SPECIAL UNIPOTENT REPRESENTATIONS OF REAL CLASSICAL GROUPS: COUNTING

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Abstract. Let G be a real reductive group in the Harish-Chandra class. We derive some consequences of theory of coherent continuation representations, primitive ideals and cells to the counting of irreducible representations of G with a given infinitesimal character and a given bound in the complex associated variety. When G is a real classical group (including the real metaplectic group), we give a precise count for the number of special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. Here $\check{\mathcal{O}}$ is a nilpotent adjoint orbit in the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group).

Contents

1. Introduction and the main results

Let G be a real reductive group in the Harish-Chandra class (which may be linear or non-linear). Write $\mathfrak{g}$ for the complexified Lie algebra of G and let ${}^a\mathfrak{h}$ denote the universal Cartan subalgebra of $\mathfrak{g}$ (also called the abstract Cartan subalgbra in [Vog82]). Let $\lambda \in {}^a\mathfrak{h}^*$ (a superscript $*$ indicates the dual space). By Harish-Chandra isomorphism, it determines an algebraic character $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$. Here $\mathcal{Z}(\mathfrak{g})$ denotes the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible Casselman-Wallach representations of G (see [Wal92, Chapter 11]), and by $\mathrm{Irr}_\lambda(G)$ its subset consisting of the representations with infinitesimal character χ_λ (or simply λ). The latter set has finite cardinality.

Let $\mathrm{Nil}(\mathfrak{g}^*)$ denote the set of nilpotent elements in $\mathfrak{g}^*$. It has only finitely many orbits under the coadjoint action of the inner automorphism group $\mathrm{Inn}(\mathfrak{g})$ of $\mathfrak{g}$. Let S be an $\mathrm{Inn}(\mathfrak{g})$-stable Zariski closed subset of $\mathrm{Nil}(\mathfrak{g}^*)$. Put

$$\mathrm{Irr}_{\lambda,\mathsf{S}}(G) := \{\, \pi \in \mathrm{Irr}_\lambda(G) \mid \mathrm{AV}_\mathbb{C}(\pi) \subseteq \mathsf{S} \,\}.$$

Here $\mathrm{AV}_\mathbb{C}(\pi)$ denotes the complex associated variety of π, namely the associated variety of the annihilate ideal of π. It is an $\mathrm{Inn}(\mathfrak{g})$-stable Zariski closed subset of $\mathrm{Nil}(\mathfrak{g}^*)$. An interesting problem of representation theory is to count the finite set $\mathrm{Irr}_{\lambda,\mathsf{S}}(G)$. The coherent continuation representation (of the integral Weyl group) provides a powerful

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tool for this problem. The first goal of the paper is to give a systematic treatment of the problem in this general set-up, by building on earlier ideas of several authors including Vogan [Vog81], Joseph [Jos80a, Jos80b], King [Kin81], Barbasch-Vogan [BV85], Casian [Cas86], as well as Soergel [Soe90].

1.1. **The coherent continuation representation.** Write $\mathrm{Rep}(G)$ for the category of Casselman-Wallach representations of G, and write $\mathcal{K}(G)$ for the Grothendieck group of this category. Throughout this article we take $\mathbb{C}$ as the coefficient ring to define Grothendieck groups. When no confusion is possible, for every object O in an abelian category, we still use the same symbol to indicate the Grothendieck group element represented by the object O.

Write $\mathrm{Rep}_{\lambda,\mathsf{S}}(G)$ for the full subcategory of $\mathrm{Rep}(G)$ whose objects are the representations that have generalized infinitesimal character λ and whose complex associated variety is contained in S. Write $\mathcal{K}_{\lambda,\mathsf{S}}(G)$ for the Grothendieck group of this category. Then

$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = \dim \mathcal{K}_{\lambda,\mathsf{S}}(G) \qquad (\sharp \text{ indicates the cardinality of a finite set}).$$

We also have that

$$\mathcal{K}_{\mathsf{S}}(G) = \bigoplus_{\mu \in W \backslash {}^a\mathfrak{h}^*} \mathcal{K}_{\mu,\mathsf{S}}(G) \qquad (W \text{ denotes the Weyl group}),$$

where $\mathcal{K}_{\mathsf{S}}(G)$ is the Grothendieck group of $\mathrm{Rep}_{\mathsf{S}}(G)$, and latter is the category of Casselman-Wallach representations of G whose complex associated variety is contained in S.

Let $\mathcal{R}(\mathfrak{g})$ be the Grothendieck group of the category of finite-dimensional algebraic representations of $\mathrm{Inn}(\mathfrak{g})$. It is a commutative $\mathbb{C}$-algebra under the tensor product of representations. Write

$$\Delta \subseteq Q \quad (\subseteq {}^a\mathfrak{h}^*)$$

for the root system and the root lattice of $\mathfrak{g}$, respectively. By pulling back through the adjoint representation $G \to \mathrm{Inn}(\mathfrak{g})$, every algebraic representation of $\mathrm{Inn}(\mathfrak{g})$ is viewed as a representation of G. Under the tensor product of representations, $\mathcal{K}_{\mathsf{S}}(G)$ is naturally a $\mathcal{R}(\mathfrak{g})$-module.

Put

$$[\lambda] := \lambda + Q \subseteq {}^a\mathfrak{h}^*,$$

and write $W_{[\lambda]}$ for its stabilizer in W. Then $W_{[\lambda]}$ equals the Weyl group of the root system ([Jan79, Section 1.3])

$$\Delta_{[\lambda]} := \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\} \qquad (\alpha^\vee \text{ denotes the coroot corresponding to } \alpha).$$

We will often refer to $W_{[\lambda]}$ as the integral Weyl group.

