}_U(V)$  extends to a smooth oscillator representation of  $\bar{J}(U, V)$ .*

*Proof.* Let  $\omega$  be a smooth oscillator representation of  $\bar{J}(U, V)$ . By Lemma 1.4, every smooth oscillator representation of  $\bar{J}_U(V)$  is of the form  $\chi \otimes \omega|_{\bar{J}_U(V)}$  for some character  $\chi$  of  $\bar{J}_U(V)$  which is trivial on  $H(U \otimes_D V)$ . The lemma then follows as  $\chi$  extends to a character of  $\bar{J}(U, V)$ . □

**Lemma 3.6.** *Two genuine smooth oscillator representations  $\omega_1$  and  $\omega_2$  of  $\bar{J}_U(V)$  are isomorphic if and only if  $(V, \omega_1)$  and  $(V, \omega_2)$  are isomorphic as enhanced oscillator representations of  $\bar{G}(U)$ .*

*Proof.* The “only if” part is trivial. To prove the “if” part, assume that  $(V, \omega_1)$  and  $(V, \omega_2)$  are isomorphic as enhanced oscillator representations. This amounts to saying that there is an element  $g \in G(V)$  and a linear isomorphism  $\varphi : \omega_1 \rightarrow \omega_2$  such that the diagram

$$\begin{array}{ccc} \omega_1 & \xrightarrow{\varphi} & \omega_2 \\ h \downarrow & & \downarrow g_U(h) \\ \omega_1 & \xrightarrow{\varphi} & \omega_2 \end{array}$$

commutes for every  $h \in \bar{J}_U(V)$ , where  $g_U : \bar{J}_U(V) \rightarrow \bar{J}_U(V)$  is the automorphism induced by  $g : V \rightarrow V$ .

Using Lemma 3.5, we extend  $\omega_2$  to a representation of  $\bar{J}(U, V)$ , which we still denote by  $\omega_2$ . Let  $\bar{g}$  be an element of  $\bar{G}(V)$  which lifts  $g$ . Then  $g_U(h) = \bar{g}h\bar{g}^{-1}$  for every  $h \in \bar{J}_U(V)$ . Therefore the diagram

$$\begin{array}{ccc} \omega_1 & \xrightarrow{\bar{g}^{-1} \circ \varphi} & \omega_2 \\ h \downarrow & & \downarrow h \\ \omega_1 & \xrightarrow{\bar{g}^{-1} \circ \varphi} & \omega_2 \end{array}$$

commutes for every  $h \in \bar{J}_U(V)$ , and consequently,  $\omega_1$  and  $\omega_2$  are isomorphic as representations of  $\bar{J}_U(V)$ . □

Recall the monoid  $\widehat{\mathcal{W}}_U^+$  from the Introduction. Note that  $\bar{J}_U(0) = \bar{G}(U) \times F$ . The map

$$(27) \quad (G(U))^* \rightarrow \widehat{\mathcal{W}}_U^+, \quad \chi \mapsto (0, \bar{\chi} \otimes \psi)$$

is clearly an injective monoid homomorphism, where  $\bar{\chi} \in (\bar{G}(U))^*$  denotes the pullback of  $\chi$  through the covering homomorphism  $\bar{G}(U) \rightarrow G(U)$ . Using this homomorphism, we view  $(G(U))^*$  as a submonoid of  $\widehat{\mathcal{W}}_U^+$ . Define a monoid homomorphism

$$(28) \quad \widehat{q}_U^+ : \widehat{\mathcal{W}}_U^+ \rightarrow \widehat{\mathcal{W}}_0^+, \quad (V, \omega) \mapsto V.$$

It is surjective by Lemma 3.4.

**Lemma 3.7.** *For each  $V \in \widehat{\mathcal{W}}_0^+$ , the fiber  $(\widehat{q}_U^+)^{-1}(V)$  is a principal homogeneous space of  $(G(U))^*$  under the additive structure on  $\widehat{\mathcal{W}}_U^+$ , that is, the map*

$$(29) \quad + : (G(U))^* \times (\widehat{q}_U^+)^{-1}(V) \rightarrow (\widehat{q}_U^+)^{-1}(V)$$

*is a well-defined simply transitive action of  $(G(U))^*$ .*

*Proof.* The map (29) is clearly a well-defined group action. Lemma 1.4 implies that this action is transitive. Lemma 1.4 and Lemma 3.6 further imply that it is simply transitive. □

**Lemma 3.8.** *The monoid  $\widehat{\mathcal{W}}_U^+$  is cancellative; namely, for all  $\sigma_1, \sigma_2, \sigma_3 \in \widehat{\mathcal{W}}_U^+$ , if  $\sigma_1 + \sigma_3 = \sigma_2 + \sigma_3$ , then  $\sigma_1 = \sigma_2$ .*

*Proof.* Applying the homomorphism  $\widehat{q}_U^+$  to the equality  $\sigma_1 + \sigma_3 = \sigma_2 + \sigma_3$  in the lemma, Lemma 3.1 implies that

$$\widehat{q}_U^+(\sigma_1) = \widehat{q}_U^+(\sigma_2).$$

Then Lemma 3.7 implies that  $\sigma_2 = \chi + \sigma_1$  for some  $\chi \in (G(U))^*$ . Therefore

$$\chi + (\sigma_1 + \sigma_3) = \sigma_2 + \sigma_3 = \sigma_1 + \sigma_3,$$

and Lemma 3.7 implies that  $\chi$  is trivial. This proves that  $\sigma_2 = \sigma_1$ . □

**Lemma 3.9.** *Let  $\sigma_1 = (V_1, \omega_1)$ ,  $\sigma_2 = (V_2, \omega_2) \in \widehat{\mathcal{W}}_U^+$ . If  $V_2$  is isometrically isomorphic to a subspace of  $V_1$ , then there exists a unique element  $\sigma_3 \in \widehat{\mathcal{W}}_U^+$  such that the equality*

$$\sigma_2 + \sigma_3 = \sigma_1$$

*holds in  $\widehat{\mathcal{W}}_U^+$ .*

*Proof.* The uniqueness is a direct consequence of Lemma 3.8. For existence, take an element  $\sigma'_3 = (V_3, \omega)$  so that  $V_2 + V_3 = V_1$  in  $\widehat{\mathcal{W}}_0^+$ . Then Lemma 3.7 implies that

$$\chi + (\sigma_2 + \sigma'_3) = \sigma_1$$

for some  $\chi \in (G(U))^*$ . The lemma follows by putting  $\sigma_3 := \chi + \sigma'_3$ . □

Recall from the Introduction the hyperbolic plane  $\mathbb{H} \in \widehat{\mathcal{W}}_0^+$  and the hyperbolic plane  $\mathbb{H}_U \in \widehat{\mathcal{W}}_U^+$ . Lemma 3.9 immediately implies the following lemma.

**Lemma 3.10.** *For every  $\sigma \in \widehat{\mathcal{W}}_U^+$ , there exists a unique element  $\sigma^\vee \in \widehat{\mathcal{W}}_U^+$  such that the equality*

$$\sigma + \sigma^\vee = (\dim \sigma) \mathbb{H}_U$$

*holds in  $\widehat{\mathcal{W}}_U^+$ .*

As in Section 3.1, denote by  $\widehat{\mathcal{W}}_U$  the Grothendieck group of the commutative monoid  $\widehat{\mathcal{W}}_U^+$ , and view  $\widehat{\mathcal{W}}_U^+$  as a submonoid of  $\widehat{\mathcal{W}}_U$  (by Lemma 3.8). The surjective monoid homomorphism  $\widehat{q}_U^+$  uniquely extends to a surjective group homomorphism

$$(30) \quad \widehat{q}_U : \widehat{\mathcal{W}}_U \rightarrow \widehat{\mathcal{W}}_0.$$

Recall that  $\mathcal{W}_U$  denotes the set of Witt towers in  $\widehat{\mathcal{W}}_U^+$ , which is a quotient monoid of  $\widehat{\mathcal{W}}_U^+$ .

Using Lemmas 3.7–3.10, it is routine to prove the following proposition. We omit the details.

**Proposition 3.11.** *The following hold true:*

(a) *The sequence*

$$(31) \quad 1 \rightarrow (G(U))^* \rightarrow \widehat{\mathcal{W}}_U \xrightarrow{\widehat{q}_U} \widehat{\mathcal{W}}_0 \rightarrow 1$$

*is exact, and  $\widehat{q}_U^{-1}(\widehat{\mathcal{W}}_0^+) = \widehat{\mathcal{W}}_U^+$ .*

(b) *Every element  $\mathbf{t} \in \mathcal{W}_U$  is uniquely of the form*

$$\mathbf{t} = \{\sigma_{\mathbf{t}} + m \mathbb{H}_U \mid m = 0, 1, 2, \dots\},$$

*where  $\sigma_{\mathbf{t}}$  is an anisotropic element of  $\widehat{\mathcal{W}}_U^+$ , as defined in the Introduction.*

(c) *The inclusion map  $\widehat{\mathcal{W}}_U^+ \rightarrow \widehat{\mathcal{W}}_U$  descends to a bijection  $\mathcal{W}_U \rightarrow \widehat{\mathcal{W}}_U / (\mathbb{Z} \cdot \mathbb{H}_U)$ . Consequently, the monoid  $\mathcal{W}_U$  is an abelian group.*



(d) *The sequence*

$$1 \rightarrow (\mathbf{G}(U))^* \xrightarrow{j_U} \mathcal{W}_U \xrightarrow{q_U} \mathcal{W}_0 \rightarrow 1$$

is exact, where  $j_U$  is the restriction of the quotient map  $\widehat{\mathcal{W}}_U^+ \rightarrow \mathcal{W}_U$  to  $(\mathbf{G}(U))^*$ , and  $q_U$  is the descent of  $\widehat{q}_U^+$ .

Extending the notion of the split rank of an element of  $\widehat{\mathcal{W}}_U^+$ , we define the split rank of each  $\sigma \in \widehat{\mathcal{W}}_U$ , which is denoted by  $\text{rank } \sigma$ , to be the integer  $m$  such that  $\sigma - m\mathbb{H}_U$  is an anisotropic element of  $\widehat{\mathcal{W}}_U^+$ . It is clear that

$$\sigma \in \widehat{\mathcal{W}}_U^+ \text{ if and only if } \text{rank } \sigma \geq 0.$$

**3.4. Restrictions of enhanced oscillator representations.** Let  $U'$  be an  $\epsilon$ -Hermitian right D-vector space with a D-linear isometric embedding

$$\varphi : U \hookrightarrow U'.$$

For convenience, we identify  $U$  with a subspace of  $U'$  via  $\varphi$ . Denote by  $U^\perp$  the orthogonal complement of  $U$  in  $U'$ . Then the induced embedding  $\mathbf{G}(U) \times \mathbf{G}(U^\perp) \hookrightarrow \mathbf{G}(U')$  uniquely lifts to a “genuine” homomorphism

$$(32) \quad \bar{\mathbf{G}}(U) \times \bar{\mathbf{G}}(U^\perp) \rightarrow \bar{\mathbf{G}}(U').$$

Here “genuine” means that when  $U$  is a symplectic space, the homomorphism (32) maps both  $\varepsilon_U$  and  $\varepsilon_{U^\perp}$  to  $\varepsilon_{U'}$ .

For every  $-\epsilon$ -Hermitian left D-vector space  $V$ , combining (32) with the homomorphism

$$\begin{aligned} \mathbf{H}(U \otimes_{\mathbf{D}} V) \times \mathbf{H}(U^\perp \otimes_{\mathbf{D}} V) &\rightarrow \mathbf{H}(U' \otimes_{\mathbf{D}} V), \\ ((w, t), (w', t')) &\mapsto (w + w', t + t'), \end{aligned}$$

we obtain a homomorphism

$$(33) \quad \bar{\mathbf{J}}_U(V) \times \bar{\mathbf{J}}_{U^\perp}(V) \rightarrow \bar{\mathbf{J}}_{U'}(V).$$

For each genuine smooth oscillator representation  $\omega'$  of  $\bar{\mathbf{J}}_{U'}(V)$ , its restriction through (33) is uniquely of the form

$$(34) \quad \omega'|_{\bar{\mathbf{J}}_U(V) \times \bar{\mathbf{J}}_{U^\perp}(V)} \cong \omega'|_U \otimes \omega'|_{U^\perp},$$

where  $\omega'|_U$  is a genuine smooth oscillator representation of  $\bar{\mathbf{J}}_U(V)$ , and  $\omega'|_{U^\perp}$  is a genuine smooth oscillator representation of  $\bar{\mathbf{J}}_{U^\perp}(V)$ . In turn this defines a monoid homomorphism (the restriction map)

$$(35) \quad \widehat{r}_\varphi^+ : \widehat{\mathcal{W}}_{U'}^+ \rightarrow \widehat{\mathcal{W}}_U^+, \quad (V, \omega') \mapsto (V, \omega'|_U).$$

It extends to a group homomorphism

$$(36) \quad \widehat{r}_\varphi : \widehat{\mathcal{W}}_{U'} \rightarrow \widehat{\mathcal{W}}_U$$

and descends to a group homomorphism

$$(37) \quad r_\varphi : \mathcal{W}_{U'} \rightarrow \mathcal{W}_U.$$

**Lemma 3.12.** *The homomorphism  $\widehat{r}_\varphi^+ : \widehat{\mathcal{W}}_{U'}^+ \rightarrow \widehat{\mathcal{W}}_U^+$  depends only on  $U$  and  $U'$ ; that is, it does not depend on the D-linear isometric embedding  $\varphi$ . Consequently, both  $\widehat{r}_\varphi$  and  $r_\varphi$  do not depend on  $\varphi$ .*

*Proof.* If  $U' = U$ , then  $\varphi$  induces an inner automorphism  $\bar{J}_U(V) \rightarrow \bar{J}_U(V)$ , and consequently  $\widehat{r}_\varphi^+$  is the identity map. In general, the lemma follows by using Witt’s extension theorem [Sch, Theorem 9.1] and applying the above argument to  $U'$ .  $\square$

In view of Lemma 3.12, we also use  $\widehat{r}_U^{U'}$  and  $r_U^{U'}$  to denote  $\widehat{r}_\varphi$  and  $r_\varphi$ , respectively. The following lemma is an obvious consequence of Proposition 3.11.

**Lemma 3.13.** *One has that*

$$\widehat{W}_U = \widehat{r}_U^{U'}(\widehat{W}_{U'}) + (G(U))^* \quad \text{and} \quad \mathcal{W}_U = r_U^{U'}(\mathcal{W}_{U'}) + (G(U))^*.$$

The assignments

$$U \mapsto \widehat{W}_U^+, \quad \varphi \mapsto \widehat{r}_\varphi^+$$

form a contravariant functor from the category of  $\epsilon$ -Hermitian right  $D$ -vector spaces (the morphisms in this category are  $D$ -linear isometric embeddings) to the category of commutative monoids. Write  $\widehat{W}_\infty^+$  for the inverse limit of this functor:

$$\widehat{W}_\infty^+ := \varprojlim_U \widehat{W}_U^+.$$

Likewise, respectively using the homomorphisms of (36) and (37), we form the inverse limits

$$\widehat{W}_\infty := \varprojlim_U \widehat{W}_U \quad \text{and} \quad \mathcal{W}_\infty := \varprojlim_U \mathcal{W}_U.$$

Using the exact sequence of part (a) of Proposition 3.11, the second assertion of Theorem 3.3 easily implies the following lemma.

