REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 1, Pages 190–206 (July 17, 1997)
S 1088-4165(97)00031-9

ON CERTAIN SMALL REPRESENTATIONS OF INDEFINITE ORTHOGONAL GROUPS

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ABSTRACT. For any $n \in \mathbb{N}$ such that $2n \leq \min(p, q)$, we construct a representation π_n of $O(p, q)$ with $p+q$ even as the kernel of a commuting set of $\frac{n(n+1)}{2}$ number of $O(p, q)$-invariant differential operators in the space of C^∞ functions on an isotropic cone with a distinguished $GL_n(\mathbb{R})$-homogeneity degree. By identifying π_n with a certain representation constructed via the formalism of the theta correspondence, we show (except when $p = q = 2n$) that the space of K-finite vectors of π_n is the $(\mathfrak{g}, K)$-module of an irreducible unitary representation of $O(p, q)$ with Gelfand-Kirillov dimension $n(p + q - 2n - 1)$. Our construction generalizes the work of Binegar and Zierau (*Unitarization of a singular representation of* $SO_e(p, q)$, Commun. Math. Phys. **138** (1991), 245–258) for $n = 1$.

1. Construction of the representation π_n

Let $V = \mathbb{R}^{p,q} \simeq \mathbb{R}^p \oplus \mathbb{R}^q$ be the real vector space equipped with the quadratic form

$$(v, w) = \sum_{t=1}^{p} x_t y_t - \sum_{t=p+1}^{p+q} x_t y_t,$$

where $v = \begin{pmatrix} x_1 \\ \vdots \\ x_{p+q} \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_{p+q} \end{pmatrix} \in \mathbb{R}^{p,q}$. Let $G = O(p, q)$ be the isometry group.

Fix a natural number n. We always assume $p + q$ even and $2n \leq \min(p, q)$. Let V^n be the direct sum of n copies of V, which can be identified with the space $M_{p+q,n}(\mathbb{R})$ of $(p + q) \times n$ real matrices.

Let

$$X = \big(x_{ij}\big)_{1\leq i\leq p+q, 1\leq j\leq n} \in M_{p+q,n}(\mathbb{R}),$$

and $\mathcal{X}$ be the following cone

$$\mathcal{X} = \{X \in M_{p+q,n}(\mathbb{R}) | X^t I_{p,q} X = 0_n, \text{ the } n \times n \text{ zero matrix}\}.$$

Received by the editors September 4, 1996 and, in revised form, January 9, 1997.
1991 *Mathematics Subject Classification.* Primary 22E45, 22E46.
Key words and phrases. Orthogonal groups, isotropic cones, theta correspondence, Howe quotient, Gelfand-Kirillov dimension, nilpotent orbits.

Let $X=(X_1,\cdots X_n)$ so that X_i is the ith column vector of X, and for $1\le i,j\le n$, let r_{ij} be quadratic forms defined by

$$r_{ij}(X)=(X_i,X_j)=\sum_{t=1}^{p}x_{ti}x_{tj}-\sum_{t=p+1}^{p+q}x_{ti}x_{tj}, \tag{1.1}$$

then $X\in\mathcal{X}$ if and only if $r_{ij}(X)=0$ for $1\le i\le j\le n$. We shall thus call $\mathcal{X}$ the isotropic cone.

Define the following subset of $\mathcal{X}$

$$\mathcal{X}^{00}=\{X\in\mathcal{X}|\,\operatorname{rank}X=n\}.$$

It is dense in $\mathcal{X}$, since $2n\le\min(p,q)$.

The group $G=O(p,q)$ acts on $V^n\simeq M_{p+q,n}(\mathbb{R})$ and the action is identified with left multiplication:

$$X\mapsto gX,\quad g\in O(p,q),\ X\in M_{p+q,n}(\mathbb{R}).$$

The group $H=GL_n(\mathbb{R})$ acts on $M_{p+q,n}(\mathbb{R})$ on the right by

$$X\mapsto Xh^{-1},\quad h\in GL_n(\mathbb{R}),\ X\in M_{p+q,n}(\mathbb{R}).$$

Clearly the actions of G and H on $M_{p+q,n}(\mathbb{R})$ commute, and both actions preserve the isotropic cone $\mathcal{X}$, and $\mathcal{X}^{00}$. Our hypothesis $2n\le\min(p,q)$ implies that $O(p,q)$ acts on $\mathcal{X}^{00}$ transitively (Witt's Lemma). Thus $\mathcal{X}^{00}$ is a homogeneous space of $O(p,q)$.

Define the following set of differential operators

$$\Delta_{ij}=\sum_{t=1}^{p}\frac{\partial^2}{\partial x_{ti}\partial x_{tj}}-\sum_{t=p+1}^{p+q}\frac{\partial^2}{\partial x_{ti}\partial x_{tj}}, \tag{1.2}$$

$$E_{ij}=\sum_{t=1}^{p+q}x_{ti}\frac{\partial}{\partial x_{tj}}, \tag{1.3}$$

for $1\le i,j\le n$. Then $\{r_{ij},\Delta_{ij},E_{ij}\}_{1\le i,j\le n}$ generate the algebra of $O(p,q)$-invariant polynomial differential operators on $M_{p+q,n}(\mathbb{R})$.

For $\alpha\in\mathbb{C}$, let $C^\infty(\mathcal{X}^{00},\alpha)_\pm$ be the space of C^∞ functions f on $\mathcal{X}^{00}$ such that

$$f(Xh)=|\det h|^\alpha(\frac{\det h}{|\det h|})^\epsilon f(X),\quad h\in GL_n(\mathbb{R}),\ X\in\mathcal{X}^{00},$$

where $\epsilon=0,1$ respectively. Likewise we define $C^\infty(U,\alpha)_\pm$ for any open conical neighborhood U of $\mathcal{X}^{00}$, namely any open neighborhood of $\mathcal{X}^{00}$ which is stable under $GL_n(\mathbb{R})$.

Since the action of G commutes with the H action, G preserves $C^\infty(\mathcal{X}^{00},\alpha)_\pm$.

Let U be an open conical neighborhood of $\mathcal{X}^{00}$. Observe that the induced action of $GL_n(\mathbb{R})$ on $C^\infty(U)$ is given by

$$(h\cdot F)(X)=F(Xh),\quad F\in C^\infty(U),\ h\in GL_n(\mathbb{R}),\ X\in U.$$

Let e_{ij} be the $n\times n$ matrix with one at the (i,j) entry and zeros elsewhere. Together they form a basis for the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of $GL_n(\mathbb{R})$. Then the derived action of $\mathfrak{gl}_n(\mathbb{R})$ on $C^\infty(U)$ is given by

$$e_{ij}\mapsto E_{ij}.$$

Clearly if $F \in C^\infty(U,\alpha)_\pm$, then

$$E_{ij}F = \alpha\delta_{ij}F, \qquad 1 \le i,j \le n.$$

Let $f \in C^\infty(\mathcal{X}^{00},\alpha)_\pm$. We would like to apply the differential operator Δ_{kl} to it. To this end, we first extend f to a conical neighborhood U of $\mathcal{X}^{00}$ in a conical way, namely we require the extension $\tilde{f}$ to belong to $C^\infty(U,\alpha)_\pm$. This can always be done. Now we can apply Δ_{kl} to the extension $\tilde{f}$ and then restrict $\Delta_{kl}(\tilde{f})$ back to $\mathcal{X}^{00}$. The problem is of course that the restriction may depend on the choice of the extension. We shall show for $\alpha = n+1-\frac{p+q}{2}$, $\Delta_{kl}(\tilde{f})|_{\mathcal{X}^{00}}$ is independent of the extension $\tilde{f}$. It clearly suffices to demonstrate the following

Proposition 1.1. *Let U be a conical neighborhood of $\mathcal{X}^{00}$ in $M_{p+q,n}(\mathbb{R})$. If $F \in C^\infty(U, n+1-\frac{p+q}{2})_\pm$ and $F|_{\mathcal{X}^{00}} = 0$, then $\Delta_{kl}(F)|_{\mathcal{X}^{00}} = 0$ for $1 \le k,l \le n$.*

Proof. Since $2n \le \min(p,q)$, we see that the $\frac{n(n+1)}{2}$ polynomials r_{ij} $(1 \le i \le j \le n)$ are algebraically independent. In fact the Jacobian matrix $J(r_{11}, r_{12}, ..., r_{nn})$ at any point of $\mathcal{X}^{00}$ has full rank $\frac{n(n+1)}{2}$ ([T], Lemma 2.9). Thus they are functionally independent. Using this fact and the fact $F|_{\mathcal{X}^{00}} = 0$, we can write

$$F = \sum_{1\le i,j\le n} r_{ij}F_{ij},$$

where $F_{ij} = F_{ji} \in C^\infty(U')$, for a small conical neighbourhood $U' \subset U$ of $\mathcal{X}^{00}$. Denote by I the ideal in $C^\infty(U')$ generated by $\{r_{ij}\}_{1\le i\le j\le n}$, then each F_{ij} is unique modulo I.