Definition 1.1. *Let $\mathcal{K}$ be a $\mathcal{R}(\mathfrak{g})$-module equipped with a family $\{\mathcal{K}_\mu\}_{\mu\in[\lambda]}$ of subspaces such that $\mathcal{K}_{w\cdot\mu} = \mathcal{K}_\mu$ for all $w \in W_{[\lambda]}$ and $\mu \in [\lambda]$. A $\mathcal{K}$-valued coherent family on $[\lambda]$ is a map*

$$\Phi : [\lambda] \to \mathcal{K}$$

satisfying the following two conditions:

- *for all $\mu \in [\lambda]$, $\Phi(\mu) \in \mathcal{K}_\mu$;*
- *for all finite-dimensional algebraic representations F of* $\mathrm{Inn}(\mathfrak{g})$ *and all $\mu \in [\lambda]$,*

$$F \cdot (\Phi(\mu)) = \sum_{\nu} \Phi(\mu + \nu),$$

where ν runs over all weights of F, counted with multiplicities.

In the notation of Definition 1.1, let $\mathrm{Coh}_{[\lambda]}(\mathcal{K})$ denote the vector space of all $\mathcal{K}$-valued coherent families on $[\lambda]$. It is a representation of $W_{[\lambda]}$ under the action

$$(w \cdot \Phi)(\mu) = \Phi(w^{-1} \cdot \mu), \qquad \text{for all } \; w \in W_{[\lambda]},\; \mu \in [\lambda].$$

This is called a coherent continuation representation. When specifying a coherent continuation representation $\mathrm{Coh}_{[\lambda]}(\mathcal{K})$, we will often explicitly describe $\mathcal{K}$ as a Grothendieck group, while the $\mathcal{R}(\mathfrak{g})$-module structure and the $W_{[\lambda]}$-invariant family $\{\mathcal{K}_\mu\}_{\mu\in[\lambda]}$ are the ones which are clear from the context. For example, $\mathcal{K}_{\mathsf{S}}(G)$ is a $\mathcal{R}(\mathfrak{g})$-module as described previously, and it is equipped with the family $\{\mathcal{K}_{\mu,\mathsf{S}}(G)\}_{\mu\in[\lambda]}$ of subspaces. We thus have the coherent continuation representation $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))$ of $W_{[\lambda]}$.

1.2. **Counting irreducible representations with a bounded complex associated variety.** Denote by W_λ the stabilizer of λ in W. Then $W_\lambda \subseteq W_{[\lambda]}$. Write 1_{W_λ} for the trivial representation of W_λ.

Our starting point is the following theorem of Vogan. We will provide a proof due to lack of a convenient reference.

Theorem 1.2 (Vogan)**.** *The equality*

$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = [1_{W_\lambda} : \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))]$$

holds.

Here and henceforth, $[\ :\ ]$ indicates the multiplicity of the first (irreducible) representation in the second one. Theorem 1.2 implies that

$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = \sum_{\sigma\in\mathrm{Irr}(W_{[\lambda]})} [1_{W_\lambda} : \sigma] \cdot [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))].$$

Thus it suffices to understand the multiplicity $[\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(G)]$ for every $\sigma \in \mathrm{Irr}(W_{[\lambda]})$.

Let $\sigma \in \mathrm{Irr}(W_{[\lambda]})$. Define the nilpotent orbit

$$\mathcal{O}_\sigma := \mathrm{Springer}^{-1}(j_{W_{[\lambda]}}^{W}\sigma_0) \subseteq \mathrm{Nil}(\mathfrak{g}^*),$$

where σ_0 denotes the special irreducible representation of $W_{[\lambda]}$ that lies in the same double cell as σ, $j_{W_{[\lambda]}}^{W}\sigma_0 \in \mathrm{Irr}(W)$ denotes the j-induction of σ_0, and "Springer" indicates the Springer correspondence. See [Car93, Chapter 11] or Section 2.6 for the notion of j-induction, and Section 2.7 on special representations and double cells.

Let

$$\mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]}) := \left\{\, \sigma \in \mathrm{Irr}(W_{[\lambda]}) \;\middle|\; \mathcal{O}_\sigma \subseteq \mathsf{S} \,\right\}. \tag{1.1}$$

Theorem 1.3. *Suppose that $\sigma \in \mathrm{Irr}(W_{[\lambda]}) \setminus \mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})$. Then*

$$[\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))] = 0.$$

Consequently we have

$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = \sum_{\sigma\in\mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})} [1_{W_\lambda} : \sigma] \cdot [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))]. \tag{1.2}$$

Theorem 1.3 clearly implies that

$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) \leq \sum_{\sigma\in\mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})} [1_{W_\lambda} : \sigma] \cdot [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))]. \tag{1.3}$$

Recall the notion of a Harish-Chandra cell representation in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ (which is a subquotient of $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$). See [Vog82, Section 14] or Section 2.7.

Theorem 1.4. *Assume that for every Harish-Chandra cell representation V in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$, the set $\{\sigma \in \mathrm{Irr}(W_{[\lambda]}) \mid [\sigma : V] \neq 0\}$ is contained in a single double cell. Then*

$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = \sum_{\sigma \in \mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})} [1_{W_\lambda} : \sigma] \cdot [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))].$$

1.3. **Counting irreducible representations annihilated by a maximal primitive ideal.** Write I_λ for the maximal ideal of $\mathcal{U}(\mathfrak{g})$ with infinitesimal character λ. Its associated variety equals the Zariski closure $\overline{\mathcal{O}_\lambda}$ of an $\mathrm{Inn}(\mathfrak{g})$-orbit $\mathcal{O}_\lambda \subseteq \mathrm{Nil}(\mathfrak{g}^*)$. Note that an irreducible Casselman-Wallach representation of G lies in $\mathrm{Irr}_{\lambda,\overline{\mathcal{O}_\lambda}}(G)$ if and only if it is annihilated by I_λ.

Let

$$ {}^L\mathscr{C}_\lambda := \left\{ \sigma \in \mathrm{Irr}(W_{[\lambda]}) \;\middle|\; \sigma \text{ occurs in } (J_{W_\lambda}^{W_{[\lambda]}} \mathrm{sgn}) \otimes \mathrm{sgn} \right\}, \tag{1.4}$$

called the Lusztig left cell attached to λ. Here $J_{W_\lambda}^{W_{[\lambda]}}$ indicates the J-induction (see [Car93, Chapter 12]), and sgn denotes the sign character (of an appropriate Weyl group).