**Lemma 3.14.** *The natural homomorphisms  $\widehat{W}_\infty^+ \rightarrow \widehat{W}_U^+$ ,  $\widehat{W}_\infty \rightarrow \widehat{W}_U$  and  $\mathcal{W}_\infty \rightarrow \mathcal{W}_U$  are isomorphisms in the following cases:*

- (i)  $U$  is a symplectic space or a quaternionic Hermitian space;
- (ii)  $U$  is a non-zero Hermitian or skew-Hermitian space;
- (iii)  $U$  is a non-anisotropic symmetric bilinear space or a non-anisotropic quaternionic skew-Hermitian space.

We will explicitly determine the group  $\widehat{W}_\infty$  (and hence the monoid  $\widehat{W}_\infty^+$  and the group  $\mathcal{W}_\infty$ ) in all cases in the next section.

#### 4. KUDLA CHARACTERS AND THE KUDLA HOMOMORPHISM

We refer the reader to Section 3.1 for the notations. Define a compact abelian topological group

$$K := \begin{cases} \text{Hil}(F), & \text{if } (D, \epsilon) = (F, -1); \\ \mathbb{E}^\times / \mathbb{N}^\times, & \text{otherwise.} \end{cases}$$

In this section, we will define a (canonical) group homomorphism

$$\xi_\infty : \mathcal{W}_\infty \rightarrow K^*.$$

For  $U$  split and non-zero, the related homomorphism  $\xi_U : \mathcal{W}_U \rightarrow K^*$  regulates a certain transformation property of the ‘‘Schrödinger functional’’ (see Lemma 2.1) in various Schrödinger realizations of an enhanced oscillator representation  $(V, \omega)$ , and in turn it determines a  $\tilde{G}(U)$ -intertwining map from  $\omega$  to a degenerate principal series representation of  $\tilde{G}(U)$ . The definition is inspired by the work of Kudla [Ku2] on the splitting of metaplectic covers and the pioneer work of Kudla-Rallis [KR1] on the structure of degenerate principal series representations. We thus call

$\xi_\infty(\mathfrak{t}) \in K^*$  the Kudla character of a Witt tower  $\mathfrak{t} \in \mathcal{W}_\infty$ , and  $\xi_\infty$  the Kudla homomorphism.

Moreover we will show (Corollary 4.8) that  $\xi_\infty$  is surjective and  $\ker \xi_\infty \cong \mathbb{Z}/2\mathbb{Z}$ . Let  $\mathfrak{t}_\infty^\circ \in \mathcal{W}_\infty$  be the non-trivial element of  $\ker \xi_\infty$ . For each  $\epsilon$ -Hermitian right D-vector space  $U$ , denote by  $\mathfrak{t}_U^\circ \in \mathcal{W}_U$  the image of  $\mathfrak{t}_\infty^\circ$  under the natural homomorphism  $\mathcal{W}_\infty \rightarrow \mathcal{W}_U$ . This is the anti-split Witt tower in the Introduction.

**4.1. Some (coherent) characters on Siegel parabolic subgroups.** Let  $U$  be an  $\epsilon$ -Hermitian right D-vector space which is split in the sense that  $\dim U = 2 \operatorname{rank} U$ . For every Lagrangian subspace  $X$  of  $U$ , denote by  $P(X)$  the (Siegel) parabolic subgroup of  $G(U)$  stabilizing  $X$ , and by  $\bar{P}(X) \rightarrow P(X)$  the covering homomorphism induced by  $\bar{G}(U) \rightarrow G(U)$ .

We use  $|\cdot|_X$  to denote the following positive character on  $\bar{P}(X)$ :

$$(38) \quad \bar{P}(X) \rightarrow P(X) \xrightarrow{\text{restriction on } X} \operatorname{GL}(X) \xrightarrow{\det} E^\times \xrightarrow{|\cdot|_E^{\delta_D}} \mathbb{R}_+^\times.$$

Here and henceforth “det” stands for the reduced norm;  $|\cdot|_E$  is the normalized absolute value on  $E$ ;  $\mathbb{R}_+^\times$  denotes the multiplicative group of positive real numbers; and  $\delta_D$  is the degree of  $D$  over  $E$ , which is 2 if  $D$  is a quaternion algebra and is 1 otherwise.

Let  $\sigma = (V, \omega) \in \widehat{\mathcal{W}}_U^+$ . By Lemma 2.1, there exists a unique (up to scalar multiples) non-zero linear functional  $\lambda_{X \otimes_D V}$  on  $\omega$  which is invariant under  $X \otimes_D V \subset H(U \otimes_D V)$ . Since  $\bar{P}(X)$  normalizes  $X \otimes_D V$  in  $\bar{J}_U(V)$ , there exists a character  $\kappa_{\sigma, X}$  on  $\bar{P}(X)$  such that

$$(39) \quad \lambda_{X \otimes_D V}(h \cdot \phi) = \kappa_{\sigma, X}(h) |h|_X^{\frac{\dim \sigma}{2}} \lambda_{X \otimes_D V}(\phi), \quad h \in \bar{P}(X), \phi \in \omega.$$

**Lemma 4.1.** *Let  $Y$  be another Lagrangian subspace of  $U$ . Then*

$$\kappa_{\sigma, X}(h) = \kappa_{\sigma, Y}(h)$$

for all  $h \in \bar{P}(X) \cap \bar{P}(Y)$ .

*Proof.* Using the Jordan decomposition, we assume without loss of generality that  $h$  is semisimple; namely, the image  $h_0$  of  $h$  under the covering homomorphism  $\bar{G}(U) \rightarrow G(U)$  is semisimple. Then it is elementary to see that there is an  $h_0$ -stable Lagrangian subspace  $Y'$  of  $U$  such that

$$U = X \oplus Y' \quad \text{and} \quad Y = (X \cap Y) \oplus (Y' \cap Y).$$

Using the complete polarization

$$U \otimes_D V = (X \otimes_D V) \oplus (Y' \otimes_D V),$$

we realize  $\omega|_{H(U \otimes_D V)}$  on the space  $\mathcal{S}(X \otimes_D V)$  as in (3). Respectively write  $\mu_X$  and  $\mu_{X \cap Y}$  for Haar measures on  $X \otimes_D V$  and  $(X \cap Y) \otimes_D V$ . Then

$$\lambda_{X \otimes_D V} : \mathcal{S}(X \otimes_D V) \rightarrow \mathbb{C}, \quad \phi \mapsto \int_{X \otimes_D V} \phi \mu_X$$

is the unique (up to scalar multiples) linear functional on  $\omega$  which is invariant under  $X \otimes_D V$ . Likewise

$$\lambda_{Y \otimes_D V} : \mathcal{S}(X \otimes_D V) \rightarrow \mathbb{C}, \quad \phi \mapsto \int_{(X \cap Y) \otimes_D V} \phi|_{(X \cap Y) \otimes_D V} \mu_{X \cap Y}$$

is the unique (up to scalar multiples) linear functional on  $\omega$  which is invariant under  $Y \otimes_D V$ .

Note that there is a non-zero constant  $c_h$  such that

$$(h \cdot \phi)(u) = c_h \phi((h_0^{-1} \otimes 1)(u)) \quad \text{for all } \phi \in \mathcal{S}(X \otimes_{\mathbb{D}} V), u \in X \otimes_{\mathbb{D}} V,$$

where “1” stands for the identity map of  $V$ . With the above explicit realizations, it is then routine to verify the equality in the lemma. We omit the details.  $\square$

Put

$$(40) \quad \bar{G}(U)_{\text{split}} := \bigcup_{X \text{ is a Lagrangian subspace of } U} \bar{P}(X).$$

By Lemma 4.1, we get a well-defined map

$$\kappa_{\sigma} : \bar{G}(U)_{\text{split}} \rightarrow \mathbb{C}^{\times},$$

which sends  $h \in \bar{P}(X)$  to  $\kappa_{\sigma, X}(h)$ , for each Lagrangian subspace  $X$  of  $U$ .

**Lemma 4.2.** *The map  $\kappa_{\sigma} : \bar{G}(U)_{\text{split}} \rightarrow \mathbb{C}^{\times}$  is  $\bar{G}(U)$ -conjugation invariant.*

*Proof.* Let  $h \in \bar{P}(X)$  and let  $g \in \bar{G}(U)$ . Put  $X' := g_0(X)$ , where  $g_0$  denotes the image of  $g$  under the covering homomorphism  $\bar{G}(U) \rightarrow G(U)$ . Then the linear functional

$$\lambda' : \omega \rightarrow \mathbb{C}, \quad \phi \mapsto \lambda_{X \otimes_{\mathbb{D}} V}(g^{-1} \cdot \phi)$$

is non-zero and invariant under  $X' \otimes_{\mathbb{D}} V \subset H(U \otimes_{\mathbb{D}} V)$ . The definition of  $\kappa_{\sigma, X'}$  implies that

$$\lambda'((ghg^{-1}) \cdot (g \cdot \phi)) = \kappa_{\sigma, X'}(ghg^{-1})|ghg^{-1}|_{X'}^{\dim \sigma} \lambda'(g \cdot \phi), \quad \phi \in \omega.$$

This is equivalent to the following equality:

$$(41) \quad \lambda_{X \otimes_{\mathbb{D}} V}(h \cdot \phi) = \kappa_{\sigma, X'}(ghg^{-1})|h|_{X'}^{\frac{\dim \sigma}{2}} \lambda_{X \otimes_{\mathbb{D}} V}(\phi), \quad \phi \in \omega.$$

Therefore  $\kappa_{\sigma}(ghg^{-1}) = \kappa_{\sigma}(h)$ , by comparing (39) and (41).  $\square$

To summarize, the map  $\kappa_{\sigma}$  has the following two properties:

- P1: it is  $\bar{G}(U)$ -conjugation invariant;
- P2: its restriction to  $\bar{P}(X)$  is a continuous group homomorphism, for each Lagrangian subspace  $X$  of  $U$ .

For any abelian topological group  $A$ , denote by  $\text{Hom}(\bar{G}(U)_{\text{split}}, A)$  the group of all maps from  $\bar{G}(U)_{\text{split}}$  to  $A$  with the properties P1 and P2. Thus

$$(42) \quad \kappa_{\sigma} \in \text{Hom}(\bar{G}(U)_{\text{split}}, \mathbb{C}^{\times}).$$

It is easily verified that

$$(43) \quad \widehat{\mathcal{W}}_U^+ \rightarrow \text{Hom}(\bar{G}(U)_{\text{split}}, \mathbb{C}^{\times}), \quad \sigma \mapsto \kappa_{\sigma}$$

is a monoid homomorphism, and  $\kappa_{\mathbb{H}_U} = 1$ . Therefore the homomorphism (43) extends to a group homomorphism

$$(44) \quad \widehat{\mathcal{W}}_U \rightarrow \text{Hom}(\bar{G}(U)_{\text{split}}, \mathbb{C}^{\times}), \quad \sigma \mapsto \kappa_{\sigma},$$

and descends to a group homomorphism

$$(45) \quad \mathcal{W}_U \rightarrow \text{Hom}(\bar{G}(U)_{\text{split}}, \mathbb{C}^{\times}), \quad \mathfrak{t} \mapsto \kappa_{\mathfrak{t}}.$$

4.2. **The group  $\text{Hom}(\bar{G}(U)_{\text{split}}, \mathbb{C}^\times)$  and the Kudla homomorphism.** In this subsection, further assume that  $U$  is non-zero. We first work with the group  $G(U)$ . Put

$$G(U)_{\text{split}} := \bigcup_{X \text{ is a Lagrangian subspace of } U} P(X).$$

Similar to  $\text{Hom}(\bar{G}(U)_{\text{split}}, A)$ , we define the group  $\text{Hom}(G(U)_{\text{split}}, A)$  for every abelian topological group  $A$ .

Define a map

$$\nu_U : G(U)_{\text{split}} \rightarrow E^\times / N^\times$$

by sending  $h \in P(X)$  to  $\det(h|_X)N^\times$ , for each Lagrangian subspace  $X$  of  $U$ . We omit the proof of the following elementary lemma.

**Lemma 4.3.** *The map  $\nu_U$  is well-defined, belongs to  $\text{Hom}(G(U)_{\text{split}}, E^\times / N^\times)$ , and is surjective. The pullback through  $\nu_U$  yields a group isomorphism*

$$\text{Hom}(G(U)_{\text{split}}, A) \cong \text{Hom}(E^\times / N^\times, A)$$

for every abelian topological group  $A$ .

Now assume that  $U$  is a symplectic space. Then we have two natural maps (both two-to-one):

$$\widetilde{\text{Sp}}(U)_{\text{split}} := \bar{G}(U)_{\text{split}} \rightarrow \text{Sp}(U)_{\text{split}} := G(U)_{\text{split}}$$

and

$$K = \text{Hil}(F) \rightarrow F^\times / N^\times, \quad (a, t) \mapsto a.$$

Here the first map is obtained by restricting the metaplectic cover. In what follows we define a map

$$\tilde{\nu}_U \in \text{Hom}(\widetilde{\text{Sp}}(U)_{\text{split}}, \text{Hil}(F))$$

which lifts  $\nu_U \in \text{Hom}(\text{Sp}(U)_{\text{split}}, F^\times / N^\times)$ .

Write  $\sigma_\psi = (V_\psi, \omega_\psi)$  for the unique element of  $\widehat{\mathcal{W}}_U^+$  so that  $V_\psi$  is one dimensional and has a vector  $v_0$  such that  $\langle v_0, v_0 \rangle_{V_\psi} = 1$ . Applying the arguments of Section 4.1, we get a function  $\kappa_{\sigma_\psi} \in \text{Hom}(\widetilde{\text{Sp}}(U)_{\text{split}}, \mathbb{C}^\times)$ .

On the other hand, define a character  $\gamma_\psi$  on  $\text{Hil}(F)$  by

$$(46) \quad \gamma_\psi(a, t) := t \frac{\gamma(x \mapsto \psi(ax^2))}{\gamma(x \mapsto \psi(x^2))} \in \mathbb{C}^\times, \quad (a, t) \in \text{Hil}(F),$$

where the two  $\gamma$ 's on the right hand side of (46) stand for Weil indices (see [Weil, Section 14] or [Weis]) of non-degenerate characters (on  $F$ ) of degree two.