We compute

$$\Delta_{kl}(\sum_{ij} r_{ij}F_{ij}) = \sum_{ij} r_{ij}\Delta_{kl}(F_{ij}) + \sum_{ij}[\Delta_{kl}, r_{ij}]F_{ij}.$$

We have

$$[\Delta_{kl}, r_{ij}] = \delta_{jk}H_{il} + \delta_{il}H_{jk} + \delta_{jl}H_{ik} + \delta_{ik}H_{jl},$$

where

$$H_{ij} = E_{ij} + \frac{p+q}{2}\delta_{ij}, \qquad 1 \le i,j \le n. \tag{1.4}$$

Now since $F = \sum_{ij} r_{ij}F_{ij} \in C^\infty(U,\alpha)$, we have (as functions in $C^\infty(U')$)

$$E_{kl}(\sum_{ij} r_{ij}F_{ij}) = \alpha\delta_{kl}(\sum_{ij} r_{ij}F_{ij}), \qquad 1 \le k,l \le n. \tag{1.5}$$

A simple computation gives

$$E_{kl}(r_{ij}) = \delta_{li}r_{kj} + \delta_{lj}r_{ki}.$$

Thus we have

$$\begin{aligned} E_{kl}(\sum_{ij} r_{ij}F_{ij}) &= \sum_{ij} E_{kl}(r_{ij})F_{ij} + r_{ij}E_{kl}(F_{ij}) \\ &= \sum_{ij}(\delta_{li}r_{kj} + \delta_{lj}r_{ki})F_{ij} + r_{ij}E_{kl}(F_{ij}) \\ &= \sum_{ij} r_{ij}(\delta_{ik}F_{lj} + \delta_{ik}F_{jl} + E_{kl}(F_{ij})) \\ &= \sum_{ij} r_{ij}(2\delta_{ik}F_{lj} + E_{kl}(F_{ij})). \end{aligned}$$

Comparing the coefficients in front of r_{ij} in Equation (1.5) (after symmetrization), we obtain

$$\delta_{ik}F_{lj} + \delta_{jk}F_{li} + E_{kl}(F_{ij}) \equiv \alpha\delta_{kl}F_{ij}, \qquad (\text{mod } I).$$

Therefore we have

$$E_{kl}(F_{ij}) \equiv \alpha\delta_{kl}F_{ij} - \delta_{ik}F_{lj} - \delta_{jk}F_{li}, \qquad (\text{mod } I)$$

and so

$$H_{kl}(F_{ij}) \equiv (\alpha + \frac{p+q}{2})\delta_{kl}F_{ij} - \delta_{ik}F_{lj} - \delta_{jk}F_{li}, \qquad (\text{mod } I).$$

We continue our computation

$$\begin{aligned} \sum_{ij}[\Delta_{kl}, r_{ij}]F_{ij} &= \sum_{ij}(\delta_{jk}H_{il} + \delta_{il}H_{jk} + \delta_{jl}H_{ik} + \delta_{ik}H_{jl})F_{ij} \\ &\equiv \sum_{ij} \delta_{jk}((\alpha + \frac{p+q}{2})\delta_{il}F_{ij} - F_{lj} - \delta_{ji}F_{li}) \\ &\quad + \sum_{ij} \delta_{il}((\alpha + \frac{p+q}{2})\delta_{jk}F_{ij} - \delta_{ij}F_{kj} - F_{ki}) \\ &\quad + \sum_{ij} \delta_{jl}((\alpha + \frac{p+q}{2})\delta_{ik}F_{ij} - F_{kj} - \delta_{ij}F_{ki}) \\ &\quad + \sum_{ij} \delta_{ik}((\alpha + \frac{p+q}{2})\delta_{jl}F_{ij} - \delta_{ji}F_{lj} - F_{li}) \\ &\equiv 2(\alpha + \frac{p+q}{2})F_{lk} + 2(\alpha + \frac{p+q}{2})F_{kl} - 2nF_{lk} - 2nF_{kl} - 2F_{lk} - 2F_{kl} \\ &\equiv 4(\alpha + \frac{p+q}{2} - n - 1)F_{kl}, \qquad (\text{mod } I). \end{aligned}$$

To summarize, we have

$$\Delta_{kl}(\sum_{ij} r_{ij}F_{ij}) \equiv \sum_{ij} r_{ij}\Delta_{kl}(F_{ij}) + 4(\alpha + \frac{p+q}{2} - n - 1)F_{kl}, \qquad (\text{mod } I).$$

If $\alpha = n + 1 - \frac{p+q}{2}$, then we clearly have

$$\Delta_{kl}(\sum_{ij} r_{ij}F_{ij})|_{\mathcal{X}^{00}} = 0. \quad \square$$

Thus for $1 \le k \le l \le n$, we have the following differential operator Δ'_{kl} defined on $C^\infty(\mathcal{X}^{00}, n+1-\frac{p+q}{2})_\pm$

$$\Delta'_{kl}(f) = \Delta_{kl}(\tilde{f})|_{\mathcal{X}^{00}}, \qquad f \in C^\infty(\mathcal{X}^{00}, n+1-\frac{p+q}{2})_\pm,$$

where $\tilde{f}$ is a conical extension of f to a conical neighborhood U of $\mathcal{X}^{00}$. Since the action of G commutes with Δ_{kl}, the action of G also commutes with Δ'_{kl}.

Set $\epsilon \equiv \frac{p-q}{2}$ (mod 2). Let

$$\text{(1.6)} \qquad \mathcal{H}_n = \{f \in C^\infty(\mathcal{X}^{00}, n+1-\frac{p+q}{2})_\epsilon | \Delta'_{kl}(f) = 0, \text{ for } 1 \le k \le l \le n\}.$$

Then G acts on $\mathcal{H}_n$. We denote this representation of G by π_n, where $2n \le \min(p,q)$.

The following is our main result and its proof is given in §4 and §5.

Theorem 1.2. *For any natural number n such that $2n \le \min(p,q)$ (except for $p = q = 2n$), we have*

(a) *the K-finite vectors of π_n is the $(\mathfrak{g}, K)$-module of an irreducible unitary representation of $O(p,q)$, and*

(b) *π_n has the Gelfand-Kirillov dimension $n(p+q-2n-1)$.*

Remarks. A) For $n = 1$, our construction reduces to that of Binegar and Zierau ([BZ]), which in turn is a generalization of the work of Kostant for $p = q = 4$ ([K1], [K2]). It turns out that π_1 is the minimal representation of $O(p,q)$ ($p+q$ even, $\min(p,q) \ge 2$, except for $p = q = 2$). In [BZ] the irreducibility and the unitarizability of π_1 is proved by long and intriguing calculations involving spherical functions and Gegenbauer polynomials. Our strategy of proving the irreducibility and unitarizability of π_n is to identify it with a certain representation constructed via the formalism of the theta correspondence. Then the statement on the irreducibility and the unitarizability will be derived from the powerful result of Li on the theta lifting in the stable range ([L1]).