Proposition 1.5 ([BV85, (5.26), Proposition 5.28])**.** *The following equality of sets holds:*

$$ {}^L\mathscr{C}_\lambda = \left\{ \sigma \in \mathrm{Irr}_{\overline{\mathcal{O}_\lambda}}(W_{[\lambda]}) \;\middle|\; [1_{W_\lambda} : \sigma] \neq 0 \right\}.$$

Moreover, $[1_{W_\lambda} : \sigma] = 1$ when $\sigma \in {}^L\mathscr{C}_\lambda$.

In view of Proposition 1.5, Theorems 1.3 and 1.4 have the following consequence.

Corollary 1.6. *The equality*

$$\sharp(\mathrm{Irr}_{\lambda,\overline{\mathcal{O}_\lambda}}(G)) = \sum_{\sigma \in {}^L\mathscr{C}_\lambda} [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\overline{\mathcal{O}_\lambda}}(G))]$$

holds. Consequently,

$$\sharp(\mathrm{Irr}_{\lambda,\overline{\mathcal{O}_\lambda}}(G)) \leq \sum_{\sigma \in {}^L\mathscr{C}_\lambda} [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))]. \tag{1.5}$$

If for every Harish-Chandra cell representation V in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$, the set $\{\sigma \in \mathrm{Irr}(W_{[\lambda]} \mid [\sigma : V] \neq 0\}$ is contained in a single double cell, then the equality holds in (1.5)*.*

1.4. **Special unipotent representations of real classical groups.** We are particularly interested in counting special unipotent representations of real classical groups.

Let $\star$ be one of the 10 symbols

$$A^{\mathbb{R}},\ A^{\mathbb{H}},\ A,\ \widetilde{A},\ B,\ D,\ C,\ \widetilde{C},\ D^*,\ C^*.$$

Suppose that G is a classical Lie group of type $\star$, namely G respectively equals one of the following Lie groups:

$$\begin{gathered} \mathrm{GL}_n(\mathbb{R}),\ \mathrm{GL}_n(\mathbb{H}),\ \mathrm{U}(p,q),\ \widetilde{\mathrm{U}}(p,q), \\ \mathrm{SO}(p,q)\ (p+q \text{ is odd}),\ \mathrm{SO}(p,q)\ (p+q \text{ is even}), \\ \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{O}^*(2n),\ \mathrm{Sp}(p,q), \qquad (n,p,q \geq 0). \end{gathered}$$

Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the metaplectic double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n = 0$, and $\widetilde{\mathrm{U}}(p,q)$ is the double cover of $\mathrm{U}(p,q)$ defined by a square root of the determinant character.

Define the Langlands dual $\check{G}$ of G to be respectively the complex group

$$\begin{gathered} \mathrm{GL}_n(\mathbb{C}),\ \mathrm{GL}_{2n}(\mathbb{C}),\ \mathrm{GL}_{p+q}(\mathbb{C}),\ \mathrm{GL}_{p+q}(\mathbb{C}), \\ \mathrm{Sp}_{p+q-1}(\mathbb{C})\ (p+q \text{ is odd}),\ \mathrm{SO}_{p+q}(\mathbb{C})\ (p+q \text{ is even}), \\ \mathrm{SO}_{2n+1}(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{SO}_{2n}(\mathbb{C}),\ \text{or}\ \ \mathrm{SO}_{2p+2q+1}(\mathbb{C}). \end{gathered}$$

Write $\check{\mathfrak{g}}$ for the Lie algebra of $\check{G}$, and let $\check{\mathcal{O}} \subseteq \mathrm{Nil}(\check{\mathfrak{g}})$ be a nilpotent $\check{G}$-orbit.

Let $\lambda_{\check{\mathcal{O}}} \in \check{\mathfrak{g}}$ be half of the neutral element in any $\mathfrak{sl}_2$ triple attached to $\check{\mathcal{O}}$, as in [BV85, Section 5]. It is a semisimple element and is uniquely determined up to conjugation by $\check{G}$. Using the identification

$$\check{G}\backslash\{\text{semisimple element in } \check{\mathfrak{g}}\} = W\backslash{}^a\mathfrak{h}^*, \tag{1.6}$$

we view $\lambda_{\check{\mathcal{O}}}$ as an element of $W\backslash{}^a\mathfrak{h}^*$, and write $I_{\check{\mathcal{O}}} := I_{\star,\check{\mathcal{O}}}$ for the maximal ideal of $\mathcal{U}(\mathfrak{g})$ with infinitesimal character $\lambda_{\check{\mathcal{O}}}$. We remark that in the metaplectic case, namely $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$, both ${}^a\mathfrak{h}$ and ${}^a\mathfrak{h}^*$ are identified with $\mathbb{C}^n$ in the usual way and hence (1.6) still holds.

Following Barbasch-Vogan [BV85], define the set of the special unipotent representations of G attached to $\check{\mathcal{O}}$ by

$$\begin{aligned}\mathrm{Unip}_{\check{\mathcal{O}}}(G) :=& \mathrm{Unip}_{\star,\check{\mathcal{O}}}(G)\\ :=& \begin{cases}\{\pi \in \mathrm{Irr}(G) \mid \pi \text{ is genuine and annihilated by } I_{\check{\mathcal{O}}}\}, & \text{if } \star \in \{\widetilde{A}, \widetilde{C}\};\\ \{\pi \in \mathrm{Irr}(G) \mid \pi \text{ is annihilated by } I_{\check{\mathcal{O}}}\}, & \textit{otherwise}.\end{cases}\end{aligned}$$

Here "genuine" means that the central subgroup $\{\pm 1\}$ of G, which is the kernel of the covering homomorphism $\widetilde{\mathrm{U}}(p,q) \to \mathrm{U}(p,q)$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}) \to \mathrm{Sp}_{2n}(\mathbb{R})$, acts on π through the nontrivial character.