**Lemma 4.4.** (a) *There exists a unique element  $\tilde{\nu}_U \in \text{Hom}(\widetilde{\text{Sp}}(U)_{\text{split}}, \text{Hil}(F))$  which lifts  $\nu_U$  and makes the diagram*

$$\begin{array}{ccc} \widetilde{\text{Sp}}(U)_{\text{split}} & \xrightarrow{\tilde{\nu}_U} & \text{Hil}(F) \\ \downarrow \kappa_{\sigma_\psi} & & \downarrow \gamma_\psi \\ \mathbb{C}^\times & \xlongequal{\quad} & \mathbb{C}^\times \end{array}$$

commute. Moreover,  $\tilde{\nu}_U$  is independent of the choice of the non-trivial unitary character  $\psi$ .

(b) *The map  $\tilde{\nu}_U$  is surjective, and via pullback it yields a group isomorphism*

$$\text{Hom}(\widetilde{\text{Sp}}(U)_{\text{split}}, A) \cong \text{Hom}(\text{Hil}(F), A)$$

for every abelian topological group  $A$ .

*Proof.* The uniqueness assertion of part (a) is obvious. Let  $\psi'$  be an arbitrary non-trivial unitary character of  $F$ . Replacing the fixed character  $\psi$  by  $\psi'$  in the previous arguments, we get two functions

$$\kappa_{\sigma_{\psi'}} \in \text{Hom}(\widetilde{\text{Sp}}(V)_{\text{split}}, \mathbb{C}^\times) \quad \text{and} \quad \gamma_{\psi'} : \text{Hil}(F) \rightarrow \mathbb{C}^\times.$$

It is known that [Rao, Corollary A.5.]

$$(47) \quad \gamma_\psi(a, t) \gamma_{\psi'}(a, t) = (a, -\alpha)_F, \quad (a, t) \in \text{Hil}(F),$$

where  $\alpha \in F^\times$  is determined by the formula

$$\psi'(x) = \psi(\alpha x), \quad x \in F.$$

Let  $h \in \widetilde{\text{Sp}}(U)_{\text{split}}$  and denote by  $h_0$  its image under the map  $\widetilde{\text{Sp}}(U)_{\text{split}} \rightarrow \text{Sp}(U)_{\text{split}}$ . By using the Schrödinger model for the dual pair of  $\text{Sp}(U)$  and an even orthogonal group [Ku3, Section II.4], we have that

$$(48) \quad \kappa_{\sigma_\psi}(h) \kappa_{\sigma_{\psi'}}(h) = (\nu_U(h_0), -\alpha)_F.$$

Then it is elementary to see that (47) and (48) imply that

$$\kappa_{\sigma_{\psi'}}(h) = \gamma_{\psi'}(\nu_U(h_0), t_h),$$

where  $t_h \in \{\pm 1\}$  is independent of  $\psi'$ . Put  $\tilde{\nu}_U(h) := (\nu_U(h_0), t_h)$ . It is then routine to check that  $\tilde{\nu}_U \in \text{Hom}(\widetilde{\text{Sp}}(U)_{\text{split}}, \text{Hil}(F))$ , and  $\tilde{\nu}_U$  has all properties of part (b) of the lemma.  $\square$

We are back in the general case. Recall the compact abelian group  $K$  from the beginning of this section. Define  $\bar{\nu}_U \in \text{Hom}(\bar{G}(U)_{\text{split}}, K)$  by

$$(49) \quad \bar{\nu}_U := \begin{cases} \tilde{\nu}_U, & \text{if } U \text{ is a symplectic space;} \\ \nu_U, & \text{otherwise.} \end{cases}$$

For all cases, combining Lemmas 4.3 and 4.4, we get an isomorphism

$$(50) \quad (\bar{\nu}_U)_* : \text{Hom}(\bar{G}(U)_{\text{split}}, \mathbb{C}^\times) \xrightarrow{\sim} K^*.$$

We then have the homomorphism

$$(51) \quad \xi_U : \mathcal{W}_U \rightarrow K^*, \quad \mathbf{t} \mapsto (\bar{\nu}_U)_*(\kappa_{\mathbf{t}}).$$

**Definition 4.5.** *The Kudla homomorphism*

$$\xi_\infty : \mathcal{W}_\infty \rightarrow K^*$$

*is the composition of  $\xi_U$  with the natural isomorphism  $\mathcal{W}_\infty \rightarrow \mathcal{W}_U$  (recall that  $U$  is assumed to be split and non-zero in this subsection).*

**Lemma 4.6.** *The Kudla homomorphism  $\xi_\infty$  is independent of the non-zero split  $\epsilon$ -Hermitian right D-vector spaces  $U$ .*

*Proof.* Let  $\varphi : U \rightarrow U'$  be a D-linear isometric embedding of non-zero split  $\epsilon$ -Hermitian right D-vector spaces. Recall the restriction homomorphism  $r_\varphi : \mathcal{W}_{U'} \rightarrow \mathcal{W}_U$  from (37). It suffices to show that  $\xi_U \circ r_\varphi = \xi_{U'}$ . This is a direct consequence of the fact that the diagram

$$\begin{array}{ccc} \bar{G}(U)_{\text{split}} & \longrightarrow & \bar{G}(U')_{\text{split}} \\ \downarrow \bar{\nu}_U & & \downarrow \bar{\nu}_{U'} \\ K & \xlongequal{\quad} & K \end{array}$$

commutes, where the top horizontal arrow is obtained by restricting the homomorphism (32). □

4.3. **The group  $\widehat{\mathcal{W}}_\infty$ .** Recall the homomorphisms

$$\dim : \widehat{\mathcal{W}}_0 \rightarrow \mathbb{Z} \quad \text{and} \quad \text{disc} : \widehat{\mathcal{W}}_0 \rightarrow \Delta.$$

Still write

$$\dim : \widehat{\mathcal{W}}_\infty \rightarrow \mathbb{Z} \quad \text{and} \quad \text{disc} : \widehat{\mathcal{W}}_\infty \rightarrow \Delta$$

for their respective compositions with the natural homomorphism  $\widehat{\mathcal{W}}_\infty \rightarrow \widehat{\mathcal{W}}_0$ . Denote by

$$(52) \quad \widehat{\xi}_\infty : \widehat{\mathcal{W}}_\infty \rightarrow K^*$$

the compositions of  $\xi_\infty : \mathcal{W}_\infty \rightarrow K^*$  with the natural homomorphism  $\widehat{\mathcal{W}}_\infty \rightarrow \mathcal{W}_\infty$ .

**Theorem 4.7.** *The group  $\widehat{\mathcal{W}}_\infty$  is canonically isomorphic to the group of Table 4. Under this isomorphism, the homomorphisms  $\dim : \widehat{\mathcal{W}}_\infty \rightarrow \mathbb{Z}$  and  $\text{disc} : \widehat{\mathcal{W}}_\infty \rightarrow \Delta$  are identical to the obvious projections, and  $\widehat{\xi}_\infty : \widehat{\mathcal{W}}_\infty \rightarrow K^*$  is identical to the obvious projection except for the following two cases:*

- (i) *when  $\epsilon = 1$  and  $D$  is a quaternion algebra,  $\widehat{\xi}_\infty$  is identical to the homomorphism*

$$(53) \quad (m, \delta) \mapsto ((-1)^m \delta, \cdot)_{\mathbb{F}};$$

- (ii) *when  $\epsilon = -1$  and  $D = \mathbb{F}$ ,  $\widehat{\xi}_\infty$  is identical to the homomorphism*

$$(54) \quad (m, (\delta, t)) \mapsto \begin{cases} \left( (-1)^{\frac{m^2-m}{2}} \delta, p_{\mathbb{F}}(\cdot) \right)_{\mathbb{F}}, & \text{if } m \text{ is even;} \\ \gamma_{\psi'}, & \text{if } m \text{ is odd,} \end{cases}$$

where  $p_{\mathbb{F}} : \text{Hil}(\mathbb{F}) \rightarrow \mathbb{F}^\times / \mathbb{N}^\times$  is the projection map,  $\psi'$  is the character of  $\mathbb{F}$  given by

$$\psi'(x) := \psi \left( (-1)^{\frac{m^2-m}{2}} \delta x \right), \quad x \in \mathbb{F},$$

and  $\gamma_{\psi'}$  is defined as in (46).

The fiber product  $\mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \frac{\mathbb{E}_\pm^\times}{\mathbb{N}^\times}$  of Table 4 is the same as in Table 2. For the data in the definitions of the other fiber products in Table 4, we are given the homomorphism  $\left(\frac{\mathbb{E}^\times}{\mathbb{N}^\times}\right)^* \rightarrow \mathbb{Z}/2\mathbb{Z}$  whose kernel equals  $\left(\frac{\mathbb{E}^\times}{\mathbb{F}^\times}\right)^* \subset \left(\frac{\mathbb{E}^\times}{\mathbb{N}^\times}\right)^*$ , the homomorphism  $\mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \frac{\mathbb{E}_\pm^\times}{\mathbb{N}^\times} \rightarrow \mathbb{Z}/2\mathbb{Z}$  whose kernel equals  $2\mathbb{Z} \times \frac{\mathbb{F}^\times}{\mathbb{N}^\times}$ , and the homomorphism  $\mathbb{Z} \times \frac{\mathbb{F}^\times}{\mathbb{N}^\times} \rightarrow \mathbb{Z}/2\mathbb{Z}$  whose kernel equals  $2\mathbb{Z} \times \frac{\mathbb{F}^\times}{\mathbb{N}^\times}$ .

TABLE 4. The generalized Witt-Grothendieck group  $\widehat{\mathcal{W}}_\infty$

D	F	quadratic extension	quaternion algebra
$\epsilon = 1$	$2\mathbb{Z} \times \left(\frac{\mathbb{F}^\times}{\mathbb{N}^\times}\right)^* \times \{\pm 1\}^*$	$\left(\mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \frac{\mathbb{E}_\pm^\times}{\mathbb{N}^\times}\right) \times_{\mathbb{Z}/2\mathbb{Z}} \left(\frac{\mathbb{E}^\times}{\mathbb{N}^\times}\right)^*$	$\mathbb{Z} \times \frac{\mathbb{F}^\times}{\mathbb{N}^\times}$
$\epsilon = -1$	$\mathbb{Z} \times \text{Hil}(\mathbb{F})$	$\left(\mathbb{Z} \times \frac{\mathbb{F}^\times}{\mathbb{N}^\times}\right) \times_{\mathbb{Z}/2\mathbb{Z}} \left(\frac{\mathbb{E}^\times}{\mathbb{N}^\times}\right)^*$	$\mathbb{Z} \times \left(\frac{\mathbb{F}^\times}{\mathbb{N}^\times}\right)^*$

We prove Theorem 4.7 case by case in what follows. Since there is a canonical isomorphism  $\widehat{\mathcal{W}}_\infty \cong \widehat{\mathcal{W}}_U$ , there is no harm to replace  $\widehat{\mathcal{W}}_\infty$  by  $\widehat{\mathcal{W}}_U$  in the proof. Here  $U$  is a non-zero split  $\epsilon$ -Hermitian right D-vector space, as before. Write

$$(55) \quad \widehat{\xi}_U : \widehat{\mathcal{W}}_U \rightarrow \mathbb{K}^*$$

for the composition of  $\xi_U : \mathcal{W}_U \rightarrow \mathbb{K}^*$  with the quotient map  $\widehat{\mathcal{W}}_U \rightarrow \mathcal{W}_U$ .

Recall the homomorphism  $\mu_U : G(U) \rightarrow A_\infty$  in (25) (Section 3.2).

**Case 1:**  $U$  is a symmetric bilinear space. We have that

$$\begin{aligned} \widehat{\mathcal{W}}_U &= \mathbb{Z} \mathbb{H}_U \oplus (G(U))^* && \text{by Proposition 3.11} \\ &= 2\mathbb{Z} \oplus \left( \frac{\mathbb{F}^\times}{\mathbb{N}^\times} \times \{\pm 1\} \right)^* && \text{by Theorem 3.3} \\ &= 2\mathbb{Z} \times \left( \frac{\mathbb{F}^\times}{\mathbb{N}^\times} \right)^* \times \{\pm 1\}^*. \end{aligned}$$

Theorem 4.7 in this case then follows by noting that the diagram

$$\begin{array}{ccc} G(U)_{\text{split}} & \xrightarrow{\text{inclusion}} & G(U) \\ \downarrow \nu_U & & \downarrow \mu_U \\ \mathbb{K} = \mathbb{F}^\times / \mathbb{N}^\times & \xrightarrow{\text{inclusion}} & A_\infty = \mathbb{F}^\times / \mathbb{N}^\times \times \{\pm 1\} \end{array}$$

commutes.

**Case 2:**  $U$  is a symplectic space. We have that

$$\begin{aligned} \widehat{\mathcal{W}}_U &= \widehat{\mathcal{W}}_0 && \text{by Proposition 3.11 and Theorem 3.3} \\ &= \mathbb{Z} \times \text{Hil}(\mathbb{F}). && \text{by Theorem 3.2.} \end{aligned}$$

Note that (47) implies that (54) is a group homomorphism, and Lemma 4.4 implies that  $\widehat{\xi}_U$  and the map (54) are identical at all elements of  $\widehat{\mathcal{W}}_U^+$  of dimension one. Therefore Theorem 4.7 in this case follows.

**Case 3:**  $U$  is a Hermitian space or a skew-Hermitian space. It follows from the discussion of [HKS, Section 1] that the image of

$$(56) \quad \widehat{q}_U \times \widehat{\xi}_U : \widehat{\mathcal{W}}_U \rightarrow \widehat{\mathcal{W}}_0 \times (\mathbb{E}^\times / \mathbb{N}^\times)^*$$

is contained in the fibre product  $\widehat{\mathcal{W}}_0 \times_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{E}^\times / \mathbb{N}^\times)^*$ . In view of the exact sequence (31), Theorem 4.7 in this case follows by noting that the diagram

$$\begin{array}{ccc} (G(U))^* & \xrightarrow{\text{the inclusion}} & \widehat{\mathcal{W}}_U \\ \text{the isomorphism induced by } \mu_U \downarrow & & \downarrow \widehat{\xi}_U \\ A_\infty^* = (\mathbb{E}^\times / \mathbb{F}^\times)^* & \xrightarrow{\text{the inclusion}} & \mathbb{K}^* = (\mathbb{E}^\times / \mathbb{N}^\times)^* \end{array}$$

commutes.

**Case 4:**  $U$  is a quaternionic Hermitian space. We have that

$$\begin{aligned} \widehat{\mathcal{W}}_U &= \widehat{\mathcal{W}}_0 && \text{by Proposition 3.11 and Theorem 3.3} \\ &= \mathbb{Z} \times (\mathbb{F}^\times / \mathbb{N}^\times). && \text{by Theorem 3.2.} \end{aligned}$$

By [Ya, Section 6], we know that  $\widehat{\xi}_\infty$  equals the map (53).