B) We remark also that for $n = 1$, the infinitesimal structure of $C^\infty(\mathcal{X}^{00}, \alpha)_\pm$ as an $O(p,q)$ module is the main object of study in Howe and Tan's work [HT], and is completely determined for any α. In particular, they pointed out that when $p+q$ is even, the space

$$C^\infty(\mathcal{X}^{00}, 2-\frac{p+q}{2}) = C^\infty(\mathcal{X}^{00}, 2-\frac{p+q}{2})_+ \oplus C^\infty(\mathcal{X}^{00}, 2-\frac{p+q}{2})_-$$

contains a unitary constituent whose K-types are distributed along a single line (a ladder representation), and they went on to say that this representation should be obtained from the theta lift of the trivial representation for the dual pair $(O(p,q), SL_2(\mathbb{R}))$. Our article is partly motivated by their important observation. In this connection, we would like to point out a mistake in [BZ]. In our notation, π_1 is a submodule of $C^\infty(\mathcal{X}^{00}, 2-\frac{p+q}{2})_\epsilon$, where $\epsilon \equiv \frac{p-q}{2}$ (mod 2), instead of $\epsilon \equiv 2-\frac{p+q}{2}$ (mod 2). The latter statement which is claimed in [BZ] is not correct when p and q are both odd. See [HT] for details.

C) The representation π_n is a subrepresentation of the degenerate principal series representation

$$Ind_{(O(p-n,q-n)\cdot GL_n(\mathbb{R}))\ltimes N}^{O(p,q)} \chi,$$

where the inducing representation χ is the character $|\det|^{n+1-\frac{p+q}{2}}(\frac{\det}{|\det|})^{\frac{p-q}{2}}$ on $GL_n(\mathbb{R})$ defined in the obvious way and is trivial on $O(p-n,q-n)\ltimes N$. This degenerate series representation itself is not K-multiplicity free except in the case of $n=1$, but the subrepresentation π_n is K-multiplicity free. Moreover, π_n is not spherical except when $p=q$. We would also like to point out that π_n has ZN_k-rank $2n$ in the sense of Howe, where ZN_k is the center of the unipotent radical N_k of a certain parabolic subgroup P_k of $O(p,q)$ with $k=\min(p,q)$, the real rank of $O(p,q)$. See [H5], [L2].

D) The representations π_n can be thought of as the orthogonal analog of some unipotent representations of hermitian symmetric groups (e.g. $Sp(2n,\mathbb{R})$) constructed by Sahi ([S]). The approaches there are rather different from ours, but from the point of view of the local theta correspondence, the nature of the representations constructed are really quite similar. They are all theta lifts of unitary characters in the stable range. For more results in this direction (both stable and unstable range), we refer the reader to [LZ1], [LZ2].

E) Philosophically it is rather conceivable that the representation π_n has the GK-dimension $n(p+q-2n-1)$. The space of C^∞ functions on the isotropic cone with a fixed $GL_n(\mathbb{R})$ homogeneity degree satisfies $\frac{n(n+1)}{2}+n^2$ equations. As the kernel of $\frac{n(n+1)}{2}$ differential operators on that space, the representation π_n "should" have GK-dimension $n(p+q)-(\frac{n(n+1)}{2}+n^2)-\frac{n(n+1)}{2}=n(p+q-2n-1)$. Our argument relies on a description of K-types of π_n and thus its asymptotic behavior is readily computed.

2. Theta lifting of unitary characters in the stable range

Let (G,G') be a reductive dual pair inside some real symplectic group Sp. Let $\tilde{Sp}$ be the metaplectic two fold cover of Sp. Let ω be the oscillator representation of $\tilde{Sp}$, realized on a Hilbert space $\mathcal{Y}$. Let $\mathcal{Y}^\infty$ be the space of smooth vectors of $\mathcal{Y}$, and ω^∞ be the associated smooth representation of $\tilde{Sp}$. For a reductive subgroup E of Sp, let $\tilde{E}$ be the inverse image of E inside $\tilde{Sp}$. Denote by $\mathcal{R}(\tilde{E})$ the set of infinitesimal equivalence classes of continuous irreducible admissible representations of $\tilde{E}$ on locally convex space. Let $\mathcal{R}(\tilde{E},\omega)$ be the set of elements of $\mathcal{R}(\tilde{E})$ which are realized as quotients by $\omega^\infty(\tilde{E})$-invariant closed subspaces of $\mathcal{Y}^\infty$.

Given $\rho\in\mathcal{R}(\tilde{G}',\omega)$, let $\Omega(\rho)$ be the maximal quotient of $\mathcal{Y}^\infty$ on which $\tilde{G}'$ acts by a representation of class ρ. We have

$$\Omega(\rho)\cong\rho'\otimes\rho$$

where ρ' is a $\tilde{G}$ module. By the result of [H2], ρ' is a finitely generated admissible quasi-simple representation of $\tilde{G}$, and has a unique irreducible $\tilde{G}$ quotient, denoted by $\theta(\rho)$. The correspondence

$$\rho\mapsto\theta(\rho)$$

is usually called the (local) theta correspondence or the Howe quotient correspondence.

Let $\mathbb{Z}_2=\{\pm1\}$ be the kernel of the projection from $\tilde{Sp}$ to Sp. Let $\hat{\tilde{G}}(\epsilon)$ (resp. $\hat{\tilde{G}}'(\epsilon)$) be the subset of the unitary dual of $\tilde{G}$ (resp. $\tilde{G}'$) consisting of those representations whose restriction to $\mathbb{Z}_2$ is a multiple of the unique non-trivial character of $\mathbb{Z}_2$.

Suppose that (G, G') is a type I dual pair and it is in the stable range with G' the small member. By the result of [L1], $\hat{\tilde{G}}'(\epsilon) \subset \mathcal{R}(\tilde{G}', \omega)$, and the Howe duality correspondence gives rise to an injection

$$\hat{\tilde{G}}'(\epsilon) \hookrightarrow \hat{\tilde{G}}(\epsilon).$$

(The case of $(G, G') = (O(p,q), Sp(2n, \mathbb{R}))$ with $p = q = 2n$ is excluded for some technical reason).

Proposition 2.1. *Suppose that (G, G') is in the stable range with G' the small member. Then for any unitary character $\chi \in \hat{\tilde{G}}'(\epsilon)$, χ' is already irreducible. Thus $\chi' = \theta(\chi)$ is irreducible and unitary.*

Remark. In the p-adic case, a result of the same nature was first shown by Kudla and Rallis (Proposition 3.1, [KR2]).

The proof of Proposition 2.1 relies on Li's construction in the stable range of $\theta(\rho)$, for $\rho \in \hat{\tilde{G}}'(\epsilon)$, described below.

Let ρ be realized on a Hilbert space H_ρ. Consider the $\tilde{G} \times \tilde{G}'$ module $\omega^\infty \otimes \rho^\infty$ on the algebraic tensor product

$$\mathcal{Y}^\infty \otimes H_\rho^\infty,$$

where a superscript ∞ denotes the smooth representation or the space of smooth vectors, as usual. For the ease of notation, we shall write $\omega \otimes \rho$ for $\omega^\infty \otimes \rho^\infty$. Note that $\omega \otimes \rho$ is in fact a representation of G. The unitary structures of $\mathcal{Y}$ and H_ρ give rise to an inner product $(,)$ on $\mathcal{Y}^\infty \otimes H_\rho^\infty$.

For any $\Phi_1, \Phi_2 \in \mathcal{Y}^\infty \otimes H_\rho^\infty$, define

$$(\Phi_1, \Phi_2)_\rho = \int_{G'} (\Phi_1, (\omega \otimes \rho)(g)\Phi_2) dg.$$

The result of [L1] tells us that if (G, G') is in the stable range with G' the small member, then the above integral is convergent and defines a $\tilde{G}$-invariant Hermitian form.

Let R_ρ be the radical of the form $(,)_\rho$, which is preserved by $\tilde{G}$. Denote the representation of $\tilde{G}$ on the quotient space

$$H(\rho) = (\mathcal{Y}^\infty \otimes H_\rho^\infty)/R_\rho$$

by $\pi(\rho)$. The following is the main result of [L1].

Theorem 2.2. *Suppose that (G, G') is a type I dual pair in the stable range with G' the small member (except for $(G, G') = (O(2n, 2n), Sp(2n, \mathbb{R}))$). Then for any $\rho \in \hat{\tilde{G}}'(\epsilon)$,*

(a) *$\pi(\rho)$ is non-zero and irreducible;*
(b) *The form $(\, , \,)_\rho$ is non-negative, and therefore $\pi(\rho)$ is unitary;*
(c) *$\theta(\rho) \cong \pi(\rho^*)$.*

Now let $\rho = \chi \in \hat{\tilde{G}}'(\epsilon)$ be a unitary character, so that $H_\chi \cong \mathbb{C}$. Thus we have a canonical isomorphism

$$\mathcal{Y}^\infty \otimes H_\chi^\infty \cong \mathcal{Y}^\infty.$$

Denote by $\mathcal{J}_\chi$ the minimal $\omega^\infty(\tilde{G}')$-invariant closed subspace of $\mathcal{Y}^\infty$ such that $\tilde{G}'$ acts on the quotient by a multiple of χ. Thus we have

$$\Omega(\chi) = \mathcal{Y}^\infty/\mathcal{J}_\chi.$$

Proposition 2.1 will follow from Theorem 2.2 and the following simple lemma.