The main goal of the paper is to count the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$. In view of Corollary 1.6, we will explicitly determine both ${}^L\mathscr{C}_\lambda$ and $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ (for $\lambda = \lambda_{\check{\mathcal{O}}}$) and will express the sum $\sum_{\sigma \in {}^L\mathscr{C}_\lambda} [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))]$ as the count of certain combinatorial constructs. These combinatorial constructs provide the key linkage with the authors' second paper [BMSZ21], whose main goal is to construct all the representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$.

1.5. **The cases of general linear groups and unitary groups.** For a Young diagram $\imath$, write

$$\mathbf{r}_1(\imath) \geq \mathbf{r}_2(\imath) \geq \mathbf{r}_3(\imath) \geq \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\imath) \geq \mathbf{c}_2(\imath) \geq \mathbf{c}_3(\imath) \geq \cdots$$

for its column lengths. Denote by $|\imath| := \sum_{i=1}^{\infty} \mathbf{r}_i(\imath)$ the total size of $\imath$.

When no confusion is possible, we still use $\check{\mathcal{O}}$ to denote the Young diagram attached to the nilpotent orbit $\check{\mathcal{O}}$. Note that the Young diagram determines the nilpotent orbit unless $\check{G} = \mathrm{SO}_{4n}(\mathbb{C})$ $(n \geq 1)$ and all the row lengths are even.

Let $\mathbb{N}^+$ denote the set of positive integers. For any Young diagram $\imath$, we introduce the set $\mathrm{Box}(\imath)$ of boxes of $\imath$ as the following subset of $\mathbb{N}^+ \times \mathbb{N}^+$:

$$\mathrm{Box}(\imath) := \left\{ (i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \;\middle|\; j \leq \mathbf{r}_i(\imath) \right\}.$$

We also introduce five symbols $\bullet$, s, r, c and d, and make the following definitions.

Definition 1.7. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is the set of boxes of a Young diagram when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *when $S = \{s\}$ or $\{r\}$, every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$;*
- *when $S = \{c\}$ or $\{d\}$, every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$.*

A painted Young diagram is a pair $(\imath, \mathcal{P})$ consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$.

Definition 1.8. *Suppose that $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}$. A painting $\mathcal{P}$ on a Young diagram $\imath$ has type $\star$ if*

- *the image of $\mathcal{P}$ is contained in*
$$\begin{cases} \{\bullet, c, d\}, & \text{if } \star = A^{\mathbb{R}}; \\ \{\bullet\}, & \text{if } \star = A^{\mathbb{H}}; \\ \{\bullet, s, r\}, & \text{if } \star \in \{A, \widetilde{A}\}, \end{cases}$$
- *if $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}\}$, then $\mathcal{P}^{-1}(\bullet)$ has even number of boxes in every column of $\imath$,*
- *if $\star \in \{A, \widetilde{A}\}$, then $\mathcal{P}^{-1}(\bullet)$ has even number of boxes in every row of $\imath$.*

Denote by $\mathsf{PAP}_\star(\imath)$ the set of paintings on $\imath^t$ that has type $\star$, where $\imath^t$ is the transpose of $\imath$.

The middle letter A in PAP refers to the common A in $\{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}$.

Special unipotent representations of general linear groups are well-understood (see [Vog86, Page 450]). In particular, we have the following counting result for general linear groups.

Theorem 1.9. *Suppose that $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}\}$. Then*
$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp(\mathsf{PAP}_\star(\check{\mathcal{O}})).$$

Remark 1.10. If $\star = A^{\mathbb{R}}$, then
$$\sharp(\mathsf{PAP}_\star(\check{\mathcal{O}})) = \prod_{i \in \mathbb{N}^+} (1 + \text{the number of rows of length } i \text{ in } \check{\mathcal{O}}).$$
If $\star = A^{\mathbb{H}}$, then
$$\sharp(\mathsf{PAP}_\star(\check{\mathcal{O}})) = \begin{cases} 1, & \text{if all row lengths of } \check{\mathcal{O}} \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose that $\imath$ is a Young diagram and $\mathcal{P}$ is a painting on $\imath$ that has type A or $\widetilde{A}$. Define the signature of $\mathcal{P}$ to be the pair

$$(p_{\mathcal{P}}, q_{\mathcal{P}}) := \left(\frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(r)),\ \frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(s)) \right). \tag{1.7}$$

Example 1.11. *Suppose that*
$$\check{\mathcal{O}} = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \multicolumn{2}{c}{} \\ \cline{1-3} \ & \multicolumn{4}{c}{} \\ \cline{1-1} \ & \multicolumn{4}{c}{} \\ \cline{1-1} \ & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad \mathcal{P} = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet & r \\ \hline \bullet & \bullet & \multicolumn{3}{c}{} \\ \cline{1-2} s & r & \multicolumn{3}{c}{} \\ \cline{1-2} s & \multicolumn{4}{c}{} \\ \cline{1-1} r & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \in \mathrm{PAP}_A(\check{\mathcal{O}}).$$

Then $(p_{\mathcal{P}}, q_{\mathcal{P}}) = (6, 5)$.

Given two Young diagrams $\imath$ and $\jmath$, write $\imath \overset{r}{\sqcup} \jmath$ for the Young diagram whose multiset of nonzero row lengths equals the union of those of $\imath$ and $\jmath$. Also write $2\imath = \imath \overset{r}{\sqcup} \imath$. Similarly, we write $\imath \overset{c}{\sqcup} \jmath$ for the Young diagram whose multiset of nonzero column lengths equals the union of those of $\imath$ and $\jmath$.

For unitary groups, we have the following counting result.