**Case 5:**  $U$  is a quaternionic skew-Hermitian space. Note that the diagram

$$\begin{array}{ccc}
 (\mathbb{G}(U))^* & \xrightarrow{\text{the inclusion}} & \widehat{\mathcal{W}}_U \\
 \text{the isomorphism induced by } \mu_U \downarrow & & \downarrow \widehat{\xi}_U \\
 \mathbb{A}_\infty^* = (\mathbb{F}^\times / \mathbb{N}^\times)^* & \xlongequal{\quad} & \mathbb{K}^* = (\mathbb{F}^\times / \mathbb{N}^\times)^*
 \end{array}$$

commutes. Using Theorem 3.3 and the exact sequence (31), this implies that the homomorphism

$$\dim \times \widehat{\xi}_U : \widehat{\mathcal{W}}_U \rightarrow \mathbb{Z} \times (\mathbb{F}^\times / \mathbb{N}^\times)^*$$

is an isomorphism, and Theorem 4.7 holds in this case.

In conclusion, we have proved Theorem 4.7 in all cases. As a direct consequence of Theorem 4.7, we have the following corollary.

**Corollary 4.8.** *The Kudla homomorphism  $\xi_\infty : \mathcal{W}_\infty \rightarrow \mathbb{K}^*$  is surjective, and its kernel has order 2. The non-trivial element of the kernel has anisotropic degree (see (6))  $d_{\mathbb{D},\epsilon}$ .*

5. DEGENERATE PRINCIPAL SERIES AND THE DOUBLING METHOD

**5.1. Degenerate principal series representations.** Let  $U$  be a split  $\epsilon$ -Hermitian right  $\mathbb{D}$ -vector space, with a Lagrangian subspace  $X$ . For each character  $\chi \in (\bar{\mathbb{P}}(X))^*$ , put

$$(57) \quad \mathbb{I}(\chi) := \{f \in C^\infty(\bar{\mathbb{G}}(U)) \mid f(px) = \chi(p)f(x), p \in \bar{\mathbb{P}}(X), x \in \bar{\mathbb{G}}(U)\}.$$

Under right translations, this is a smooth representation of  $\bar{\mathbb{G}}(U)$ .

Define a group homomorphism

$$\widehat{\mathcal{W}}_U \rightarrow (\bar{\mathbb{P}}(X))^*, \quad \sigma \mapsto \chi_{\sigma,X}$$

so that

$$\chi_{\sigma,X}(h) := \kappa_{\sigma,X}(h) |h|_X^{\frac{\dim \sigma}{2}}, \quad h \in \bar{\mathbb{P}}(X)$$

for all  $\sigma \in \widehat{\mathcal{W}}_U^+$ . See Equation (39).

For each  $\sigma \in \widehat{\mathcal{W}}_U$ , we associate an important subrepresentation  $\mathbb{Q}_\sigma$  of  $\mathbb{I}(\chi_{\sigma,X})$  as follows: if  $\sigma \notin \widehat{\mathcal{W}}_U^+$ , we simply put  $\mathbb{Q}_\sigma = 0$ ; if  $\sigma = (V, \omega) \in \widehat{\mathcal{W}}_U^+$ , we define  $\mathbb{Q}_\sigma$  to be the image of the  $\bar{\mathbb{G}}(U)$ -intertwining linear map

$$\begin{aligned}
 (58) \quad \omega &\rightarrow \mathbb{I}(\chi_{\sigma,X}), \\
 \phi &\mapsto (g \mapsto \lambda_{X \otimes_{\mathbb{D}} V}(g \cdot \phi)),
 \end{aligned}$$

where the functional  $\lambda_{X \otimes_{\mathbb{D}} V}$  is as in (39). The following result is well-known (see [Ra1, Theorem II.1.1] and [MVW, Chapter 3, Theorem IV.7]).

**Proposition 5.1.** *Let  $\sigma = (V, \omega) \in \widehat{\mathcal{W}}_U^+$ . Extend  $\omega$  to a smooth oscillator representation of  $\bar{\mathbb{J}}(U, V)$  so that the functional  $\lambda_{X \otimes_{\mathbb{D}} V}$  is  $\bar{\mathbb{G}}(V)$ -invariant. Then the homomorphism (58) descends to an isomorphism  $\omega_{\bar{\mathbb{G}}(V)} \cong \mathbb{Q}_\sigma$ , where  $\omega_{\bar{\mathbb{G}}(V)}$  denotes the maximal quotient of  $\omega$  on which  $\bar{\mathbb{G}}(V)$  acts trivially.*

The first key point of this article is the following proposition, which is responsible for the upper bound in conservation relations.

**Proposition 5.2.** *Let  $\sigma_1$  and  $\sigma_2$  be two elements of  $\widehat{\mathcal{W}}_U$  such that*

$$\begin{cases} \sigma_1 - \sigma_2 \text{ represents the anti-split Witt tower in } \mathcal{W}_U; \\ \dim \sigma_1 + \dim \sigma_2 = \dim U + d_{D,\epsilon} - 2; \text{ and} \\ \dim \sigma_1 \geq \dim \sigma_2. \end{cases}$$

*Then  $I(\chi_{\sigma_1,X})/\mathbb{Q}_{\sigma_1} \cong \mathbb{Q}_{\sigma_2}$  as representations of  $\bar{G}(U)$ .*

*Proof.* The assertion follows from the work of Kudla-Rallis [KR2, Introduction], Kudla-Sweet [KS, Theorem 1.2], and Yamana [Ya, Introduction]. □

**Remarks:** (a) We say that an element  $\sigma \in \widehat{\mathcal{W}}_U$  represents an element  $\mathbf{t} \in \mathcal{W}_U$  if  $\sigma$  maps to  $\mathbf{t}$  under the natural homomorphism  $\widehat{\mathcal{W}}_U \rightarrow \mathcal{W}_U$  (we will apply this terminology to an arbitrary  $\epsilon$ -Hermitian right  $D$ -vector space, including the archimedean case).

(b) Write  $\dim U = 2r$  and put

$$(59) \quad \rho_r = \frac{2r + d_{D,\epsilon} - 2}{4}.$$

Then  $2\rho_r$  coincides with the normalized exponent of the modulus character of  $P(X)$ : the multiple of a left invariant Haar measure on  $\bar{P}(X)$  by the function  $|\cdot|_X^{2\rho_r}$  (see (38)) is a right invariant Haar measure on  $\bar{P}(X)$ . The condition  $\dim \sigma + \dim \sigma' = \dim U + d_{D,\epsilon} - 2$  of Proposition 5.2 then amounts to  $\frac{\dim \sigma}{2} + \frac{\dim \sigma'}{2} = 2\rho_r$ .

(c) Let  $\sigma_1, \sigma_2 \in \widehat{\mathcal{W}}_U$ . Assume that

$$\begin{cases} \sigma_1 - \sigma_2 \text{ represents the anti-split Witt tower in } \mathcal{W}_U; \text{ and} \\ \dim \sigma_1 + \dim \sigma_2 = \dim U + d_{D,\epsilon} - 2. \end{cases}$$

Then Proposition 1.7 implies that  $\text{rank } \sigma_1 \geq \text{rank } U$  if and only if  $\sigma_2 \notin \widehat{\mathcal{W}}_U^+$ . Therefore Proposition 5.2 implies that

$$(60) \quad \mathbb{Q}_\sigma = I(\chi_{\sigma,X}) \quad \text{for all } \sigma \in \widehat{\mathcal{W}}_U^+ \text{ such that } \text{rank } \sigma \geq \text{rank } U.$$

**5.2. The doubling method.** Now we allow  $U$  to be non-split; that is,  $U$  is an arbitrary  $\epsilon$ -Hermitian right  $D$ -vector space. Put

$$U^\square := U \oplus U^-$$

as in (14). Then

$$U^\Delta := \{(u, u) \mid u \in U\}$$

is a Lagrangian subspace of  $U^\square$ . As in Section 5.1, we have a subgroup  $\bar{P}(U^\Delta)$  of  $\bar{G}(U^\square)$ , and a representation  $I(\chi)$  of  $\bar{G}(U^\square)$  for each character  $\chi \in (\bar{P}(U^\Delta))^*$ . As in (32), we have a natural homomorphism  $\bar{G}(U) \times \bar{G}(U^-) \rightarrow \bar{G}(U^\square)$ .

The theory of local zeta integrals [PSR,LR] implies the following lemma.

**Lemma 5.3.** *Let  $\pi \in \text{Irr}(\bar{G}(U))$  and let  $\chi \in (\bar{P}(U^\Delta))^*$ . When  $U$  is a symplectic space, assume that  $\varepsilon_U$  acts through the scalar multiplication by  $\chi(\varepsilon_{U^\square})$  in  $\pi$ . Then*

$$\text{Hom}_{\bar{G}(U)}(I(\chi), \pi) \neq 0.$$

For each  $\sigma \in \widehat{\mathcal{W}}_U$ , put

$$\mathcal{R}_\sigma := \begin{cases} \{\pi \in \text{Irr}(\bar{G}(U)) \mid \text{Hom}_{\bar{G}(U)}(\omega, \pi) \neq 0\}, & \text{if } \sigma = (V, \omega) \in \widehat{\mathcal{W}}_U^+; \\ \emptyset, & \text{if } \sigma \notin \widehat{\mathcal{W}}_U^+. \end{cases}$$

The following result gives a sufficient condition for the non-vanishing of theta lifting.

**Lemma 5.4.** *Let  $\sigma^\square \in \widehat{\mathcal{W}}_{U^\square}$  and let  $\sigma := \widehat{\mathfrak{r}}_{U^\square}(\sigma^\square) \in \widehat{\mathcal{W}}_U$  (see Lemma 3.12). Then for all  $\pi \in \text{Irr}(\bar{G}(U))$ ,*

$$\text{Hom}_{\bar{G}(U)}(\mathbb{Q}_{\sigma^\square}, \pi) \neq 0 \quad \text{implies} \quad \pi \in \mathcal{R}_\sigma.$$

Here  $\mathbb{Q}_{\sigma^\square}$  is a subrepresentation of the representation  $I(\chi_{\sigma^\square, U^\Delta})$  of  $\bar{G}(U^\square)$ , as in Section 5.1.

*Proof.* Write  $\sigma^\square = (V, \omega^\square)$ . Then  $\mathbb{Q}_{\sigma^\square}$  is isomorphic to a quotient of  $(\omega^\square)|_{\bar{G}(U^\square)}$ . Therefore

$$\text{Hom}_{\bar{G}(U)}(\mathbb{Q}_{\sigma^\square}, \pi) \neq 0 \quad \text{implies} \quad \text{Hom}_{\bar{G}(U)}(\omega^\square, \pi) \neq 0.$$

Write  $\sigma = (V, \omega)$ . Then  $(\omega^\square)|_{\bar{G}(U)}$  is isomorphic to a direct sum of smooth representations which are isomorphic to  $\omega|_{\bar{G}(U)}$ . Therefore

$$\text{Hom}_{\bar{G}(U)}(\omega^\square, \pi) \neq 0 \quad \text{implies} \quad \text{Hom}_{\bar{G}(U)}(\omega, \pi) \neq 0.$$

□

On the other hand, we have Lemma 5.5.

**Lemma 5.5.** *Let  $\sigma^\square \in \widehat{\mathcal{W}}_{U^\square}^+$  and let  $\sigma := \widehat{\mathfrak{r}}_{U^\square}(\sigma^\square) \in \widehat{\mathcal{W}}_U^+$  (see Lemma 3.12). Assume that  $\sigma^\square$  is anisotropic and  $\xi_{U^\square}(\sigma^\square)$  is trivial (see (55)). Then for all  $\pi \in \text{Irr}(\bar{G}(U))$ ,*

$$\pi \in \mathcal{R}_\sigma \quad \text{implies} \quad \text{Hom}_{\bar{G}(U) \times \bar{G}(U^-)}(\mathbb{Q}_{\sigma^\square}, \pi \otimes \pi^\vee) \neq 0.$$

*Proof.* Write  $\sigma^\square = (V, \omega^\square)$ . As in (34), write

$$\omega^\square|_{\bar{J}_U(V) \times \bar{J}_{U^-}(V)} = \omega \otimes \omega^-,$$

where  $\omega$  and  $\omega^-$  are smooth oscillator representations of  $\bar{J}_U(V)$  and  $\bar{J}_{U^-}(V)$ , respectively. The triviality of  $\xi_{U^\square}(\sigma^\square)$  implies that  $\omega$  and  $\omega^-$  are the contragredient representations of each other with respect to the isomorphism

$$\begin{aligned} \bar{G}(U) \times \text{H}(U \otimes_{\mathbb{D}} V) &\rightarrow \bar{G}(U^-) \times \text{H}(U^- \otimes_{\mathbb{D}} V), \\ (g, (w, t)) &\mapsto (g, (w, -t)). \end{aligned}$$

Extend  $\omega$  and  $\omega^-$  to representations of  $\bar{J}(U, V)$  and  $\bar{J}(U^-, V)$ , respectively, so that they are the contragredient representations of each other with respect to the isomorphism

$$\begin{aligned} (\bar{G}(U) \times \bar{G}(V)) \times \text{H}(U \otimes_{\mathbb{D}} V) &\rightarrow (\bar{G}(U^-) \times \bar{G}(V)) \times \text{H}(U^- \otimes_{\mathbb{D}} V), \\ ((g, h), (w, t)) &\mapsto ((g, h), (w, -t)). \end{aligned}$$

Assume that  $\pi \in \mathcal{R}_\sigma$ . Then there is an irreducible representation  $\tau \in \text{Irr}(\bar{G}(V))$  such that (cf. [MVW, Chapter 3, IV.4])

$$(61) \quad \text{Hom}_{\bar{G}(U) \times \bar{G}(V)}(\omega, \pi \otimes \tau) \neq 0.$$

Since  $V$  is anisotropic, both  $\pi$  and  $\tau$  are unitarizable. By taking complex conjugations on the representations in (61), we have that

$$(62) \quad \text{Hom}_{\bar{G}(U^-) \times \bar{G}(V)}(\omega^-, \pi^\vee \otimes \tau^\vee) \neq 0.$$

Combining (61) and (62), we have that

$$\text{Hom}_{\bar{G}(U) \times \bar{G}(U^-)}((\omega \otimes \omega^-)_{\bar{G}(V)}, \pi \otimes \pi^\vee) \neq 0,$$

where a subscript “ $\bar{G}(V)$ ” indicates the maximal quotient on which  $\bar{G}(V)$  acts trivially. The lemma then follows, by Proposition 5.1.  $\square$

**Remark:** The lemma above is a variant of a more well-known result in the literature on local theta correspondence ([HKS, Proposition 3.1] and [Ku3, Proposition 1.5]). Note that we include the non-archimedean quaternionic case, for which MVW-involutions do not exist [LST]. To compensate this, the space  $V$  is assumed to be anisotropic, which is what we need (for Lemma 6.6).