Lemma 2.3. *We have*

$$R_{\bar{\chi}} = \mathcal{J}_\chi = \textit{closure of the span } \{(\omega(g)-\chi(g))f | g \in \tilde{G}',\ f \in \mathcal{Y}^\infty\}.$$

Proof. Let $\mathcal{I}_\chi$ = closed span of the set $\{(\omega(g)-\chi(g))f | g \in \tilde{G}',\ f \in \mathcal{Y}^\infty\}$. Then clearly $\mathcal{I}_\chi$ is a $\tilde{G}'$-invariant closed subspace of $\mathcal{Y}^\infty$ and $\tilde{G}'$ acts on the quotient by a representation of class χ. Moreover, any $\tilde{G}'$-invariant closed subspace of $\mathcal{Y}^\infty$ for which $\tilde{G}'$ acts on the quotient by a representation of class χ must contain $\mathcal{I}_\chi$. Therefore $\mathcal{J}_\chi = \mathcal{I}_\chi$.

Next we let $\Phi = (\omega(g_1)-\chi(g_1))f$, where $g_1 \in \tilde{G}'$, $f \in \mathcal{Y}^\infty$. Then for any $\Phi_2 \in \mathcal{Y}^\infty$, we have

$$\begin{aligned}(\Phi,\Phi_2)_{\bar{\chi}} &= \int_{G'} ((\omega(g_1)-\chi(g_1))f, \bar{\chi}(g)\omega(g)\Phi_2)dg \\ &= \int_{G'} (f, \bar{\chi}(g)\omega(g_1^{-1}g)\Phi_2)dg - \chi(g_1)\int_{G'}(f,\chi(g)\omega(g)\Phi_2)dg \\ &= \int_{G'} (f, \bar{\chi}(g_1 g)\omega(g)\Phi_2)dg - \chi(g_1)\int_{G'}(f,\chi(g)\omega(g)\Phi_2)dg = 0.\end{aligned}$$

This implies $\Phi \in R_{\bar{\chi}}$ and so $\mathcal{J}_\chi \subseteq R_{\bar{\chi}}$.

Now let $h \in R_{\bar{\chi}} \cap \mathcal{J}_\chi^\perp$, where $\mathcal{J}_\chi^\perp$ denotes the subspace of $\mathcal{Y}^\infty$ which is orthogonal to $\mathcal{J}_\chi$ with respect to the unitary structure of $\mathcal{Y}^\infty$. Then we have

$$\begin{aligned}0 = (h,h)_{\bar{\chi}} &= \int_{G'} (h, \bar{\chi}(g)\omega(g)h)dg = \int_{G'} (\omega(g^{-1})h, \bar{\chi}(g)h)dg \\ &= \int_{G'} (\chi(g^{-1})h, \bar{\chi}(g)h)dg = \int_{G'} (h,h)dg.\end{aligned}$$

We conclude that $h = 0$. Therefore $R_{\bar{\chi}} = \mathcal{J}_\chi$. □

3. Howe quotients $R(0)$ and their K-types

In this section we specify the dual pair (G, G') to be

$$(O(p,q), Sp(2n,\mathbb{R})) \subset Sp(2(p+q)n,\mathbb{R}).$$

Let $K \cong O(p) \times O(q)$ and $K' \cong U(n)$ be the fixed maximal compact subgroups of G and G', respectively. We may choose a maximal compact subgroup U of $Sp(2(p+q)n,\mathbb{R})$ in such a way that

$$K = U \cap G \ \text{ and } \ K' = U \cap G'.$$

We shall apply the Howe Duality Theorem in the context of $(\mathfrak{g}, \tilde{K})$-modules. Let ω be the oscillator representation of $\tilde{S}p(2(p+q)n,\mathbb{R})$ which we fix by specifying a non-trivial unitary additive character of $\mathbb{R}$. Let S be the space of $\tilde{U}$-finite vectors of ω, which consists of polynomials in a Fock model of ω. Then S is naturally a $(\mathfrak{g}', \tilde{K}')$-module. Let $R(0)$ be the maximal quotient of S on which $(\mathfrak{g}', \tilde{K}')$ acts trivially. The Howe Duality Theorem [H2] tells us that $R(0)$ is a quasi-simple $(\mathfrak{g}, \tilde{K})$-module of finite length and has a unique irreducible quotient, denoted by

$Q(0)$. Thus $Q(0)$ is the representation of $(\mathfrak{g}, \tilde{K})$ which corresponds to the trivial $(\mathfrak{g}', \tilde{K'})$ module under the Howe quotient correspondence.

We now assume that the dual pair in question is stable with G' the small member, namely $2n \leq \min(p,q)$. Furthermore, we let $p+q$ be even. Then the coverings $\tilde{G} \to G$, $\tilde{G'} \to G'$ are both trivial. By Li's result [L1] we see the Howe correspondence will then give rise to an injection $\hat{G'} \hookrightarrow \hat{G}$. On the other hand, we know by Proposition 2.1 that $R(0)$ is already irreducible and so it coincides with $Q(0)$, which is unitary. We shall record this as

Theorem 3.1. *If $2n$ is less than or equal to $\min(p,q)$ (except for $p = q = 2n$), then $R(0)$ is the $(\mathfrak{g}, K)$-module of an irreducible unitary representation of $O(p,q)$.*

Remark. In [KR1], Kudla and Rallis consider the dual pair $(O(p,q), Sp(2n,\mathbb{R})) \subseteq Sp(2(p+q)n,\mathbb{R})$ and study the Howe quotient which corresponds to the trivial representation of $O(p,q)$, among other things. Subsequently in [LZ1] and [LZ2], Lee and Zhu examine in more detail the dual pairs $(U(p,q), U(n,n)) \subseteq Sp(4(p+q)n,\mathbb{R})$ (resp. $(O(p,q), Sp(2n,\mathbb{R})) \subseteq Sp(2(p+q)n,\mathbb{R})$), and investigate the Howe quotients which correspond to various one dimensional representations of $U(p,q)$ (resp. $O(p,q)$). In particular, they were able to determine completely the structure of these Howe quotients and their relationship with some degenerate series representations of $U(n,n)$ and $Sp(2n,\mathbb{R})$. One of the objectives of the present paper is to reverse the role of the two members in the dual pair $(O(p,q), Sp(2n,\mathbb{R}))$ and study the Howe quotient which corresponds to the trivial representation of $Sp(2n,\mathbb{R})$.

We proceed to describe the K-types of $R(0)$. We first introduce some notations. As usual, we shall parameterize representations of $U(n)$ by their highest weights with respect to the upper triangular Borel subalgebra, or their corresponding Young diagrams. Following Weyl ([W], [A]) we parameterize representations of $O(p)$ by restriction from $U(p)$. Thus given a Young diagram Y with rows of length $a_1 \geq a_2 \geq ... \geq a_p \geq 0$ such that the sum of the lengths of the first two columns is less than or equal to p, the irreducible $O(p)$ representation associated to Y is defined to be the irreducible summand of the representation of $U(p)$ given by Y which contains the highest weight vector. More precisely, if the rows of Y are of the form

$$(a_1, ..., a_k, 0, ..., 0), \qquad a_k > 0, k \leq [\frac{p}{2}],$$

or

$$(a_1, ..., a_k, \underbrace{1, ..., 1}_{l}, 0, ..., 0), \qquad a_k \geq 1, 2k + l = p,$$

the $O(p)$ representation associated to Y will be denoted by $(a_1, ..., a_k, 0, ..., 0; 1)$ or respectively $(a_1, ..., a_k, 0, ..., 0; -1)$. In order to avoid repetition, we may require $k < \frac{p}{2}$ in $(a_1, ..., a_k, 0, ..., 0; -1)$ if p is even, since in that case we have $(a_1, ..., a_{\frac{p}{2}}; -1) \simeq (a_1, ..., a_{\frac{p}{2}}; 1)$ if $a_{\frac{p}{2}} \neq 0$.