Theorem 1.12. *Suppose that* $\star = A$ *or* $\widetilde{A}$ *so that* $G = \mathrm{U}(p,q)$ *or* $\widetilde{\mathrm{U}}(p,q)$, *respectively. Assume that there is a decomposition*

$$\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}} \overset{r}{\sqcup} 2\check{\mathcal{O}}'_{\mathrm{b}}$$

with the following property:

- *if* $\star = A$, *then all nonzero row lengths of* $\check{\mathcal{O}}_{\mathrm{g}}$ *have the same parity as* $p+q$, *and all nonzero row lengths of* $\check{\mathcal{O}}'_{\mathrm{b}}$ *have different parity as* $p+q$;
- *if* $\star = \widetilde{A}$, *then all nonzero row lengths of* $\check{\mathcal{O}}'_{\mathrm{b}}$ *have the same parity as* $p+q$, *and all nonzero row lengths of* $\check{\mathcal{O}}_{\mathrm{g}}$ *have different parity as* $p+q$.

Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp\left\{ \mathcal{P} \in \mathrm{PAP}_\star(\check{\mathcal{O}}_{\mathrm{g}}) \mid (p_{\mathcal{P}} + \left|\check{\mathcal{O}}'_{\mathrm{b}}\right|, q_{\mathcal{P}} + \left|\check{\mathcal{O}}'_{\mathrm{b}}\right|) = (p,q) \right\}.$$

If there is no such decomposition, then $\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = 0$.

In particular, when $\star = \widetilde{A}$ and $p+q$ is odd, the set $\mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{U}}(p,q))$ is empty.

1.6. **Orthogonal and symplectic groups: reduction to good parity.** Now we assume that $\star \in \left\{ B, D, C, \widetilde{C}, D^*, C^* \right\}$. Then there is a unique decomposition

$$\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}} \overset{r}{\sqcup} 2\check{\mathcal{O}}'_{\mathrm{b}}$$

such that $\check{\mathcal{O}}_{\mathrm{g}}$ has $\star$-good parity in the sense that all its nonzero row lengths are

$$\begin{cases} \text{even}, & \text{if } \star \in \{ B, \widetilde{C} \}; \\ \text{odd}, & \text{if } \star \in \{ C, D, D^*, C^* \}, \end{cases}$$

and $\check{\mathcal{O}}'_{\mathrm{b}}$ has $\star$-bad parity in the sense that all its nonzero row lengths are

$$\begin{cases} \text{odd}, & \text{if } \star \in \{ B, \widetilde{C} \}; \\ \text{even}, & \text{if } \star \in \{ C, D, D^*, C^* \}. \end{cases}$$

For simplicity, put

$$l := \left|\check{\mathcal{O}}'_{\mathrm{b}}\right|,$$

and

$$G'_{\mathrm{b}} := \begin{cases} \mathrm{GL}_l(\mathbb{R}), & \text{if } \star \in \{ B, C, D \}; \\ \widetilde{\mathrm{GL}}_l(\mathbb{R}), & \text{if } \star = \widetilde{C}; \\ \mathrm{GL}_{\frac{l}{2}}(\mathbb{H}), & \text{if } \star \in \{ C^*, D^* \}. \end{cases} \tag{1.8}$$

Here $\widetilde{\mathrm{GL}}_l(\mathbb{R})$ is the double cover of $\mathrm{GL}_l(\mathbb{R})$ that fits the following Cartesian diagram of Lie groups:

$$\begin{array}{ccc} \widetilde{\mathrm{GL}}_l(\mathbb{R}) & \longrightarrow & \mathrm{GL}_l(\mathbb{R}) \\ \big\downarrow & & \big\downarrow{\scriptstyle g \mapsto \text{ sign of } \det(g)} \\ \{\pm 1, \pm\sqrt{-1}\} & \xrightarrow{x \mapsto x^2} & \{\pm 1\}. \end{array} \tag{1.9}$$

Define

$$\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\widetilde{\mathrm{GL}}_l(\mathbb{R})) := \{\pi \in \mathrm{Irr}(\widetilde{\mathrm{GL}}_l(\mathbb{R})) \mid \pi \text{ is genuine and annihilated by } I_{\check{\mathcal{O}}'_{\mathrm{b}}} := I_{A^{\mathbb{R}}, \check{\mathcal{O}}'_{\mathrm{b}}}\}.$$

Here and as before, "genuine" means that the central subgroup $\{\pm 1\}$ acts through the nontrivial character. Then we have a bijective map

$$\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\mathrm{GL}_l(\mathbb{R})) \to \mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\widetilde{\mathrm{GL}}_l(\mathbb{R})), \quad \pi \mapsto \pi \otimes \tilde{\chi}_l,$$

where $\tilde{\chi}_l$ is the character given by the left vertical arrow of (1.9).

Note that G has a closed subgroup isomorphic to G'_{b} (as Lie groups) if and only if

$$\begin{cases} p, q \geq l, & \text{if } G = \mathrm{SO}(p,q); \\ p, q \geq \frac{l}{2}, & \text{if } G = \mathrm{Sp}(p,q); \\ \text{no condition}, & \text{otherwise}. \end{cases}$$

In such cases, G has a Levi subgroup that is identified with $G'_{\mathrm{b}} \times G_{\mathrm{g}}$ (or $(G'_{\mathrm{b}} \times G_{\mathrm{g}})/\{\pm 1\}$ when $\star = \widetilde{C}$), where

$$G_{\mathrm{g}} := \begin{cases} \mathrm{SO}(p-l, q-l), & \text{if } \star \in \{\, B, D\,\}; \\ \mathrm{O}^*(2n-2l), & \text{if } \star = D^*; \\ \mathrm{Sp}_{2n-2l}(\mathbb{R}), & \text{if } \star = C; \\ \widetilde{\mathrm{Sp}}_{2n-2l}(\mathbb{R}), & \text{if } \star = \widetilde{C}; \\ \mathrm{Sp}(p-\frac{l}{2}, q-\frac{l}{2}), & \text{if } \star = C^*. \end{cases} \tag{1.10}$$