**5.3. Non-vanishing of theta lifting.** Concerning non-vanishing of theta lifting, we have Proposition 5.6.

**Proposition 5.6.** *Let  $\sigma_1$  and  $\sigma_2$  be two elements of  $\widehat{\mathcal{W}}_U$  such that*

$$\begin{cases} \sigma_1 - \sigma_2 \text{ represents the anti-split Witt tower in } \mathcal{W}_U; \text{ and} \\ \dim \sigma_1 + \dim \sigma_2 = 2 \dim U + d_{D,\epsilon} - 2. \end{cases}$$

Then

$$\mathcal{R}_{\sigma_1} \cup \mathcal{R}_{\sigma_2} = \{ \pi \in \text{Irr}(\bar{G}(U)) \mid \pi \text{ is genuine with respect to } \sigma_1 \text{ (and } \sigma_2) \}.$$

*Proof.* By Lemma 3.13 and without loss of generality, we assume that there exist  $\sigma_i^\square \in \widehat{\mathcal{W}}_{U^\square}$  such that  $\sigma_i = \widehat{r}_U^{U^\square}(\sigma_i^\square)$  ( $i = 1, 2$ ), and  $\sigma_1^\square - \sigma_2^\square$  represents the anti-split Witt tower in  $\mathcal{W}_{U^\square}$ .

By Proposition 5.2, we either have a short exact sequence

$$0 \rightarrow \mathcal{Q}_{\sigma_1^\square} \rightarrow \text{I}(\chi_{\sigma_1^\square, U^\Delta}) \rightarrow \mathcal{Q}_{\sigma_2^\square} \rightarrow 0$$

or have a short exact sequence

$$0 \rightarrow \mathcal{Q}_{\sigma_2^\square} \rightarrow \text{I}(\chi_{\sigma_2^\square, U^\Delta}) \rightarrow \mathcal{Q}_{\sigma_1^\square} \rightarrow 0.$$

Then Lemma 5.3 implies that

$$\text{Hom}_{\bar{G}(U)}(\mathcal{Q}_{\sigma_1^\square}, \pi) \neq 0 \quad \text{or} \quad \text{Hom}_{\bar{G}(U)}(\mathcal{Q}_{\sigma_2^\square}, \pi) \neq 0,$$

for all  $\pi \in \text{Irr}(\bar{G}(U))$  which is genuine with respect to  $\sigma_1$  and  $\sigma_2$ . By Lemma 5.4,

$$\pi \in \mathcal{R}_{\sigma_1} \quad \text{or} \quad \pi \in \mathcal{R}_{\sigma_2}.$$

This proves the proposition.  $\square$

The upper bound in Theorem 1.10 is an easy consequence of Proposition 5.6.

**Corollary 5.7.** *Let  $\pi \in \text{Irr}(\bar{G}(U))$ . Let  $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{W}_U$  be two elements so that  $\mathbf{t}_1 - \mathbf{t}_2 = \mathbf{t}_U^\circ$ . Assume that  $\pi$  is genuine with respect to  $\mathbf{t}_1$  (and hence genuine with respect to  $\mathbf{t}_2$ ). Then*

$$n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) \leq 2 \dim U + d_{D,\epsilon}.$$

*Proof.* Write  $\tilde{\mathbf{t}}_i \subset \widehat{\mathcal{W}}_U$  for the inverse image of  $\mathbf{t}_i$  under the natural homomorphism  $\widehat{\mathcal{W}}_U \rightarrow \mathcal{W}_U$  ( $i = 1, 2$ ). Then

$$n_{\mathbf{t}_i}(\pi) = \min\{\dim \sigma_i \mid \sigma_i \in \tilde{\mathbf{t}}_i, \pi \in \mathcal{R}_{\sigma_i}\}.$$

Assume that

$$n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) > 2 \dim U + d_{D,\epsilon}.$$

Then Proposition 1.7 implies that

$$n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) = 2 \dim U + d_{D,\epsilon} + 2k$$

for some integer  $k > 0$ . Therefore there exist  $\sigma_1 \in \tilde{\mathbf{t}}_1$  and  $\sigma_2 \in \tilde{\mathbf{t}}_2$  so that

$$\begin{cases} \pi \notin \mathcal{R}_{\sigma_1} \text{ and } \pi \notin \mathcal{R}_{\sigma_2}; \text{ and} \\ \dim \sigma_1 + \dim \sigma_2 = 2 \dim U + d_{D,\epsilon} - 2. \end{cases}$$

This contradicts Proposition 5.6. □

**Remarks:** (a) Let  $\sigma_1$  and  $\sigma_2$  be as in Proposition 5.6. Proposition 1.7 implies that  $\sigma_1$  is in the stable range (that is,  $\text{rank } \sigma_1 \geq \dim U$ ) if and only if  $\sigma_2 \notin \widehat{\mathcal{W}}_U^+$ . Therefore Proposition 5.6 implies that [Ku3, Propositions 4.3 and 4.5]

$$(63) \quad \mathcal{R}_\sigma = \{ \pi \in \text{Irr}(\bar{G}(U)) \mid \pi \text{ is genuine with respect to } \sigma \}$$

for all  $\sigma \in \widehat{\mathcal{W}}_U^+$  in the stable range.

(b) It is easy to see that Theorem 1.10 (the conservation relations) is equivalent to the following: for all  $\sigma_1, \sigma_2 \in \widehat{\mathcal{W}}_U$  as in Proposition 5.6, we have that

$$\mathcal{R}_{\sigma_1} \sqcup \mathcal{R}_{\sigma_2} = \{ \pi \in \text{Irr}(\bar{G}(U)) \mid \pi \text{ is genuine with respect to } \sigma_1 \text{ (and } \sigma_2) \}.$$

For  $\dim \sigma_1 = \dim \sigma_2$ , the above assertion is called theta dichotomy in the literature [KR3, HKS]. The theta dichotomy was established by Harris [Ha, Theorem 2.1.7] (for unitary-unitary dual pairs), and by Zorn [Zo, Theorem 1.1] and Gan-Gross-Prasad [GGP, Theorem 11.1] (for orthogonal-symplectic dual pairs). For a related work of Prasad, see [Pra].

### 6. NON-OCCURRENCE OF THE TRIVIAL REPRESENTATION BEFORE STABLE RANGE

Let  $U$  be an  $\epsilon$ -Hermitian right  $D$ -vector space. Recall that  $\mathbf{t}_U^\circ \in \mathcal{W}_U$  denotes the anti-split Witt tower (Section 4). The main purpose of this section is to show the following proposition, which is the second key point of this article and which is responsible for the lower bound in conservation relations.

**Proposition 6.1.** *One has that*

$$n_{\mathbf{t}_U^\circ}(1_U) \geq 2 \dim U + d_{D,\epsilon}.$$

Here and as before  $1_U \in \text{Irr}(\bar{G}(U))$  stands for the trivial representation.

Proposition 6.1 is proved in [Ra1, Appendix], [KR3, Lemma 4.2], and [GG, Theorem 2.9], respectively, for orthogonal groups, symplectic groups, and unitary groups. Only the quaternionic case is new. Because of the lack of MVW-involutions, the approach of [KR3] and [GG], which uses the doubling method, does not work for this case. We will follow the idea of Rallis ([Ra1, Ra2], which treat the case of orthogonal groups) to provide a uniform proof of Proposition 6.1.

By the argument of Section 2.1, Proposition 6.1 implies the following proposition.

**Proposition 6.2.** *Let  $\pi \in \text{Irr}(\bar{G}(U))$ . Let  $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{W}_U$  be two elements so that  $\mathbf{t}_1 - \mathbf{t}_2 = \mathbf{t}_U^\circ$ . Assume that  $\pi$  is genuine with respect to  $\mathbf{t}_1$  (and hence genuine with respect to  $\mathbf{t}_2$ ). Then*

$$n_{\mathbf{t}_1}(\pi) + n_{-\mathbf{t}_2}(\pi^\vee) \geq 2 \dim U + d_{D,\epsilon}.$$

As demonstrated in Section 2.1, Corollary 5.7 and Proposition 6.2 then imply Theorem 1.10.

Proposition 6.1 is clear when  $U = 0$ . For the rest of this section, assume that  $U \neq 0$ , and put  $d := \dim U > 0$ . Write  $\sigma_U^\circ = (V^\circ, \omega_U^\circ)$  for the anisotropic element of  $\mathfrak{t}_U^\circ$ . We view  $\omega_U^\circ$  as a representation of  $G(U)$  (when  $U$  is a symplectic space, the restriction of  $\omega_U^\circ$  to  $\bar{G}(U)$  descends to a representation of  $G(U)$ ). Likewise, view  $1_U$  as the trivial representation of  $G(U)$ . Then Proposition 6.1 amounts to the following proposition.

**Proposition 6.3.** *One has that*

$$(64) \quad \text{Hom}_{G(U)}(\mathcal{S}(U^{d-1}) \otimes \omega_U^\circ, 1_U) = 0,$$

where  $U^{d-1}$  carries the diagonal action of  $G(U)$ , and  $\mathcal{S}(U^{d-1})$  carries the induced action of  $G(U)$ .

**6.1. Non-occurrence of  $1_U$  in  $\omega_U^\circ$ .** Let  $V$  be a  $-\epsilon$ -Hermitian left  $D$ -vector space. We start with the following observation, which is easily seen from the Schrödinger model of an oscillator representation. See [Li2], and for a fuller treatment see [Ho2, Part II, Section 3].

**Lemma 6.4.** *Let  $\omega$  be a smooth oscillator representation of  $J_U(V)$ . Assume that  $U$  is split with a Lagrangian subspace  $X$ , and  $V$  is anisotropic. Then every linear functional on  $\omega$  is  $N(X)$ -invariant if and only if it is invariant under  $X \otimes_{\mathbb{D}} V \subset H(U \otimes_{\mathbb{D}} V)$ , where  $N(X)$  denotes the unipotent radical of  $P(X)$ .*

Consequently, we have Lemma 6.5.

**Lemma 6.5.** *Let  $\omega$  be a smooth oscillator representation of  $J_U(V)$ . Assume that  $U$  is split and non-zero, and  $V$  is anisotropic and non-zero. Then*

$$\text{Hom}_{G(U)}(\omega, 1_U) = 0.$$

*Proof.* Let  $X$  be a Lagrangian subspace of  $U$ . Assume that there is a non-zero element  $\lambda \in \text{Hom}_{G(U)}(\omega, 1_U)$ . Then Lemma 6.4 implies that  $\lambda$  is a scalar multiple of  $\lambda_{X \otimes_{\mathbb{D}} V}$ . This contradicts the equality (39), as all Kudla characters are unitary. □

**Lemma 6.6.** *If  $d = 1$ , then*

$$(65) \quad \text{Hom}_{G(U)}(\omega_U^\circ, 1_U) = 0.$$

*Proof.* Note that  $d = 1$  implies that  $U$  is not a symplectic space, and  $\bar{G}(U) = G(U)$  is a compact group. Introduce the space  $U^\square := U \oplus U^-$  and its Lagrangian subspace  $U^\Delta$  as in Section 5.2. Write  $\sigma^\square$  for the anisotropic element of the anti-split Witt tower  $\mathfrak{t}_{U^\square}^\circ$ . By Lemma 5.5, it suffices to show that

$$(66) \quad \text{Hom}_{G(U) \times G(U^-)}(\mathbb{Q}_{\sigma^\square}, 1_U \otimes 1_{U^-}) = 0.$$

Note that (66) is implied by the following:

$$(67) \quad \text{Hom}_{G(U)}(\mathbb{Q}_{\sigma^\square}, 1_U) = 0.$$

As a simple instance of Proposition 5.2, we have an exact sequence of representations of  $G(U^\square)$ ,

$$(68) \quad 0 \rightarrow \mathbb{Q}_{\sigma^\square} \rightarrow I(\chi_{\sigma^\square, U^\Delta}) \rightarrow 1_{U^\square} \rightarrow 0.$$

Note that

$$(\mathbf{I}(\chi_{\sigma\Box, U\Delta}))|_{G(U)} \cong C^\infty(G(U)),$$

where  $G(U)$  acts on  $C^\infty(G(U))$  by right translations. By (68), we have a decomposition (recall that  $G(U)$  is compact)

$$C^\infty(G(U)) \cong (\mathbf{Q}_{\sigma\Box})|_{G(U)} \oplus 1_U,$$

and hence (67) follows by the uniqueness of the Haar measure. □

**Lemma 6.7.** *One has that*

$$(69) \quad \text{Hom}_{G(U)}(\omega_U^\circ, 1_U) = 0.$$

*Proof.* When  $U$  is a symplectic space, this is a special case of Lemma 6.5. Now assume that  $U$  is not a symplectic space. Then there is an orthogonal decomposition  $U = U_1 \oplus U_2$  of  $\epsilon$ -Hermitian space such that  $\dim U_1 = 1$ . Therefore

$$\text{Hom}_{G(U)}(\omega_U^\circ, 1_U) \subset \text{Hom}_{G(U_1)}(\omega_{U_1}^\circ, 1_{U_1}) = \text{Hom}_{G(U_1)}(\omega_{U_1}^\circ \otimes \omega_{U_2}^\circ, 1_{U_1}) = 0,$$

by Lemma 6.6. □

**6.2. Vanishing on small orbits in the null cone.** Put

$$\Gamma := \{(v_1, v_2, \dots, v_{d-1}) \in U^{d-1} \mid \langle v_i, v_j \rangle_U = 0, \ i, j = 1, 2, \dots, d-1\},$$

which is referred to as the null cone in  $U^{d-1}$ . Write

$$\Gamma := \bigsqcup_{i=0}^{\text{rank } U} \Gamma_i,$$

where

$$\Gamma_i := \{(v_1, v_2, \dots, v_{d-1}) \in \Gamma \mid v_1, v_2, \dots, v_{d-1} \text{ span a subspace of } U \text{ of dimension } i\}.$$

Then each  $\Gamma_i$  is a single  $G(U) \times \text{GL}_{d-1}(\mathbb{D})$ -orbit. Here  $\text{GL}_{d-1}(\mathbb{D})$  acts on  $U^{d-1}$  by

$$(70) \quad h \cdot (v_1, v_2, \dots, v_{d-1}) := (v_1, v_2, \dots, v_{d-1})h^{-1}.$$

**Lemma 6.8.** *If  $2i < d$ , then*

$$\text{Hom}_{G(U)}(\mathcal{S}(\Gamma_i) \otimes \omega_U^\circ, 1_U) = 0.$$

*Proof.* It suffices to show that for each compact open subgroup  $L$  of  $\text{GL}_{d-1}(\mathbb{D})$ ,

$$\text{Hom}_{G(U) \times L}(\mathcal{S}(\Gamma_i) \otimes \omega_U^\circ, 1_U) = 0,$$

where  $\omega_U^\circ$  and  $1_U$  are viewed as representations of  $G(U) \times L$  so that  $L$  acts trivially. Let  $O_i$  be a  $G(U) \times L$ -orbit in  $\Gamma_i$  (which is open). We only need to show that

$$\text{Hom}_{G(U) \times L}(\mathcal{S}(O_i) \otimes \omega_U^\circ, 1_U) = 0.$$

By Frobenius reciprocity [BZ, Chapter I, Proposition 2.29], one has that

$$\text{Hom}_{G(U) \times L}(\mathcal{S}(O_i) \otimes \omega_U^\circ, 1_U) = \text{Hom}_{(G(U) \times L)_\mathbf{v}}(\delta_\mathbf{v} \otimes \omega_U^\circ, 1_U),$$

where

$$\mathbf{v} = (v_1, v_2, \dots, v_{d-1}) \in O_i,$$

$(G(U) \times L)_\mathbf{v}$  is the stabilizer of  $\mathbf{v}$  in  $G(U) \times L$ , and  $\delta_\mathbf{v}$  is a certain positive character on  $(G(U) \times L)_\mathbf{v}$ .