Proposition 3.2. *Let $(G, G') = (O(p,q), Sp(2n,\mathbb{R}))$ be the reductive dual pair in $Sp(2(p+q)n,\mathbb{R})$, ω be the oscillator representation of $\tilde{S}p(2(p+q)n,\mathbb{R})$. Then the K-types in $R(0)$ are of the form*

$$(a_1, a_2, ..., a_k, 0, ..., 0; \epsilon_1) \otimes (b_1, b_2, ..., b_l, 0, ..., 0; \epsilon_2),$$

satisfying the following equality of n-tuples:

$$\frac{p}{2}\mathbf{1}_n + (a_1, ..., a_k, \underbrace{1, ..., 1}_{\frac{1-\epsilon_1}{2}(p-2k)}, 0, ..., 0) = \frac{q}{2}\mathbf{1}_n + (b_1, ..., b_l, \underbrace{1, ..., 1}_{\frac{1-\epsilon_2}{2}(q-2l)}, 0, ..., 0),$$

where $\epsilon_i = \pm 1$, $\mathbf{1}_n = (1, ..., 1)$, *and* $k \leq [\frac{p}{2}]$, $k + \frac{1-\epsilon_1}{2}(p-2k) \leq n$, *and* $l \leq [\frac{q}{2}]$, $l + \frac{1-\epsilon_2}{2}(q-2l) \leq n$. *Moreover, each such K-type occurs with multiplicity one.*

In particular, we have

(a) *if* $p = q$, *then K-types in* $R(0)$ *are of the form*

$$(a_1, a_2, ..., a_k, 0, ..., 0; \epsilon) \otimes (a_1, a_2, ..., a_k, 0, ..., 0; \epsilon),$$

where $\epsilon = \pm 1$, $k \leq [\frac{p}{2}]$, *and* $k + \frac{1-\epsilon}{2}(p-2k) \leq n$, *and*

(b) *if* $2n \leq \min(p, q)$, *then K-types in* $R(0)$ *are of the form*

$$(a_1, a_2, ..., a_n, 0, ..., 0; 1) \otimes (b_1, b_2, ..., b_n, 0, ..., 0; 1),$$

where $a_1 \geq ... \geq a_n \geq 0$, $b_1 \geq ... \geq b_n \geq 0$, *and* $a_i + \frac{p}{2} = b_i + \frac{q}{2}$ *for* $1 \leq i \leq n$.

Proof. Suppose $\sigma = \sigma_1 \otimes \sigma_2$ is a K-type. Let S_σ be the σ-isotypic submodule of S so that we have

$$S = \sum_{\sigma \in \Sigma \subset \hat{K}} S_\sigma,$$

where Σ is a known subset of $\hat{K}$.

Let $\mathcal{J}$ be the span of $\{\omega(X)v, (\omega(k') - 1)v | v \in S, X \in \mathfrak{g}', k' \in \tilde{K}'\}$ so that

$$R(0) = S/\mathcal{J}.$$

Since $\mathfrak{g}'$, $\tilde{K}'$ preserve S_σ, we have

$$S/\mathcal{J} \simeq \sum_{\sigma \in \Sigma \subset \hat{K}} S_\sigma/(S_\sigma \cap \mathcal{J}).$$

Recall the seesaw dual pair [Ku]

$$\begin{array}{ccc} K = O(p) \times O(q) & & Sp(2n, \mathbb{R}) \\ | \bigcap & & | \bigcap \\ G = O(p,q) & & Sp(2n, \mathbb{R}) \times Sp(2n, \mathbb{R}), \end{array}$$

where $Sp(2n, \mathbb{R}) \hookrightarrow Sp(2n, \mathbb{R}) \times Sp(2n, \mathbb{R})$ is the diagonal embedding.

By the standard result of Howe ([H2]), we have

$$S_\sigma \simeq \sigma \otimes \pi(\sigma)$$

for some irreducible $(\mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{sp}(2n, \mathbb{R}), \tilde{L})$ module $\pi(\sigma)$, where $L = U(n) \times U(n)$.

Since $\mathfrak{g}'$, $\tilde{K}'$ act only on the second factor of the decomposition $S_\sigma \simeq \sigma \otimes \pi(\sigma)$, we see that $S_\sigma/(S_\sigma \cap \mathcal{J}) \simeq \sigma \otimes \pi(\sigma)^0$, where $\pi(\sigma)^0$ is the maximal quotient of $\pi(\sigma)$ on which $(\mathfrak{sp}(2n, \mathbb{R}), \tilde{U}(n))$ acts trivially.

Thus we conclude that a K-type σ occurs in $R(0)$ if and only if $\pi(\sigma)^0 \neq 0$, and the dimension of $\pi(\sigma)^0$ gives the multiplicity of σ in $R(0)$.

We know ([KV], [A]) that if

$$\sigma_1 = (a_1, a_2, ..., a_k, 0, ..., 0; \epsilon_1),$$
$$\sigma_2 = (b_1, b_2, ..., b_l, 0, ..., 0; \epsilon_2),$$

then $\pi(\sigma) = \pi_1 \otimes \pi_2$, where π_1 (resp. π_2) is the lowest (resp. highest) weight $(\mathfrak{sp}(2n,\mathbb{R}), \tilde{U}(n))$-module with the lowest (resp. highest) $\tilde{U}(n)$-type τ_1 (resp. τ_2), and the highest weights of τ_1 and τ_2 are respectively

$$\frac{p}{2}\mathbf{1}_n + (a_1, ..., a_k, \underbrace{1, ..., 1}_{\frac{1-\epsilon_1}{2}(p-2k)}, 0, ..., 0),$$

$$-\frac{q}{2}\mathbf{1}_n + (0, ..., 0, \underbrace{-1, ..., -1}_{\frac{1-\epsilon_2}{2}(q-2k)}, -b_l, ..., -b_1).$$

Here $k \leq [\frac{p}{2}]$, $k + \frac{1-\epsilon_1}{2}(p-2k) \leq n$, and $l \leq [\frac{q}{2}]$, $l + \frac{1-\epsilon_2}{2}(q-2l) \leq n$.

For $\pi(\sigma)$ to have a non-zero maximal quotient such that $(\mathfrak{sp}(2n,\mathbb{R}), \tilde{U}(n))$ acts trivially, it is equivalent to the fact that there exists a non-zero $(\mathfrak{sp}(2n,\mathbb{R}), \tilde{U}(n))$-equivariant map $\pi_1 \otimes \pi_2 \mapsto \mathbb{C}$. Clearly it is equivalent to the condition $\pi_2 \simeq \pi_1^*$, and in this case $\pi(\sigma)^0$ is one dimensional. This happens if and only if $\tau_2 \simeq \tau_1^*$ as representations of $\tilde{U}(n)$, namely if and only if

$$\frac{p}{2}\mathbf{1}_n + (a_1, ..., a_k, \underbrace{1, ..., 1}_{\frac{1-\epsilon_1}{2}(p-2k)}, 0, ..., 0) = \frac{q}{2}\mathbf{1}_n + (b_1, ..., b_l, \underbrace{1, ..., 1}_{\frac{1-\epsilon_2}{2}(q-2l)}, 0, ..., 0).$$

Assume $p = q$. Then if $\epsilon_1 = \epsilon_2 = \epsilon$, the above condition clearly implies $k = l$ and $a_i = b_i$ for all i. If $\epsilon_1 = 1$ and $\epsilon_2 = -1$, then we get $k = q - l$. Since $k \leq [\frac{p}{2}] = [\frac{q}{2}]$, $l \leq [\frac{q}{2}]$, we must have $q = 2d$ and $k = l = d$. This contradicts our convention to avoid repetition of labeling representations of $O(q)$, since $\epsilon_2 = -1$, and $q = 2d$, but $l = d$. Similarly it is also impossible to have $\epsilon_1 = -1$ and $\epsilon_2 = 1$.