Theorem 1.13. *If G has a closed subgroup isomorphic to G'_{b}, then parabolic induction yields a bijection*

$$\begin{array}{rccc} \mathfrak{I}\colon & \mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G) \\ & (\pi', \pi_0) & \mapsto & \pi' \rtimes \pi_0. \end{array}$$

Otherwise,

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \emptyset.$$

Combining with the counting result for general linear groups (Theorem 1.9), we list the more specific results as follows:

(a) Assume that $\star \in \{B, D\}$ so that $G = \mathrm{SO}(p,q)$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \begin{cases} \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) \times \sharp(\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\mathrm{GL}_l(\mathbb{R}))), & \text{if } p, q \geq l; \\ 0, & \text{otherwise}. \end{cases}$$

(b) Assume that $\star = C^*$ so that $G = \mathrm{Sp}(p,q)$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \begin{cases} \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})), & \text{if } p, q \geq \frac{l}{2}; \\ 0, & \text{otherwise}. \end{cases}$$

(c) Assume that $\star \in \{C, \widetilde{C}\}$ so that $G = \mathrm{Sp}_{2n}(\mathbb{R})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) \times \sharp(\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\mathrm{GL}_l(\mathbb{R}))).$$

(d) Assume that $\star = D^*$ so that $G = \mathrm{O}^*(2n)$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})).$$

1.7. **Orthogonal and symplectic groups: the case of good parity.** We now assume that $\check{\mathcal{O}}$ has $\star$-good parity, namely $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$. By Theorem 1.13, the counting problem in general is reduced to this case.

Definition 1.14. *A $\star$-pair is a pair $(i, i+1)$ of consecutive positive integers such that*

$$\begin{cases} i \text{ is odd}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \text{ is even}, & \text{if } \star \in \{B, D, D^*\}. \end{cases}$$

A $\star$-pair $(i, i+1)$ is said to be

- *vacant in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$;*

- *tailed in $\check{\mathcal{O}}$, if* $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ *is positive and odd;*
- *primitive in $\check{\mathcal{O}}$, if* $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ *is positive and even.*

Denote $\mathrm{PP}_\star(\check{\mathcal{O}})$ *the set of all $\star$-pairs that are primitive in $\check{\mathcal{O}}$.*

We attach to $\check{\mathcal{O}}$ a pair of Young diagrams

$$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) := (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})), \tag{1.11}$$

as follows.

The case when $\star = B$. In this case,

$$\mathbf{c}_1(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\jmath_{\check{\mathcal{O}}})) = \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}\right).$$

The case when $\star = \widetilde{C}$. In this case, for all $i \geq 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_i(\jmath_{\check{\mathcal{O}}})) = \left(\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}\right).$$

The case when $\star = \{C, C^*\}$. In this case, for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_i(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i-1, 2i) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

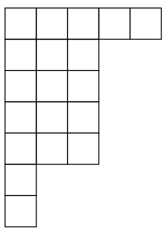
The case when $\star \in \{D, D^*\}$. In this case,

$$\mathbf{c}_1(\imath_{\check{\mathcal{O}}}) = \begin{cases} 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0, \end{cases}$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{\mathcal{O}}; \\ \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0\right), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}\right), & \text{otherwise.} \end{cases}$$

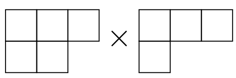
Example 1.15. *Suppose that $\star = C$, and $\check{\mathcal{O}}$ is the following Young diagram which has $\star$-good parity.*



Then

$$\mathrm{PP}_\star(\check{\mathcal{O}}) = \{(1,2), (5,6)\}$$

and

$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) =$  .

Here and henceforth, when no confusion is possible, we write $\alpha \times \beta$ for a pair (α, β). We will also write $\alpha \times \beta \times \gamma$ for a triple (α, β, γ).

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.16. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- $\mathcal{P}^{-1}(\bullet) = \mathcal{Q}^{-1}(\bullet)$*;*
- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, r, c, d\}, & \text{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \text{if } \alpha = D; \\ \{\bullet, s, c\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \text{if } \alpha = C^*; \\ \{\bullet, s\}, & \text{if } \alpha = D^*, \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, s\}, & \text{if } \alpha = C; \\ \{\bullet\}, & \text{if } \alpha = D; \\ \{\bullet, r, d\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \text{if } \alpha = C^*; \\ \{\bullet, r\}, & \text{if } \alpha = D^*. \end{cases}$$

For any painted bipartition τ as in Definition 1.8, we write

$$\imath_\tau := \imath, \;\; \mathcal{P}_\tau := \mathcal{P}, \;\; \jmath_\tau := \jmath, \;\; \mathcal{Q}_\tau := \mathcal{Q}, \;\; \alpha_\tau := \alpha, \;\; |\tau| := |\imath| + |\jmath|\,,$$

and

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise}. \end{cases}$$

We further define a pair (p_τ, q_τ) of natural numbers given by the following recipe.

- If $\star_\tau \in \{B, D, C^*\}$, then (p_τ, q_τ) is given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\begin{cases} p_\tau := (\#\bullet) + 2(\#r) + (\#c) + (\#d) + (\#B^+); \\ q_\tau := (\#\bullet) + 2(\#s) + (\#c) + (\#d) + (\#B^-). \end{cases}$$

  Here

$$\#\bullet := \#(\mathcal{P}^{-1}(\bullet)) + \#(\mathcal{Q}^{-1}(\bullet))$$

  and the other terms are similarly defined.
- If $\star_\tau \in \{C, \widetilde{C}, D^*\}$, then $p_\tau := q_\tau := |\tau|$.