Since  $2i < d$ , there is a non-zero non-degenerate subspace  $U_0$  of  $U$  which is perpendicular to  $v_1, v_2, \dots, v_{d-1}$ . Note that  $G(U_0) \subset (G(U) \times L)_\mathbf{v}$  and  $\delta_\mathbf{v}$  is trivial on  $G(U_0)$ . Therefore,

$$\begin{aligned} \text{Hom}_{(G(U) \times L)_\mathbf{v}}(\delta_\mathbf{v} \otimes \omega_U^\circ, 1_U) &\subset \text{Hom}_{G(U_0)}(\omega_U^\circ, 1_U) \\ &= \text{Hom}_{G(U_0)}(\omega_{U_0}^\circ \otimes \omega_{U_0^\perp}^\circ, 1_{U_0}) \\ &= 0 \quad (\text{by Lemma 6.7}). \end{aligned}$$

□

**6.3. A homogeneity calculation for the main orbits in the null cone.** Let  $E^\times$  act on  $U^{d-1}$  by

$$a \cdot x := xa^{-1}, \quad a \in E^\times, x \in U^{d-1}.$$

Denote by  $\mathcal{O}_D$  the ring of integers in  $D$ ,

$$\mathcal{O}_D := \{a \in D \mid |\det a|_E \leq 1\}.$$

For each  $i = 0, 1, \dots, \text{rank } U$ , using the decomposition

$$\text{GL}_{d-1}(D) = \text{GL}_{d-1}(\mathcal{O}_D) \left\{ \begin{bmatrix} g & 0 \\ * & h \end{bmatrix} \in \text{GL}_{d-1}(D) \mid g \in \text{GL}_i(D), h \in \text{GL}_{d-1-i}(D) \right\},$$

it is easy to see that  $\Gamma_i$  is a homogeneous space for the action of  $G(U) \times \text{GL}_{d-1}(\mathcal{O}_D)$ . Consequently,  $\Gamma_i$  is a homogeneous space for the action of  $G(U) \times \text{GL}_{d-1}(\mathcal{O}_D) \times E^\times$ .

In the rest of this subsection, assume that  $U$  is split, and write  $d = 2r > 0$ . Put

$$\rho_r := \frac{2r + d_{D,\epsilon} - 2}{4},$$

as in (59). Recall that  $\delta_D$  (the degree of  $D$  over  $E$ ) equals 2 if  $D$  is a quaternion algebra, and equals 1 otherwise. The following lemma is an easy consequence of [Lo, Theorem 33D]. We omit the details.

**Lemma 6.9.** *Up to the scalar multiple, there exists a unique positive Borel measure  $\mu_{\Gamma_r}$  on  $\Gamma_r$  such that*

$$(g, h, a) \cdot \mu_{\Gamma_r} = |a|_E^{2\delta_D^2 r \rho_r} \mu_{\Gamma_r}, \quad (g, h, a) \in G(U) \times \text{GL}_{d-1}(\mathcal{O}_D) \times E^\times,$$

where  $(g, h, a) \cdot \mu_{\Gamma_r}$  denotes the push-forward of  $\mu_{\Gamma_r}$  through the action of  $(g, h, a)$  on  $\Gamma_r$ .

We will use the following convention for the rest of this section: given a group  $G$  acting on two sets  $A$  and  $B$ , then for every  $g \in G$  and every map  $\varphi : A \rightarrow B$ ,  $g \cdot \varphi : A \rightarrow B$  is the map defined by

$$(g \cdot \varphi)(a) := g \cdot (\varphi(g^{-1} \cdot a)), \quad a \in A.$$

If no action of  $G$  is specified on a set  $C$ , we consider  $C$  to carry the trivial action of  $G$ .

Note that each  $G(U)$ -orbit in  $\Gamma$  is  $E^\times$ -stable. We shall examine  $G(U)$ -orbits in  $\Gamma_r$ .

**Lemma 6.10.** *Let  $O_r$  be a  $G(U)$ -orbit in  $\Gamma_r$ . Then the space  $\text{Hom}_{G(U)}(\omega_U^\circ, C^\infty(O_r))$  is one dimensional and every element  $\lambda$  of the space satisfies*

$$a \cdot \lambda = |a|_E^{-2\delta_D^2 r \rho_1} \lambda, \quad a \in E^\times. \quad \left( \rho_1 = \frac{d_{D,\epsilon}}{4} \right).$$



*Proof.* Fix an element  $\mathbf{v} = (v_1, v_2, \dots, v_{d-1}) \in O_r$ . Denote by  $X$  the Lagrangian subspace of  $U$  spanned by  $v_1, v_2, \dots, v_{d-1}$ . Fix a Lagrangian subspace  $Y$  of  $U$  which is complementary to  $X$ . For every  $a \in E^\times$ , denote by  $m_a \in G(U)$  the element which stabilizes both  $X$  and  $Y$ , and acts on  $X$  through the scalar multiplication by  $a$ .

The stabilizer of  $\mathbf{v}$  in  $G(U)$  equals  $N(X)$ , the unipotent radical of the parabolic subgroup  $P(X)$ . Therefore

$$O_r = G(U)/N(X).$$

The corresponding action of  $E^\times$  on  $G(U)/N(X)$  is given by

$$a \cdot (gN(X)) = gm_a^{-1}N(X), \quad a \in E^\times.$$

By Frobenius reciprocity,

$$(71) \quad \text{Hom}_{G(U)}(\omega_U^\circ, C^\infty(O_Z)) = \text{Hom}_{N(X)}(\omega_U^\circ, \mathbb{C}).$$

It is easy to check that, under the identification (71), the action of  $E^\times$  on the left hand side corresponds to the following action on the right hand side:

$$a \cdot \lambda := \lambda \circ (\omega_U^\circ(m_a^{-1})), \quad a \in E^\times, \lambda \in \text{Hom}_{N(X)}(\omega_U^\circ, \mathbb{C}).$$

By Lemma 6.4,  $\text{Hom}_{N(X)}(\omega_U^\circ, \mathbb{C})$  is spanned by  $\lambda_{X \otimes V^\circ}$ , and (39) implies that

$$\lambda_{X \otimes V^\circ} \circ (\omega_U^\circ(m_a^{-1})) = |a|_{\mathbb{E}}^{-2\delta_D^2 r \rho_1} \lambda_{X \otimes V^\circ}, \quad a \in E^\times.$$

This proves the lemma. □

**Lemma 6.11.** *Every element  $\lambda \in \text{Hom}_{G(U)}(\mathcal{S}(\Gamma_r) \otimes \omega_U^\circ, 1_U)$  satisfies*

$$a \cdot \lambda = |a|_{\mathbb{E}}^{2r\delta_D^2(\rho_r - \rho_1)} \lambda, \quad a \in E^\times.$$

*Proof.* Without loss of generality, assume that  $\lambda$  is fixed by an open subgroup of  $\text{GL}_{d-1}(\mathcal{O}_D)$ . Then  $\lambda$  naturally corresponds to an element of  $\text{Hom}_{G(U)}(\omega_U^\circ, C^\infty(\Gamma_r)\mu_{\Gamma_r})$ , where  $\mu_{\Gamma_r}$  is as in Lemma 6.9. Therefore the lemma follows from Lemmas 6.9 and 6.10, and by considering the following product of restriction maps:

$$C^\infty(\Gamma_r) \hookrightarrow \prod_{O_r \text{ is a } G(U)\text{-orbit in } \Gamma_r} C^\infty(O_r).$$

□

**6.4. The Fourier transform.** For every  $\epsilon$ -Hermitian right  $D$ -vector space  $U'$ , define the Fourier transform

$$\mathcal{F}_{U'} : \mathcal{S}(U') \rightarrow \mathcal{S}(U')$$

by

$$(\mathcal{F}_{U'}(\phi))(x) := \int_{U'} \phi(y) \psi\left(\frac{\langle x, y \rangle_{U'} + \langle x, y \rangle_{U'}^i}{2}\right) dy, \quad \phi \in \mathcal{S}(U'), x \in U',$$

where  $dy$  is a fixed Haar measure on  $U'$ . It is easy to check that

$$(72) \quad \mathcal{F}_{U'}(g \cdot \phi) = g \cdot (\mathcal{F}_{U'}(\phi)), \quad g \in G(U'), \phi \in \mathcal{S}(U'),$$

and

$$(73) \quad \mathcal{F}_{U'}(a \cdot \phi) = |a|_{\mathbb{E}}^{-\dim_E U'} (a^{-1})^\iota \cdot (\mathcal{F}_{U'}(\phi)), \quad a \in E^\times, \phi \in \mathcal{S}(U'),$$

where the action of  $E^\times$  on  $\mathcal{S}(U')$  is given by

$$(a \cdot \phi)(x) := \phi(ax), \quad a \in E^\times, \phi \in \mathcal{S}(U'), x \in U'.$$

We refer the reader to the notation of the last subsection. For every linear functional  $\lambda$  on  $\mathcal{S}(U^{d-1}) \otimes \omega_U^\circ$ , define its Fourier transform to be the linear functional

$$\widehat{\lambda} : \mathcal{S}(U^{d-1}) \otimes \omega_U^\circ \rightarrow \mathbb{C}, \quad \phi \otimes \phi' \mapsto \lambda(\mathcal{F}_{U^{d-1}}(\phi) \otimes \phi').$$

Using extension by zero, we get an inclusion

$$\mathcal{S}(U^{d-1} \setminus \Gamma) \subset \mathcal{S}(U^{d-1}).$$

(Recall that  $\Gamma$  is the null cone in  $U^{d-1}$ .)

**Lemma 6.12.** *Let  $\lambda \in \text{Hom}_{G(U)}(\mathcal{S}(U^{d-1}) \otimes \omega_U^\circ, 1_U)$ . If both  $\lambda$  and  $\widehat{\lambda}$  vanish on the subspace  $\mathcal{S}(U^{d-1} \setminus \Gamma) \otimes \omega_U^\circ$ , then  $\lambda = 0$ .*

*Proof.* If  $U$  is not split, then the lemma follows from Lemma 6.8. Now assume that  $U$  is split. Then Lemma 6.8 and Lemma 6.11 imply that

$$(74) \quad a \cdot \lambda = |a|_{\mathbb{E}}^{2r\delta_D^2(\rho_r - \rho_1)} \lambda, \quad a \in \mathbb{E}^\times.$$

It is easily checked that (74) and (73) imply that

$$(75) \quad a \cdot \widehat{\lambda} = |a|_{\mathbb{E}}^{\delta_D^2 d(d-1) - 2r\delta_D^2(\rho_r - \rho_1)} \widehat{\lambda}, \quad a \in \mathbb{E}^\times.$$

Note that (72) implies that  $\widehat{\lambda} \in \text{Hom}_{G(U)}(\mathcal{S}(U^{d-1}) \otimes \omega_U^\circ, 1_U)$ . Similar to (74), we have that

$$(76) \quad a \cdot \widehat{\lambda} = |a|_{\mathbb{E}}^{2r\delta_D^2(\rho_r - \rho_1)} \widehat{\lambda}, \quad a \in \mathbb{E}^\times.$$

Since  $d = 2r$  and  $\rho_r - \rho_1 = \frac{r-1}{2}$ , we will have

$$\delta_D^2 d(d-1) - 2r\delta_D^2(\rho_r - \rho_1) \neq 2r\delta_D^2(\rho_r - \rho_1),$$

and so we conclude that  $\lambda = 0$  by comparing (75) and (76). □

**Corollary 6.13.** *If  $U$  is a symplectic space of dimension 2, then  $\text{Hom}_{G(U)}(\mathcal{S}(U) \otimes \omega_U^\circ, 1_U) = 0$ .*

*Proof.* In this case,  $\Gamma = U$ . Therefore this is a special case of Lemma 6.12. □

**6.5. Reduction to the null cone and conclusion of the proof.** By Lemma 6.6 and Corollary 6.13, Proposition 6.3 holds when  $d = 1$ , or when  $U$  is a symplectic space and  $d = 2$ . We prove Proposition 6.3 by induction on  $d$ . So assume that  $d \geq 4$  when  $U$  is a symplectic space, and  $d \geq 2$  in all other cases, and assume that Proposition 6.3 holds when  $d$  is smaller.

Let  $U_0$  be a non-zero non-degenerate subspace of  $U$  of dimension  $d_0$ , where  $d_0 = 2$  if  $U$  is a symplectic space, and  $d_0 = 1$  otherwise. Denote by  $U_0^\perp$  the orthogonal complement of  $U_0$  in  $U$ . Put

$$B_0 := \begin{cases} \{(v_1, v_2) \in (U_0)^2 \mid v_1, v_2 \text{ is a basis of } U_0\}, & \text{if } U \text{ is a symplectic space;} \\ U_0 \setminus \{0\}, & \text{otherwise.} \end{cases}$$

Then

$$S_0 := B_0 \times U^{d-d_0-1} \subset U^{d-1}$$

is stable under  $G(U_0^\perp) \subset G(U)$ , and the map

$$(77) \quad G(U) \times S_0 \rightarrow U^{d-1}, \quad (g, \mathbf{v}) \mapsto g \cdot \mathbf{v}$$

is  $G(U) \times G(U_0^\perp)$ -equivariant, where  $G(U) \times G(U_0^\perp)$  acts on  $G(U) \times S_0$  by

$$(g, h) \cdot (x, \mathbf{v}) := (gxh^{-1}, h \cdot \mathbf{v}),$$

and acts on  $U^{d-1}$  by

$$(g, h) \cdot \mathbf{v} := g \cdot \mathbf{v}.$$

**Lemma 6.14.** *One has that*

$$(78) \quad \text{Hom}_{\mathbf{G}(U) \times \mathbf{G}(U_0^\perp)}(\mathcal{S}(\mathbf{G}(U) \times S_0) \otimes \omega_U^\circ, 1_U) = 0,$$

where the representations  $\omega_U^\circ$  and  $1_U$  of  $\mathbf{G}(U)$  are extended to the group  $\mathbf{G}(U) \times \mathbf{G}(U_0^\perp)$  by the trivial action of  $\mathbf{G}(U_0^\perp)$ .