Assume $2n \leq p$. If $\epsilon_1 = -1$, then $k \leq [\frac{p}{2}]$, and $k + (p - 2k) \leq n$. Clearly $k \leq n \leq \frac{p}{2}$ and $p - k \leq n \leq \frac{p}{2}$, thus $k \geq \frac{p}{2}$. So we must have $p = 2d$ and $k = d$. This again contradicts our convention. Therefore we have $\epsilon_1 = 1$. Similarly $\epsilon_2 = 1$. The rest of the assertion is then clear. □

4. Identification of π_n with the Howe quotient

Let $X = \begin{pmatrix} Y \\ Z \end{pmatrix} \in \mathcal{X}^{00}$, where $Y \in M_{p,n}(\mathbb{R})$, $Z \in M_{q,n}(\mathbb{R})$. Then first we have

$$\begin{pmatrix} Y \\ Z \end{pmatrix}^t I_{p,q} \begin{pmatrix} Y \\ Z \end{pmatrix} = 0_n,$$

namely $Y^tY = Z^tZ$. Since $X \in \mathcal{X}^{00}$ has rank n, the rank of $X^tX = Y^tY + Z^tZ = 2Y^tY$ is also n. We conclude that both Y and Z have rank n.

Let $S^{p,n}$ be the following Stiefel manifold

$$S^{p,n} = \{V \in M_{p,n}(\mathbb{R}) | V^tV = I_n\}.$$

$O(p)$ acts transitively on $S^{p,n}$ by the left matrix multiplication and we have $S^{p,n} \simeq O(p)/O(p-n)$ as an $O(p)$ homogeneous space. Similarly we define the Stiefel manifold $S^{q,n}$.

Let P_n be the set of $n \times n$ positive definite symmetric matrices. We can write

$$Y = VR^{\frac{1}{2}}, \quad \text{where} \quad V \in S^{p,n},\ R \in P_n,$$

$$Z = WS^{\frac{1}{2}}, \quad \text{where} \quad W \in S^{q,n},\ S \in P_n.$$

Then it is clear that $X = \begin{pmatrix} Y \\ Z \end{pmatrix} \in \mathcal{X}^{00}$ if and only if $R = S$. So we have the following "polar coordinates" decomposition of $\mathcal{X}^{00}$

$$X = \begin{pmatrix} V \\ W \end{pmatrix} R^{\frac{1}{2}}, \quad V \in S^{p,n},\, W \in S^{q,n},\, R \in P_n.$$

We fix $R \in P_n$. Given any neighborhood U of $\mathcal{X}^{00}$ and $f \in C^\infty(U)$, define

$$J_R(f) = \int_{S^{p,n}\times S^{q,n}} f(\begin{pmatrix} V \\ W \end{pmatrix} R^{\frac{1}{2}}) dV dW, \tag{4.1}$$

where dV, dW are the rotation-invariant probability measures on $S^{p,n}$ and $S^{q,n}$ respectively. It is easy to see that

(a) $J_R(f)$ depends only on $f|_{\mathcal{X}^{00}}$, and

(b) $J_R(f_1(\Delta_{kl} f_2)) = J_R((\Delta_{kl} f_1) f_2)$, for $f_1, f_2 \in C^\infty(U)$.

Recall that we have the oscillator representation ω of $\tilde{S}p(2(p+q)n, \mathbb{R})$ associated to the dual pair $(G, G') = (O(p,q), Sp(2n,\mathbb{R})) \subset Sp(2(p+q)n, \mathbb{R})$. The oscillator representation ω is realised on $L^2(V^n)$ via the Schrodinger model. This induces an action of $\tilde{S}p(2(p+q)n, \mathbb{R})$ on $\mathcal{S} = \mathcal{S}(V^n)$, the space of Schwartz functions on V^n. It is well-known that $\mathcal{S}$ is the space of smooth vectors of ω, and it carries the standard Frechet topology. Define

$$\mathcal{S}_0 = \operatorname{span}\{(\omega(\tilde{K'}) - 1)\mathcal{S}, \omega(\mathfrak{g}')\mathcal{S}\}$$

where $K' \cong U(n)$ is a maximal compact subgroup of $G' = Sp(2n, \mathbb{R})$, and $\mathfrak{g}'$ is the Lie algebra of $G' = Sp(2n, \mathbb{R})$ as before. Let $\bar{\mathcal{S}_0}$ be its closure in the Frechet topology. Clearly $\bar{\mathcal{S}_0}$ is also the closed span of $\{(\omega(\tilde{G'}) - 1)\mathcal{S}\}$.

Let $\Omega(0) = \mathcal{S}/\bar{\mathcal{S}_0}$. Note that the underlying Harish-Chandra module of $\Omega(0)$ is the Howe quotient $R(0)$. It is irreducible and unitary (except for $p = q = 2n$) (cf. §3).

Denote by $\mathcal{X}_{sing} = \mathcal{X} - \mathcal{X}^{00}$, the complement of $\mathcal{X}^{00}$ in $\mathcal{X}$. It is the singular part of the isotropic cone. Note that $\mathcal{X}^{00}$ is contained in $V^n - \mathcal{X}_{sing}$ as a closed subset.

Let $C^\infty(V^n - \mathcal{X}_{sing})$ be the space of smooth functions on $V^n - \mathcal{X}_{sing}$. Define a subspace N_0 of $C^\infty(V^n - \mathcal{X}_{sing})$ as follows: we say $f \in N_0$ if there exist two finite sets of functions $\{h_i\}$, $\{F_j\}$ in $C^\infty(V^n - \mathcal{X}_{sing})$ such that $f = \sum_i (\omega(k_i) - 1)h_i + \sum_j \omega(g_j) F_j$ in a neighborhood of $\mathcal{X}^{00}$, where $k_i \in \tilde{K'}$, and $g_j \in \mathfrak{g}'$.

We endow $C^\infty(V^n - \mathcal{X}_{sing})$ with the topology induced by the semi-norms $||f||_{pd,R} = J_R(|pd(f)|^2)$, where pd runs over the set of polynomial coefficient differential operators on V^n, R runs over the set P_n of $n \times n$ positive definite symmetric matrices, and J_R is as in Equation (4.1). Denote by $\bar{N}_0$ the closure of N_0 in this topology. Let $C^\infty(V^n - \mathcal{X}_{sing})/\bar{N}_0$ be the coset space of $C^\infty(V^n - \mathcal{X}_{sing})$ with respect to $\bar{N}_0$.

Recall that $\epsilon \equiv \frac{p-q}{2}$ (mod 2). We define an $O(p,q)$-equivariant map

$$T: \quad C^\infty(\mathcal{X}^{00}, n+1-\frac{p+q}{2})_\epsilon \longrightarrow C^\infty(V^n - \mathcal{X}_{sing})/\bar{N}_0 \tag{4.2}$$

as follows. For $f \in C^\infty(\mathcal{X}^{00}, n+1-\frac{p+q}{2})_\epsilon$, extend f to $\tilde{f} \in C^\infty(V^n - \mathcal{X}_{sing})$ such that $\tilde{f}|_U \in C^\infty(U, n+1-\frac{p+q}{2})_\epsilon$, where U is an open conical neighborhood of $\mathcal{X}^{00}$. Define Tf to be the coset of $\tilde{f}$ in $C^\infty(V^n - \mathcal{X}_{sing})/\bar{N}_0$. The map T is well-defined since any two extensions $\tilde{f}_1$ and $\tilde{f}_2$ of f differ in a neighborhood U' of $\mathcal{X}^{00}$ by an element of the form $\sum_{ij} r_{ij} f_{ij}$ where $f_{ij} \in C^\infty(U')$, and so they differ by an element of $N_0 \subset \bar{N}_0$ and are thus in the same equivalence class. Since $O(p,q)$ commutes with the actions of $H = GL_n(\mathbb{R})$, $\omega(\tilde{K}')$ and $\omega(\mathfrak{g}')$, we see that T is $O(p,q)$-equivariant.

Recall that $\mathcal{H}_n$ is the subspace of $C^\infty(\mathcal{X}^{00}, n+1-\frac{p+q}{2})_\epsilon$, on which $O(p,q)$ acts by the representation π_n.

Proposition 4.1.

$$T|_{\mathcal{H}_n}\colon\ \mathcal{H}_n \longrightarrow C^\infty(V^n - \mathcal{X}_{sing})/\bar{N}_0$$

is an $O(p,q)$-equivariant embedding.