We also define a classical group

$$G_\tau := \begin{cases} \mathrm{SO}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases}$$

Define

$$\mathsf{PBP}_\star(\check{\mathcal{O}}) := \{\, \tau \text{ is a painted bipartition} \mid \star_\tau = \star, \text{ and } (\imath_\tau, \jmath_\tau) = (\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) \,\}, \tag{1.12}$$

and

$$\mathsf{PBP}_G(\check{\mathcal{O}}) := \{\, \tau \in \mathsf{PBP}_\star(\check{\mathcal{O}}) \mid G_\tau = G \,\}.$$

Example 1.17. *Suppose that $\star = B$ and*

$$\check{\mathcal{O}} = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ & \ \\ \cline{1-2} \ & \ \\ \cline{1-2} \end{array}$$

Then

$$\tau := \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet \\ \cline{1-1} c \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & \bullet & d \\ \hline \bullet \\ \cline{1-1} d \\ \cline{1-1} \end{array} \times B^+ \in \mathrm{PBP}_\star(\check{\mathcal{O}}),$$

and

$$G_\tau = \mathrm{SO}(10, 9).$$

We now state our final result on the counting of special unipotent representations.

Theorem 1.18. *Assume that $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and $\check{\mathcal{O}}$ has $\star$-good parity. Then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) \leq \begin{cases} \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})), & \text{if } \star \in \{C^*, D^*\}; \\ 2^{\sharp(\mathrm{PP}_\star(\check{\mathcal{O}}))} \cdot \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})), & \text{if } \star \in \{B, C, D, \widetilde{C}\}. \end{cases}$$

In [BMSZ21], the authors construct a set of representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ whose cardinality equals the upper bound in Theorem 1.18, when $\check{\mathcal{O}}$ has $\star$-good parity. See [BMSZ21, Theorem 4.1]. Thus the equality holds in Theorem 1.18.

1.8. **The case of complex classical groups.** Special unipotent representations of complex classical groups are well-understood ([BV85], [Bar89]). We briefly review their counting and constructions in what follows. As the methods of this paper and [BMSZ21] work for complex classical groups, we will present the results in the complex case parallel to those of this paper and [BMSZ21]. For this subsection, we introduce five more symbols $A^{\mathbb{C}}, B^{\mathbb{C}}, D^{\mathbb{C}}, C^{\mathbb{C}}$, and $\widetilde{C}^{\mathbb{C}}$, and let $\star$ be one of them. Let G be a complex classical group of type $\star$, namely $G = \mathrm{GL}_n(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathrm{SO}_{2n}(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, or $\mathrm{Sp}_{2n}(\mathbb{C})$ $(n \geq 0)$, respectively. The Langlands dual $\check{G}$ of G is respectively defined to be $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$, or $\mathrm{Sp}_{2n}(\mathbb{C})$. Let $\check{\mathcal{O}}$ be a $\check{G}$-orbit in $\mathrm{Nil}(\check{\mathfrak{g}})$ where $\check{\mathfrak{g}}$ is the Lie algebra of $\check{G}$. As in the real case we have a maximal ideal $I_{\check{\mathcal{O}}} := I_{\star,\check{\mathcal{O}}}$ of $\mathcal{U}(\mathfrak{g}_0)$, where $\mathfrak{g}_0$ is the Lie algebra of G (viewed as a complex Lie group).

Write $\overline{\mathfrak{g}}_0$ for the complex Lie algebra equipped with a conjugate linear isomorphism $^- : \mathfrak{g}_0 \to \overline{\mathfrak{g}_0}$. The latter induces a conjugate linear isomorphism $^- : \mathcal{U}(\mathfrak{g}_0) \to \mathcal{U}(\overline{\mathfrak{g}_0})$. Note that $\mathfrak{g}_0 \times \overline{\mathfrak{g}_0}$ equals the complexified Lie algebra $\mathfrak{g}$ of G. Define the set of special unipotent representations of G attached to $\check{\mathcal{O}}$ by

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) := \mathrm{Unip}_{\star,\check{\mathcal{O}}}(G) := \{\pi \in \mathrm{Irr}(G) \mid \pi \text{ is annihilated by } I_{\check{\mathcal{O}}} \otimes \mathcal{U}(\overline{\mathfrak{g}_0}) + \mathcal{U}(\mathfrak{g}_0) \otimes \overline{I_{\check{\mathcal{O}}}}\,\}.$$

If $\star = A^{\mathbb{C}}$ so that $G = \mathrm{GL}_n(\mathbb{C})$, then $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is a singleton whose unique element is given by the normalized parabolic induction $\mathrm{Ind}_P^G 1_P$, where P is the standard parabolic subgroup whose Levi component equals

$$\mathrm{GL}_{\mathbf{r}_1(\check{\mathcal{O}})}(\mathbb{C}) \times \mathrm{GL}_{\mathbf{r}_2(\check{\mathcal{O}})}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{\mathbf{r}_{\mathbf{c}_1(\check{\mathcal{O}})}(\check{\mathcal{O}})}(\mathbb{C}),$$

and 1_P denotes the trivial representation of P.

Now suppose that $\star \in \{B^{\mathbb{C}}, D^{\mathbb{C}}, C^{\mathbb{C}}, \widetilde{C}^{\mathbb{C}}\}$. As in the real case, write

$$\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}} \overset{r}{\sqcup} 2\check{\mathcal{O}}'_{\mathrm{b}} \quad \text{and} \quad l := \left|\check{\mathcal{O}}'_{\mathrm{b}}\right|$$

so that G has a Levi subgroup that is identified with $G'_{\mathrm{b}} \times G_{\mathrm{g}}$, where $G'_{\mathrm{b}} = \mathrm{GL}_l(\mathbb{C})$ and

$$G_{\mathrm{g}} := \begin{cases} \mathrm{SO}_{2n-2l+1}(\mathbb{C}), & \text{if } \star = B^{\mathbb{C}}; \\ \mathrm{SO}_{2n-2l}(\mathbb{C}), & \text{if } \star = D^{\mathbb{C}}; \\ \mathrm{Sp}_{2n-2l}(\mathbb{C}), & \text{if } \star \in \{C^{\mathbb{C}}, \widetilde{C}^{\mathbb{C}}\}. \end{cases}$$

Define the set $\mathrm{PP}_\star(\check{\mathcal{O}}_{\mathrm{g}})$ as in the real case. Then by the work of Barbasch-Vogan [BV85, Corollary 5.29] (integral case) and Moeglin-Renard [MR17] (general case), we have that

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) = 2^{\sharp(\mathrm{PP}_\star(\check{\mathcal{O}}_{\mathrm{g}}))}.$$

As in the real case, every representation in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is obtained through irreducible parabolic induction via those of $\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$ (see Theorem 1.13), and every representation in $\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$ is obtained through iterated theta lifting (see [Bar17, Theorem 3.5.1], [Mœg17] and [BMSZ21]).