*Proof.* Frobenius reciprocity [BZ, Chapter I, Proposition 2.29] implies that the left hand side of (78) equals

$$(79) \quad \text{Hom}_{\mathbf{G}(U_0^\perp)}(\mathcal{S}(S_0) \otimes (\omega_U^\circ)|_{\mathbf{G}(U_0^\perp)}, 1_{U_0^\perp}).$$

Note that

$$\mathcal{S}(S_0) = \mathcal{S}(B_0 \times U_0^{d-d_0-1}) \otimes \mathcal{S}((U_0^\perp)^{d-d_0-1}),$$

and

$$\omega_U^\circ = \omega_{U_0}^\circ \otimes \omega_{U_0^\perp}^\circ.$$

By the induction assumption, we have

$$\text{Hom}_{\mathbf{G}(U_0^\perp)}(\mathcal{S}((U_0^\perp)^{d-d_0-1}) \otimes \omega_{U_0^\perp}^\circ, 1_{U_0^\perp}) = 0,$$

and therefore the space (79) vanishes as well. □

**Lemma 6.15.** *One has that*

$$\text{Hom}_{\mathbf{G}(U)}(\mathcal{S}(U^{d-1} \setminus \Gamma) \otimes \omega_U^\circ, 1_U) = 0.$$

*Proof.* Note that  $\mathbf{G}(U) \cdot S_0$  is open in  $U^{d-1}$ , and the push-forward of measures through the map (77) induces a  $\mathbf{G}(U) \times \mathbf{G}(U_0^\perp)$ -equivariant surjective linear map

$$\mathcal{S}(\mathbf{G}(U) \times S_0) (\mu_{\mathbf{G}(U)} \otimes \mu_{S_0}) \rightarrow \mathcal{S}(\mathbf{G}(U) \cdot S_0) \mu_{\mathbf{G}(U) \cdot S_0},$$

where  $\mu_{\mathbf{G}(U)}$  is a Haar measure on  $\mathbf{G}(U)$ ,  $\mu_{S_0}$  is the restriction of a Haar measure on  $U_0^{d_0} \times U^{d-d_0-1}$  to  $S_0$ , and  $\mu_{\mathbf{G}(U) \cdot S_0}$  is the restriction of a Haar measure on  $U^{d-1}$  to  $\mathbf{G}(U) \cdot S_0$ . (This is because the map (77) is a submersion between locally analytic manifolds over  $\mathbb{F}$  [Schn].) Consequently, there exists a  $\mathbf{G}(U) \times \mathbf{G}(U_0^\perp)$ -equivariant surjective linear map

$$\mathcal{S}(\mathbf{G}(U) \times S_0) \rightarrow \mathcal{S}(\mathbf{G}(U) \cdot S_0).$$

Therefore Lemma 6.14 implies that

$$\text{Hom}_{\mathbf{G}(U)}(\mathcal{S}(\mathbf{G}(U) \cdot S_0) \otimes \omega_U^\circ, 1_U) = \text{Hom}_{\mathbf{G}(U) \times \mathbf{G}(U_0^\perp)}(\mathcal{S}(\mathbf{G}(U) \cdot S_0) \otimes \omega_U^\circ, 1_U) = 0.$$

This further implies that

$$\text{Hom}_{\mathbf{G}(U)}(\mathcal{S}(\mathbf{G}(U) \cdot (h \cdot S_0)) \otimes \omega_U^\circ, 1_U) = 0$$

for all  $h \in \text{GL}_{d-1}(\mathbb{D})$ . The lemma then follows by noting that

$$\bigcup_{U_0, h} \mathbf{G}(U) \cdot (h \cdot S_0) = U^{d-1} \setminus \Gamma,$$

where  $U_0$  runs through all non-degenerate subspaces of  $U$  of dimension  $d_0$ , and  $h$  runs through all elements of  $\text{GL}_{d-1}(\mathbb{D})$ . □

Finally, Proposition 6.3 follows by combining Lemma 6.12 and Lemma 6.15.

7. THE ARCHIMEDEAN CASE

**7.1. The generalized Witt-Grothendieck groups.** In the non-archimedean case, we work with the class of smooth representations of totally disconnected locally compact topological groups. For the archimedean case, we shall replace this by moderate growth smooth Fréchet representations of almost linear Nash groups. Recall that a Nash group is said to be almost linear if it has a Nash representation with a finite kernel. See [Su2] for details on almost linear Nash groups. For the definition of moderate growth smooth Fréchet representations of almost linear Nash groups, see [du1, Definition 1.4.1] or [Su3, Section 2].

Let  $(F, D, \epsilon)$ ,  $U$ , and  $V$  be as in Section 1.1. Recall that  $\psi : F \rightarrow \mathbb{C}^\times$  is a fixed non-trivial unitary character. In this section, assume that  $F$  is archimedean. Then the groups  $\bar{G}(U)$ ,  $\bar{G}(V)$ ,  $H(U \otimes_D V)$ ,  $\bar{J}_U(V)$ , and  $\bar{J}(U, V)$  are all naturally almost linear Nash groups. Denote by  $\text{Irr}(\bar{G}(U))$  the set of all isomorphism classes of irreducible Casselman-Wallach representations of  $\bar{G}(U)$ . Recall that a moderate growth smooth Fréchet representation of  $\bar{G}(U)$  is called a Casselman-Wallach representation if its Harish-Chandra module has a finite length. The reader may consult [Cass] and [Wa2, Chapter 11] for more information about Casselman-Wallach representations.

Replacing smooth representations in the non-archimedean case by moderate growth smooth Fréchet representations, we define in the archimedean case the notion of smooth oscillator representations as in Definition 1.3, and then enhanced oscillator representations of  $\bar{G}(U)$  as in Definition 1.8. The monoid  $\widehat{\mathcal{W}}_U^+$ , the groups  $\widehat{\mathcal{W}}_U$  and  $\mathcal{W}_U$ , and the inverse limits  $\widehat{\mathcal{W}}_\infty^+$ ,  $\widehat{\mathcal{W}}_\infty$ , and  $\mathcal{W}_\infty$ , are defined exactly as in the non-archimedean case.

Also when  $U$  is split and non-zero, define the group  $\text{Hom}(\bar{G}(U)_{\text{split}}, \mathbb{C}^\times)$  as in Section 4.1. Then there is a natural isomorphism

$$\text{Hom}(\bar{G}(U)_{\text{split}}, \mathbb{C}^\times) \cong K^*,$$

where

$$K := \begin{cases} \mathbb{R}^\times / \mathbb{R}_+^\times, & \text{if } U \text{ is a real symmetric bilinear space;} \\ \{1\}, & \text{if } U \text{ is a complex symmetric bilinear space;} \\ \text{Hil}(\mathbb{R}) \cong \mathbb{Z}/4\mathbb{Z}, & \text{if } U \text{ is a real symplectic space;} \\ \{\pm 1\}, & \text{if } U \text{ is a complex symplectic space;} \\ E^\times / \mathbb{R}_+^\times, & \text{if } D \text{ is a quadratic extension;} \\ \{1\}, & \text{if } D \text{ is a quaternion algebra.} \end{cases}$$

Here and as before,  $E$  denotes the center of  $D$ , and  $\text{Hil}(\mathbb{R})$  is defined as in (20).

Similar to Theorem 4.7, for  $F = \mathbb{R}$ , the group  $\widehat{\mathcal{W}}_\infty$  is canonically isomorphic to the group of Table 5.

In Table 5,  $E_-^\times := \{a \in E^\times \mid a^t = -a\}$ ; and for the data in the definitions of the fiber products, we are given the homomorphisms  $\bigoplus_{\varpi \in E_-^\times / \mathbb{R}_+^\times} \mathbb{Z}\varpi \rightarrow \mathbb{Z}/2\mathbb{Z}$  and  $\bigoplus_{\varpi \in \mathbb{R}^\times / \mathbb{R}_+^\times} \mathbb{Z}\varpi \rightarrow \mathbb{Z}/2\mathbb{Z}$  which map the free generators to  $1 + 2\mathbb{Z}$ , and the homomorphism  $(\frac{E_-^\times}{\mathbb{R}_+^\times})^* \rightarrow \mathbb{Z}/2\mathbb{Z}$  whose kernel equals  $(\frac{E_-^\times}{\mathbb{R}_+^\times})^* \subset (\frac{E^\times}{\mathbb{R}_+^\times})^*$ .

Identify  $\widehat{\mathcal{W}}_\infty$  with the group of Table 5. Then the homomorphism  $\widehat{\xi}_\infty : \widehat{\mathcal{W}}_\infty \rightarrow K^*$  (as in (52)) is identical to the obvious projection map except for the case when

TABLE 5. The group  $\widehat{\mathcal{W}}_\infty$  (for  $F = \mathbb{R}$ )

D	F	quadratic extension	quaternion algebra
$\epsilon = 1$	$2\mathbb{Z} \times \left(\frac{\mathbb{R}^\times}{\mathbb{R}_+^\times}\right)^* \times \{\pm 1\}^*$	$\left(\bigoplus_{\varpi \in \frac{\mathbb{E}^\times}{\mathbb{R}_+^\times}} \mathbb{Z}\varpi\right) \times_{\mathbb{Z}/2\mathbb{Z}} \left(\frac{\mathbb{E}^\times}{\mathbb{R}_+^\times}\right)^*$	$\mathbb{Z}$
$\epsilon = -1$	$\bigoplus_{\varpi \in \frac{\mathbb{R}^\times}{\mathbb{R}_+^\times}} \mathbb{Z}\varpi$	$\left(\bigoplus_{\varpi \in \frac{\mathbb{R}^\times}{\mathbb{R}_+^\times}} \mathbb{Z}\varpi\right) \times_{\mathbb{Z}/2\mathbb{Z}} \left(\frac{\mathbb{E}^\times}{\mathbb{R}_+^\times}\right)^*$	$\bigoplus_{\varpi \in \frac{\mathbb{R}^\times}{\mathbb{R}_+^\times}} \mathbb{Z}\varpi$

$\epsilon = -1$  and  $D = F$ . In this exceptional case, the homomorphism

$$(80) \quad \widehat{\xi}_\infty : \widehat{\mathcal{W}}_\infty = \bigoplus_{\varpi \in \mathbb{R}^\times / \mathbb{R}_+^\times} \mathbb{Z}\varpi \rightarrow K^*$$

maps the free generator  $\varpi_a := a\mathbb{R}_+^\times$  ( $a \in \mathbb{R}^\times$ ) to  $\gamma_{\psi_a}$ , where  $\psi_a$  denotes the character

$$F = \mathbb{R} \rightarrow \mathbb{C}^\times, \quad x \mapsto \psi(ax),$$

and  $\gamma_{\psi_a}$  is the character on  $\text{Hil}(\mathbb{R})$  defined as in (46). By the explicit calculation of the Weil indices of real quadratic spaces [Weil, Section 26], we know that the kernel of the homomorphism (80) equals

$$(81) \quad \{a\varpi_1 + b\varpi_{-1} \mid a, b \in \mathbb{Z}, a - b \in 4\mathbb{Z}\}.$$

For  $F \cong \mathbb{C}$  (then  $D = F$ ), it is easy to see that there is a canonical isomorphism:

$$\widehat{\mathcal{W}}_\infty \cong \begin{cases} 2\mathbb{Z} \times \{\pm 1\}^*, & \text{if } \epsilon = 1; \\ \mathbb{Z}, & \text{if } \epsilon = -1, \end{cases}$$

and  $\widehat{\xi}_\infty : \widehat{\mathcal{W}}_\infty \rightarrow K^*$  is the unique surjective homomorphism.

In all cases (for  $F = \mathbb{R}$  or  $F \cong \mathbb{C}$ ), the Kudla homomorphism  $\xi_\infty : \mathcal{W}_\infty \rightarrow K^*$  (as in Section 4) is surjective.

**7.2. Conservations relations.** The archimedean analogue of the following basic results remain true: the smooth version of the Stone–von Neumann Theorem [du2], Howe Duality Conjecture [Ho3], non-vanishing of theta lifting in the stable range [PP], and Kudla’s persistence principle. For each  $\mathfrak{t} \in \mathcal{W}_U$  and for each  $\pi \in \text{Irr}(\widehat{G}(U))$  which is genuine (as in Section 1.5) with respect to  $\mathfrak{t}$ , define the first occurrence index  $n_{\mathfrak{t}}(\pi)$  as in (8).

On the first occurrences, three different phenomena occur in the archimedean case. As in the non-archimedean case, we will need to use some results on the structure of the degenerate principal series of  $\widehat{G}(U)$  for  $U$  split. We refer the reader to Proposition 5.2 for the relevant notations.

**Case 1:**  $U$  is a real or complex symmetric bilinear space. Then the kernel of the Kudla homomorphism  $\xi_\infty : \mathcal{W}_\infty \rightarrow K^*$  has order 2. Define the anti-split Witt tower  $\mathfrak{t}_U^o \in \mathcal{W}_U$  as in the non-archimedean case. It corresponds to the sign character of the orthogonal group  $O(U)$ . The same results as Proposition 5.2 and Proposition 6.1 also hold in this case (see [LZ2, Section 4], [LZ3, Theorem 1], and [Pr2, Appendix C]). Then the argument as in the non-archimedean case shows that the same conservation relations hold.

**Theorem 7.1.** *Let  $U$  be a real or complex symmetric bilinear space. Let  $\mathfrak{t}_1$  and  $\mathfrak{t}_2$  be two Witt towers in  $\mathcal{W}_U$  with difference  $\mathfrak{t}_U^\circ$ . Then for every  $\pi \in \text{Irr}(G(U))$  one has that*

$$n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) = 2 \dim U.$$

**Case 2:**  $U$  is a complex symplectic space or a real quaternionic Hermitian space. Then  $G(U)$  is a perfect group, and  $\mathcal{W}_U$  has two elements, namely, the Witt tower of even dimensional enhanced oscillator representations and the Witt tower of odd dimensional enhanced oscillator representations. Concerning the degenerate principal series, we have Proposition 7.2.

**Proposition 7.2** ([LZ3, Theorem 1, Case I] and [Ya, Corollary 10.5, (2)]). *Let  $U$  be a complex symplectic space or a real quaternionic Hermitian space. Assume that  $U$  is split and let  $X$  be a Lagrangian subspace of  $U$ . Then  $Q_\sigma = I(\chi_{\sigma, X})$  for all  $\sigma \in \widehat{\mathcal{W}}_U$  such that  $\dim \sigma \geq \text{rank } U$ .*

It turns out that there is no conservation relation in the case under consideration. Instead, using Proposition 7.2, the argument as in Section 5 yields the following theorem.