Proof. Let $f \in \mathcal{H}_n$, namely $f \in C^\infty(\mathcal{X}^{00}, n+1-\frac{p+q}{2})_\epsilon$, and $\Delta_{kl}\tilde{f}|_{\mathcal{X}^{00}} = 0$ for $1 \le k \le l \le n$.

Suppose $Tf = 0$, i.e. $\tilde{f} \in \bar{N}_0$. Then there exists a net $\{\alpha\}$ and a set of functions $f_\alpha \in N_0$ such that $f_\alpha \to \tilde{f}$ in the topology of $C^\infty(V^n - \mathcal{X}_{sing})$. Since $\tilde{f} \in C^\infty(\mathcal{X}^{00}, n+1-\frac{p+q}{2})_\epsilon$, we have $\tilde{f}(Xb) = (\frac{\det b}{|\det b|})^{\frac{p-q}{2}}\tilde{f}(X)$, for $X \in U$, a conical neighborhood of $\mathcal{X}^{00}$. Here b is any element of $P' \cap K' \cong O(n)$, where P' is the maximal parabolic subgroup of $Sp(2n,\mathbb{R})$ preserving V^n. This implies in particular that $\tilde{f}|_U$ is invariant under the (two-element) kernel of the covering map $\tilde{K}' \to K'$.

Since $f_\alpha \in N_0$, there is a neighborhood U_α of $\mathcal{X}^{00}$ such that

$$f_\alpha|_{U_\alpha} = \sum_i (\omega(k_i^\alpha) - 1)h_i^\alpha + \sum_j \omega(g_j^\alpha)F_j^\alpha,$$

where $h_i^\alpha, F_j^\alpha \in C^\infty(U_\alpha)$, and $k_i^\alpha \in \tilde{K}'$, $g_j^\alpha \in \mathfrak{g}'$. By intersecting if necessary, we may assume $U_\alpha \subset U$.

Since $\tilde{f}|_{U_\alpha}$ is invariant under the (two-element) kernel of the covering map $\tilde{K}' \to K'$, we may assume (by averaging if necessary) that $f_\alpha|_{U_\alpha}$ has the same property. Thus we can write

$$f_\alpha|_{U_\alpha} = \sum_i (\omega(k_i^\alpha) - 1)h_i^\alpha + \sum_j \omega(g_j^\alpha)F_j^\alpha,$$

where $k_i^\alpha \in K'$, instead of $\tilde{K}'$. Since $K' \cong U(n)$ is connected, we may further assume that $h_i^\alpha = 0$, namely $f_\alpha|_{U_\alpha} = \sum_j \omega(g_j^\alpha)F_j^\alpha$, $g_i^\alpha \in \mathfrak{g}'$, $F_j^\alpha \in C^\infty(U_\alpha)$.

We have

$$\omega(\mathfrak{g}') = \operatorname{span}\{r, \Delta, H\},$$

where $r = \operatorname{span}\{r_{ij}\}_{1\le i,j\le n}$, $\Delta = \operatorname{span}\{\Delta_{ij}\}_{1\le i,j\le n}$, $H = \operatorname{span}\{H_{ij}\}_{1\le i,j\le n}$. See Equations (1.1), (1.2), (1.3), (1.4) in §1 for the definitions of r_{ij}, Δ_{ij} and H_{ij}. Since H is the commutator algebra of r and Δ, we may write

$$f_\alpha|_{U_\alpha} = \sum_{i,j} \Delta_{ij}(h_{ij}^\alpha) + \sum_{i,j} r_{ij} f_{ij}^\alpha,$$

where $f_{ij}^\alpha, h_{ij}^\alpha \in C^\infty(U_\alpha)$.

For any $R \in P_n$, we have $J_R(\tilde{f}\Delta_{ij}(h_{ij}^\alpha)) = J_R((\Delta_{ij}\tilde{f})h_{ij}^\alpha) = 0$, since $\Delta_{ij}\tilde{f}|_{\mathcal{X}^{00}} = 0$. This implies that $J_R(\tilde{f}f_\alpha) = 0$. Since $f_\alpha \to \tilde{f}$ in the topology of $C^\infty(V^n - \mathcal{X}_{sing})$, we have $J_R(\tilde{f}(\tilde{f} - f_\alpha)) \to 0$, and so $J_R(\tilde{f}^2) = J_R(\tilde{f}f_\alpha) + J_R(\tilde{f}(\tilde{f} - f_\alpha)) \to 0$. Thus we have $J_R(\tilde{f}^2) = 0$. This implies that $\tilde{f}|_{S^{p,n}\times S^{q,n}\times\{R\}} = 0$. Since $R \in P_n$ is arbitrary, we see that $f = \tilde{f}|_{\mathcal{X}^{00}} = 0$. ◻

We now prove part (a) of Theorem 1.2. We have the restriction map r: $\mathcal{S} \mapsto C^\infty(V^n - \mathcal{X}_{sing})$, and the image is clearly dense in the given topology of $C^\infty(V^n - \mathcal{X}_{sing})$. Moreover, since the topology in the image induced by the Frechet topology of $\mathcal{S}$ is stronger than the subspace topology of $C^\infty(V^n - \mathcal{X}_{sing})$, and since $r(\mathcal{S}_0) \subset N_0$, we obtain an $O(p,q)$-equivariant map

$$\Omega(0) = \mathcal{S}/\bar{\mathcal{S}}_0 \mapsto C^\infty(V^n - \mathcal{X}_{sing})/\bar{N}_0 \tag{4.3}$$

which has a dense image. On the left-hand side of (4.3), the K-isotypic subspaces are finite dimensional (§2), and since the mapping (4.3) has dense image, the K-isotypic subspaces on the right-hand side must also be finite dimensional, and the mapping (4.3) must carry the space of K-finite vectors of $\Omega(0)$ onto the space of K-finite vectors of $C^\infty(V^n - \mathcal{X}_{sing})/\bar{N}_0$. Since the former is irreducible, the underlying Harish-Chandra module of $C^\infty(V^n - \mathcal{X}_{sing})/\bar{N}_0$ is also irreducible and is isomorphic to $R(0)$. Since $\mathcal{H}_n \neq 0$ (for example $|\Delta_n|^{n+1-\frac{p+q}{2}}(\frac{\Delta_n}{|\Delta_n|})^{\frac{p-q}{2}} \in \mathcal{H}_n$, see [KV] for the definition of Δ_n here), we conclude from Proposition 4.1 that the Harish-Chandra module of π_n must be isomorphic to $R(0)$.

5. Gelfand-Kirillov dimension of π_n

We first recall the definition of Gelfand-Kirillov dimension ([V]) for a finitely generated $\mathcal{U}(\mathfrak{g})$-module V, where $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$ and $\mathcal{U}(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$. Choose a finite dimensional subspace V_0 so that $V = \mathcal{U}(\mathfrak{g})V_0$. For each positive integer k, let $\mathcal{U}_k(\mathfrak{g})$ be the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by products of at most k elements in $\mathfrak{g}$. We set $d_{V,V_0}(k) = \dim(\mathcal{U}_k(\mathfrak{g})V_0)$. Then there is a polynomial ϕ of degree at most $\dim\mathfrak{g}$ such that $d_{V,V_0}(k) = \phi(k)$ for large k. Moreover, the leading term which we write as $\frac{c(V)}{(\mathrm{Dim}V)!}k^{\mathrm{Dim}V}$ is independent of the choice of V_0. We say that Dim$V$ is the Gelfand-Kirillov dimension of V.