**Theorem 7.3.** *Let  $U$  be a complex symplectic space or a real quaternionic Hermitian space. Let  $\sigma = (V, \omega) \in \widehat{\mathcal{W}}_U^+$  and let  $\pi \in \text{Irr}(\widehat{G}(U))$ . Assume that  $\pi$  is genuine with respect to  $\sigma$ . If  $\dim \sigma \geq \dim U$ , then*

$$\text{Hom}_{\widehat{G}(U)}(\omega, \pi) \neq 0.$$

Consequently, for each  $\mathfrak{t} \in \mathcal{W}_U$  such that  $\pi$  is genuine with respect to  $\mathfrak{t}$ , one has that

$$n_{\mathfrak{t}}(\pi) \leq \begin{cases} \dim U, & \text{if } \dim U \in \dim \mathfrak{t}; \\ \dim U + 1, & \text{otherwise.} \end{cases}$$

Here  $\dim \mathfrak{t} \in \mathbb{Z}/2\mathbb{Z}$  denotes the parity of the dimension of an element of  $\mathfrak{t}$ .

**Case 3:**  $U$  is a real symplectic space, a complex Hermitian or skew-Hermitian space, or a real quaternionic skew-Hermitian space. When  $U$  is a complex Hermitian space, let  $\varpi_+$  and  $\varpi_-$  be the two different elements of the set  $E_-^\times/\mathbb{R}_+^\times$ ; otherwise, let  $\varpi_+$  and  $\varpi_-$  be the two different elements of the set  $\mathbb{R}^\times/\mathbb{R}_+^\times$ . Identify  $\widehat{\mathcal{W}}_\infty$  with the group of Table 5. Then

$$(82) \quad \ker \widehat{\xi}_\infty = \{a\varpi_+ + b\varpi_- \mid a, b \in \mathbb{Z}, a - b \in d_{D, \epsilon} \mathbb{Z}\},$$

where  $d_{D, \epsilon}$  is as in (4). Denote by  $\mathbb{H}_\infty$  the hyperbolic plane in  $\widehat{\mathcal{W}}_\infty$ , namely, the element of  $\widehat{\mathcal{W}}_\infty$  whose image under the natural homomorphism  $\widehat{\mathcal{W}}_\infty \rightarrow \widehat{\mathcal{W}}_U$  equals  $\mathbb{H}_U$ , for every  $\epsilon$ -Hermitian right D-vector space  $U$ . Here the hyperbolic plane  $\mathbb{H}_U \in \widehat{\mathcal{W}}_U$  is defined as in the non-archimedean case. Then under the identification of  $\widehat{\mathcal{W}}_\infty$  with the group of Table 5, we have that  $\mathbb{H}_\infty = \varpi_+ + \varpi_-$ . Therefore, Equation (82) implies that

$$\ker \xi_\infty \cong d_{D, \epsilon} \mathbb{Z}.$$

Denote by  $\mathcal{K}_U \subset \mathcal{W}_U$  the image of  $\ker \xi_\infty$  under the natural homomorphism  $\mathcal{W}_\infty \rightarrow \mathcal{W}_U$ , which is also isomorphic to  $d_{D, \epsilon} \mathbb{Z}$ .

For the degenerate principal series representations, we have the following two propositions.

**Proposition 7.4** ([LZ1, Introduction], [LZ2, Section 4], [LZ4, Section 6]). *Let  $U$  be a real symplectic space, or a complex Hermitian or skew-Hermitian space. Assume that  $U$  is split and let  $X$  be a Lagrangian subspace of  $U$ . Then for all  $\sigma \in \widehat{\mathcal{W}}_U$  such that  $\dim \sigma \leq \frac{\dim U + d_{D,\epsilon} - 2}{2}$ , one has that*

$$\frac{I(\chi'_{\sigma,X})}{\sum_{\sigma'} Q_{\sigma'}} \cong Q_{\sigma}$$

as representations of  $\bar{G}(U)$ , where  $\sigma'$  in the summation runs through all elements of  $\widehat{\mathcal{W}}_U$  such that

$$(83) \quad \begin{cases} \sigma' - \sigma \text{ represents a non-zero element of } \mathcal{K}_U; \text{ and} \\ \dim \sigma' + \dim \sigma = \dim U + d_{D,\epsilon} - 2, \end{cases}$$

and  $\chi'_{\sigma,X} := \chi_{\sigma',X}$  for an arbitrary element  $\sigma' \in \widehat{\mathcal{W}}_U$  satisfying (83).

**Proposition 7.5** (cf. [Ya, Corollary 10.5]). *Let  $U$  be a real quaternionic skew-Hermitian space. Assume that  $U$  is split and let  $X$  be a Lagrangian subspace of  $U$ . Then for all integers  $m, m'$  such that  $m + m' = \dim U - 1$  and  $m' \geq m$ , one has that*

$$\frac{I(\chi_{m',X})}{\sum_{\sigma' \in \widehat{\mathcal{W}}_U, \dim \sigma' = m'} Q_{\sigma'}} \cong \sum_{\sigma \in \widehat{\mathcal{W}}_U, \dim \sigma = m} Q_{\sigma}$$

as representations of  $G(U)$ , where  $\chi_{m',X} := \chi_{\sigma',X}$  for an arbitrary element  $\sigma' \in \widehat{\mathcal{W}}_U$  of dimension  $m'$ .

On the first occurrences, we have the following theorem.

**Theorem 7.6.** *Let  $U$  be a real symplectic space, a complex Hermitian or skew-Hermitian space, or a real quaternionic skew-Hermitian space. Let  $\mathcal{T} \subset \mathcal{W}_U$  be a  $\mathcal{K}_U$ -coset. Let  $\pi \in \text{Irr}(\bar{G}(U))$  which is genuine with respect to some (and hence all) elements of  $\mathcal{T}$ . Then there are two different elements  $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}$  such that*

$$(84) \quad n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) = 2 \dim U + d_{D,\epsilon};$$

and for all different elements  $\mathbf{t}_3, \mathbf{t}_4 \in \mathcal{T}$ , one has that

$$(85) \quad n_{\mathbf{t}_3}(\pi) + n_{\mathbf{t}_4}(\pi) \geq 2 \dim U + d_{D,\epsilon} |\mathbf{t}_3 - \mathbf{t}_4|,$$

where for  $\mathbf{t} \in \mathcal{K}_U$ ,  $|\mathbf{t}|$  denotes the non-negative integer so that  $\mathbf{t}$  is a  $|\mathbf{t}|$ -multiple of a generator of  $\mathcal{K}_U$ . Consequently the following conservation relations hold:

$$\sum_{\mathcal{Q} \in \mathcal{T}/2\mathcal{K}_U} \min\{n_{\mathbf{t}}(\pi) \mid \mathbf{t} \in \mathcal{Q}\} = 2 \dim U + d_{D,\epsilon}.$$

*Proof.* Summarizing the results in [Pr2, Appendix C], [Pa, Lemma 3.1], [LPTZ, Proposition 3.38], and [LL, Theorem 1.2.1], we know that the trivial representation  $1_U$  does not occur before the stable range in every non-split Witt tower in  $\mathcal{K}_U$ , that is,

$$(86) \quad n_{\mathbf{t}}(1_U) \geq 2 \dim U + d_{D,\epsilon} |\mathbf{t}|, \quad \mathbf{t} \in \mathcal{K}_U \setminus \{0\}.$$

As in Section 2.1, Equation (86) implies that

$$(87) \quad n_{\mathbf{t}_3}(\pi) + n_{-\mathbf{t}_4}(\pi^\vee) \geq 2 \dim U + d_{D,\epsilon} |\mathbf{t}_3 - \mathbf{t}_4|,$$

for all different elements  $\mathbf{t}_3, \mathbf{t}_4 \in \mathcal{T}$ .

On the other hand, using MVW-involutions on archimedean metaplectic groups and classical groups (cf. [MVW, Pr1, Su1, LST]), one knows that for all  $\mathbf{t} \in \mathcal{T}$ ,

$$(88) \quad n_{\mathbf{t}}(\pi) = n_{-\mathbf{t}}(\pi^\vee).$$

Therefore the inequality (85) is implied by (87).

To prove the first assertion of the theorem, we first assume that  $U$  is a real symplectic space, or a complex Hermitian or skew-Hermitian space. Then there is a unique pair  $(m_1, m_2)$  of integers so that

$$\begin{cases} m_1, m_2 \in \{\dim \sigma \mid \sigma \in \widehat{\mathcal{W}}_U, \sigma \text{ represents an element of } \mathcal{T}\}, \\ m_1 + m_2 = 2 \dim U + d_{D,\epsilon} - 2, \text{ and} \\ m_1 - m_2 = 0 \text{ or } 2. \end{cases}$$

As a first step, we show that there exists  $\mathbf{t}_1 \in \mathcal{T}$  such that  $n_{\mathbf{t}_1}(\pi) \leq m_1$ . We pick any  $\mathbf{t} \in \mathcal{T}$ . If  $n_{\mathbf{t}}(\pi) \leq m_1$ , we are done. Otherwise  $n_{\mathbf{t}}(\pi) \geq m_1 + 2 \geq m_2 + 2$ , and so  $\pi \notin \mathcal{R}_{\sigma_{\mathbf{t},m_2}}$ , where  $\sigma_{\mathbf{t},m_2}$  is the element of  $\widehat{\mathcal{W}}_U$  which represents  $\mathbf{t}$  and has dimension  $m_2$ . By applying Proposition 7.4 to  $\sigma_{\mathbf{t},m_2}$ , the same proof as in Proposition 5.6 shows that there exists an element  $\sigma' \in \widehat{\mathcal{W}}_U$  such that

$$\begin{cases} \sigma' - \sigma_{\mathbf{t},m_2} \text{ represents a non-zero element of } \mathcal{K}_U; \\ \dim \sigma' = m_1; \text{ and} \\ \pi \in \mathcal{R}_{\sigma'}. \end{cases}$$

Consequently, we have

$$\min\{n_{\mathbf{t}'}(\pi) \mid \mathbf{t}' \in \mathcal{T}, \mathbf{t}' \neq \mathbf{t}\} \leq m_1.$$

We may thus find some  $\mathbf{t}_1 \in \mathcal{T}$  such that  $n_{\mathbf{t}_1}(\pi) \leq m_1$ .

Write  $k = n_{\mathbf{t}_1}(\pi)$ , and consider  $\sigma_{\mathbf{t}_1,k-2}$ , the element of  $\widehat{\mathcal{W}}_U$  which represents  $\mathbf{t}_1$  and has dimension  $k - 2$ . Then  $\pi \notin \mathcal{R}_{\sigma_{\mathbf{t}_1,k-2}}$ , and

$$k - 2 \leq m_1 - 2 \leq m_2 \leq \frac{2 \dim U + d_{D,\epsilon} - 2}{2}.$$

Similarly, applying Proposition 7.4 to  $\sigma_{\mathbf{t}_1,k-2}$ , the same proof as in Proposition 5.6 shows that there exists an element  $\sigma' \in \widehat{\mathcal{W}}_U$  such that

$$\begin{cases} \sigma' - \sigma_{\mathbf{t}_1,k-2} \text{ represents a non-zero element of } \mathcal{K}_U; \\ \dim \sigma' + (k - 2) = 2 \dim U + d_{D,\epsilon} - 2; \text{ and} \\ \pi \in \mathcal{R}_{\sigma'}. \end{cases}$$

Consequently, we have

$$\min\{n_{\mathbf{t}'}(\pi) \mid \mathbf{t}' \in \mathcal{T}, \mathbf{t}' \neq \mathbf{t}_1\} \leq 2 \dim U + d_{D,\epsilon} - k.$$

In other words, there is an element  $\mathbf{t}_2 \in \mathcal{T} \setminus \{\mathbf{t}_1\}$  such that

$$n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) \leq 2 \dim U + d_{D,\epsilon}.$$

In view of the inequality (85), this proves the first assertion of the theorem, in the case when  $U$  is a real symplectic space, or a complex Hermitian or skew-Hermitian space.



Now assume that  $U$  is a real quaternionic skew-Hermitian space. Then  $\mathcal{T} = \mathcal{W}_U$ . Recall that for each  $\mathbf{t} \in \mathcal{W}_U$ ,  $\dim \mathbf{t} \in \mathbb{Z}/2\mathbb{Z}$  denotes the parity of the dimension of an element of  $\mathbf{t}$ . Put

$$n_0(\pi) := \min\{n_{\mathbf{t}}(\pi) \mid \mathbf{t} \in \mathcal{W}_U, \dim \mathbf{t} \text{ is even}\}$$

and

$$n_1(\pi) := \min\{n_{\mathbf{t}}(\pi) \mid \mathbf{t} \in \mathcal{W}_U, \dim \mathbf{t} \text{ is odd}\}.$$

In view of the inequality (85), for the first assertion of the theorem, it suffices to show that

$$n_0(\pi) + n_1(\pi) \leq 2 \dim U + 1 \quad (d_{D,\epsilon} = 1).$$

Assume by contradiction that

$$n_0(\pi) + n_1(\pi) \geq 2 \dim U + 3.$$

Then there are integers  $m, m'$  such that

$$(89) \quad \begin{cases} m \text{ is even and } m' \text{ is odd;} \\ m + m' = 2 \dim U - 1; \\ m < n_0(\pi); \text{ and} \\ m' < n_1(\pi); \end{cases}$$

Using Proposition 7.5, the same proof as Proposition 5.6 shows that there exists an element  $\sigma \in \widehat{\mathcal{W}}_U$  such that

$$\begin{cases} \dim \sigma = m \text{ or } m'; \text{ and} \\ \pi \in \mathcal{R}_\sigma. \end{cases}$$

Therefore either  $n_0(\pi) \leq m$  or  $n_1(\pi) \leq m'$ . This contradicts the two inequalities of (89).

The last assertion of the theorem is easily implied by (84) and (85). □

**Remarks:** (a) When  $U$  is a complex symmetric bilinear space, the conservation relations were proved by Adams-Barbasch [AB] using the explicit duality correspondence. A. Paul proved the conservation relations for archimedean unitary-unitary dual pair correspondence [Pa, Theorem 1.4], for a discrete series representation, or for a representation irreducibly induced from a discrete series representation.

(b) Let  $G$  be a topological group. An involutive continuous automorphism  $\tau$  of  $G$  is called an MVW-involution if  $\tau(g)$  and  $g^{-1}$  are conjugate in  $G$ , for all  $g$  in an open dense subset of  $G$ . MVW-involutions of  $\bar{G}(U)$  do not exist in general when  $U$  is a non-archimedean quaternionic Hermitian space or a non-archimedean quaternionic skew Hermitian space [LST, Proposition 1.3] (MVW-involutions of  $\bar{G}(U)$  do exist in all the other cases). Nonetheless the equality (88) is still valid for this case, in view of the equalities in (12).

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HUA LOO-KENG KEY LABORATORY OF MATHEMATICS, INSTITUTE OF MATHEMATICS, AMSS,  
CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, P.R. CHINA

*E-mail address:* [sun@math.ac.cn](mailto:sun@math.ac.cn)

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, BLOCK S17, 10 LOWER  
KENT RIDGE ROAD, SINGAPORE 119076

*E-mail address:* [matzhucb@nus.edu.sg](mailto:matzhucb@nus.edu.sg)