Recall that $G = O(p,q)$ and $K = O(p) \times O(q)$, and let $\mathfrak{g} = O(p+q, \mathbb{C})$ be the complexified Lie algebra of G. Let $V = \pi_n$ with $2n \le \min(p,q)$. Since $V \cong \Omega(0)$, V admits the following K-isotypic decomposition

$$V|_K \simeq \sum_{\lambda \in R} V_\lambda,$$

where R is the set $\{\lambda = (\lambda_1, ..., \lambda_n) | \lambda_1 \ge ... \ge \lambda_n \ge 0\}$, and for $\lambda = (\lambda_1, ..., \lambda_n) \in R$, V_λ denotes the representation of $O(p) \times O(q)$ given by $(\lambda_1, ..., \lambda_n, 0, ..., 0; 1) \otimes (\lambda_1 + \frac{p-q}{2}, ..., \lambda_n + \frac{p-q}{2}, 0, ..., 0; 1)$ (cf. §3). Now for each $\lambda = (\lambda_1, ..., \lambda_n) \in R$, we let $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Let $V_0 = V_{(0,\dots,0)}$ and $V_k = \sum_{\lambda \in R, |\lambda| = k} V_\lambda$, for a positive integer k. Now $\mathfrak{g}$ admits a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the complexified Lie algebra of K, and $\mathfrak{p} = \{\begin{pmatrix} 0 & a \\ a^t & 0 \end{pmatrix} : a \in M_{p,q}(\mathbb{C})\}$. We have

$$\mathfrak{p}(V_0) = V_1, \quad \text{and} \quad \mathfrak{p}(V_j) = V_{j-1} \oplus V_{j+1}, \quad (j > 1).$$

It follows from this that $\mathcal{U}_k(\mathfrak{g})V_0 = \sum_{j \le k} V_j$. Therefore if $\dim V_k$ is a polynomial in k of degree $s-1$, then $\dim(\mathcal{U}_k(\mathfrak{g})V_0)$ is a polynomial in k of degree s and so the Gelfand-Kirillov dimension of V is equal to s.

Lemma 5.1 (Lemma 6.4, [LZ1]). *Let $\phi(\alpha_1, ..., \alpha_n)$ be a polynomial of total degree d. Write $\phi = \phi_h + \phi'$, where ϕ_h is homogeneous of total degree d and ϕ' is of* *total degree less than or equal to $d-1$. Suppose that the coefficients of ϕ_h are non-negative. Then for any given positive integers $c_1, ..., c_n$,*

$$\psi(k) = \sum_{\begin{cases}\alpha_1, ..., \alpha_p \geq 0 \\ \sum_{i=1}^n c_i\alpha_i = k\end{cases}} \phi(\alpha_1, ..., \alpha_n)$$

is a polynomial in k of degree $d+n-1$.

We are now ready to prove part (b) of Theorem 1.2.

Proposition 5.2. *The Gelfand-Kirillov dimension of π_n is equal to $n(p+q-2n-1)$.*

Proof. We make the following change of variables

$$\mu_i = \lambda_i - \lambda_{i+1}, \quad 1 \leq i \leq n-1, \quad \mu_n = \lambda_n,$$

and we shall write $V(\mu)$ for V_λ. Observe that $\lambda \in R$ and $|\lambda| = k$ if and only if $\mu_1, ..., \mu_n \geq 0$, and $\sum_{i=1}^n i\mu_i = k$. Thus

$$\sum_{\lambda \in R, |\lambda| = k} \dim V_\lambda = \sum_{\begin{cases}\mu_1, ..., \mu_n \geq 0 \\ \sum_{i=1}^n i\mu_i = k\end{cases}} \dim V(\mu).$$

We shall only examine the case where $p = 2p'$, $q = 2q'$ are both even. The other case (p, q both odd) is similar. By the Weyl dimension formula, we have

$$\dim V_\lambda = 2^{p'+q'-2} \prod_{1 \leq i < j \leq p'} \frac{(l_i - l_j)(l_i + l_j)}{(2p'-2)!(2p'-4)! \cdots 2!} \times \prod_{1 \leq i < j \leq q'} \frac{(m_i - m_j)(m_i + m_j)}{(2q'-2)!(2q'-4)! \cdots 2!}$$

where $l_i = m_i = \lambda_i + p' - i$ for $1 \leq i \leq n$, $l_i = p' - i$ for $n < i \leq p'$ and $m_i = q' - i$ for $n < i \leq q'$. In the new variables, we see that $\dim V(\mu)$ is a polynomial in $\mu_1, ..., \mu_n$ of total degree

$$2[(p'-1) + ... + (p'-n)] + 2[(q'-1) + ... + (q'-n)] = n(p+q-2n-2).$$

Moreover for $i < j$, we have $\lambda_i - \lambda_j = \sum_{l=i}^{j-1} \mu_l$, $\lambda_i + \lambda_j = \sum_{l=i}^{j-1} \mu_i + 2\sum_{l=j}^n \mu_l$, and so the positivity condition on the coefficients of the homogeneous part of $\dim V(\mu)$ specified in Lemma 5.1 is satisfied. Therefore

$$\sum_{\begin{cases}\mu_1, ..., \mu_n \geq 0 \\ \sum_{i=1}^n i\mu_i = k\end{cases}} \dim V(\mu)$$

is a polynomial in k of degree

$$n(p+q-2n-2) + n - 1 = n(p+q-2n-1) - 1.$$

Thus $\dim V_k = \sum_{\lambda \in R, |\lambda| = k} \dim V_\lambda$ is a polynomial in k of the same degree. It follows from this and the discussion before Lemma 5.1 that the Gelfand-Kirillov dimension of π_n is equal to $n(p+q-2n-1)$. □

6. A Final Remark

It is well believed, as stated by Kirillov and Kostant, that coadjoint orbits are closely related to unitary representations. In this section we remark on the relation between the small representations we constructed and some nilpotent orbits.

Let G be a complex simple Lie group and B a Borel subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G. We may identify coadjoint orbits in $\mathfrak{g}^*$ with adjoint orbits in $\mathfrak{g}$ by using the Killing form. McGovern studied a certain family of nilpotent adjoint orbits and the ring of regular functions on them [M]. These orbits can be characterized as nilpotent orbits which contain open B-orbits. We call these orbits spherical nilpotent orbits. The ring of regular functions $R(\mathcal{O})$ on a spherical nilpotent orbit is multiplicity-free under the adjoint action of G. All spherical orbits are contained in the closure of the single largest spherical orbit, which is called the model orbit by McGovern.

Now we specify G to be $O(2m, \mathbb{C})$, the complex orthogonal group of type D_m. By a theorem of Gerstenhaber, the nilpotent orbits of G in $\mathfrak{g} = \mathfrak{so}(2m, \mathbb{C})$ are in one to one correspondence with partitions of $2m$ such that the even parts have even multiplicity (cf. Theorem 5.1.6 in [CM]). Though it is not needed in this paper, we note that for $G = SO(2m, \mathbb{C})$, there are two orbits corresponding to a partition with only even parts, which are labeled by I and II. There are two families of spherical orbits (cf. Table 8 of [M]). One family is corresponding to the partitions $(3, 2^{2k}, 1^{2l+1})$ such that $k = 0, 1, \cdots, [\frac{m}{2}] - 1$ and $l = m - 2k - 2$. The other family is corresponding to the partitions $(2^{2k}, 1^{2l})$ such that $k = 0, 1, 2, \cdots, [\frac{m}{2}]$ and $l = m - 2k$.

The second family of orbits $\mathcal{O}_k$ with partition $(2^{2k}, 1^{2l})$ has dimension equal to $k(2m - 2k - 1)$. It is conceivable that the orbit $\mathcal{O}_k$ is related to the small representation π_k for $k = 0, 1, \cdots, [\frac{m}{2}]$. Denote by $G_0 = O(p, q)$ with $p + q = 2m$ a real form of G and $\mathfrak{g}_0$ the Lie algebra of G_0. The intersection of $\mathcal{O}_k$ with $\mathfrak{g}_0$ is a nilpotent G_0-orbit inside $\mathfrak{g}_0$ if and only if $2k \leq \min(p, q)$. This coincides with the condition $2n \leq \min(p, q)$ we needed for the lifting to get unitary representations π_n. When $m = 2n$ is even and $G = O(2n, 2n)$, the largest spherical orbit or the model orbit $\mathcal{O}_n$ corresponding to the partitions (2^{2n}) may or may not be related to the representation π_n. This is exactly the exceptional case when $p = q = 2n$, which has been excluded from the discussion in the previous sections. It remains an interesting problem to realize π_k as geometric quantization of $\mathcal{O}_k$. For $k = 1$ the quantization of the minimal orbit corresponding to the partition $(2^2, 1^{2m-4})$ has been done by Brylinski and Kostant. They also succeeded in quantizing the minimal nilpotent orbit for other simple groups [BK].

Acknowledgments

We would like to thank David Vogan, Eng Chye Tan and Toshiyuki Kobayashi for helpful discussions. Jing-Song Huang would like to acknowledge the generous supports of NSF grant no. DMS-9306138 and RGC–CERG grant no. HKUST 588/94P. We would also like to thank the referee for critical comments.

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