Here are some words on the contents and the organization of this article. In Section 2, we develop some generalities on the coherent continuation representation, which lead to the proofs of Theorems 1.2, 1.3 and 1.4. The generalities include coherent continuation representations for highest weight modules, primitive ideals and Goldie rank polynomials, as well as cell representations in the coherent continuation setting. As mentioned earlier, we build on previous works of several authors. In Section 3, we give explicit formulas for the coherent continuation representation $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$, based on an unpublished result of Barbasch and Vogan. Sections 4 to 6 are devoted to the main concern of the article, which is to give a precise count of special unipotent representations of all real classical groups, using results of Sections 2 and 3. We first deal with the general linear groups and the unitary groups, and then real classical groups of type BCD. All answers are given in terms of combinatorial constructs described earlier in this section. It is worthwhile to note, while the algebraic theory developed in Sections 2 and 3 yields ultimately an upper bound of the count, we are unable to demonstrate the precise count using the algebraic theory alone, due to a certain technical issue on the relationship of a Harish-Chandra cell and a Lusztig double cell, which we have formerly stated as Conjecture 2.20. In the case at hand, namely for the real classical groups, we rely on the analytic theory of theta lifting to construct the right number of special unipotent representations ([BMSZ21]), thus arriving at the precise count. It will be clearly desirable to demonstrate the precise count, without recourse to the analytic theory.

2. Generalities on the coherent continuation representation

We retain the notation of Sections 1.1-1.3. The main purpose of this section is to prove Theorems 1.2, 1.3 and 1.4.

2.1. **Basic properties of the representation** $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$**.** We define a basal vector space to be a complex vector space V equipped with a basis $\mathcal{B} \subseteq V$, and call elements of $\mathcal{B}$ the basal elements in V. A subspace of a basal space V is called a basal subspace if it is spanned by a set of basal elements of V. For example, if $\mathcal{K}$ is the Grothendieck group of an abelian category in which all objects have finite length, then $\mathcal{K}$ is a basal vector space with the irreducible objects as the basis. In particular, $\mathcal{K}(G)$ is a basal vector space.

Recall that an element $\nu \in {}^a\mathfrak{h}^*$ is said to be regular if

$$\langle \nu, \alpha^\vee \rangle \neq 0 \qquad \text{for all } \alpha \in \Delta,$$

and is said to be dominant if

$$\langle \nu, \alpha^\vee \rangle \notin -\mathbb{N}^+ \qquad \text{for all } \alpha \in \Delta^+.$$

Here $\Delta^+ \subseteq \Delta$ denotes the set of positive roots.

Write $\mathrm{Rep}_\lambda(G)$ for the category of Casselman-Wallach representations of G of generalized infinitesimal character λ, and write $\mathcal{K}_\lambda(G)$ for its Grothedieck group. Then $\mathcal{K}_\lambda(G)$ is a basal vector space with the basis $\mathrm{Irr}_\lambda(G) \subseteq \mathcal{K}_\lambda(G)$. By evaluating at an element $\nu \in [\lambda]$, we get a linear map

$$\mathrm{ev}_\nu \,:\, \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G)) \longrightarrow \mathcal{K}_\nu(G).$$

We present a number of lemmas, all of which may be found in (or easily deduced from) [Vog81, Vog82].

Lemma 2.1. *Let $\nu \in [\lambda]$. The map ev_ν is surjective, and it is bijective when ν is regular.*

Proof. The surjectivity is due to Schmid and Zuckerman, see [Vog81, Theorem 7.2.7]. The injectivity (for ν regular) is due to Schmid, see [Vog81, Proposition 7.2.23]. □

Lemma 2.2. *There is a unique basis $\mathcal{B}_{[\lambda]}(G)$ of $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ such that*

$$\mathrm{Irr}_\nu(G) \subseteq \mathrm{ev}_\nu(\mathcal{B}_{[\lambda]}(G)) \subseteq \mathrm{Irr}_\nu(G) \sqcup \{0\},$$

for any dominant element $\nu \in [\lambda]$.

Proof. In view of Lemma 2.1, this is implied by [Vog81, Corollary 7.3.23]. □

By Lemma 2.2, $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ is a basal vector space, with the basis $\mathcal{B}_{[\lambda]}(G)$. By Lemma 2.1, for any element $\nu \in [\lambda]$ that is regular and dominant, the evaluation map yields a bijection

$$\mathrm{ev}_\nu : \mathcal{B}_{[\lambda]}(G) \xrightarrow{\sim} \mathrm{Irr}_\nu(G).$$

Recall from the introductory section the root system

$$\Delta_{[\lambda]} := \{\, \alpha \in \Delta \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \,\} \subseteq {}^a\mathfrak{h}^*.$$

Write $\Delta^+_{[\lambda]}$ for the set of positive roots in $\Delta_{[\lambda]}$. For each $\alpha \in \Delta$, write $s_\alpha \in W$ for the reflection attached to α.

Lemma 2.3. *(*[Vog82, Corollary 7.3.23]*) Let Φ be a basal element in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$, and let ν be a dominant element in $[\lambda]$. Then $\Phi(\nu) = 0$ if and only if*

$$s_\alpha \cdot \Phi = -\Phi \quad \textit{for some simple root } \alpha \textit{ of } \Delta^+_{[\lambda]} \textit{ such that } \langle \nu, \alpha^\vee \rangle = 0.$$

Lemma 2.4. *(*[Vog82, Corollary 7.3.23]*) Let Φ_1, Φ_2 be two basal elements in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ such that $\Phi_1(\nu) = \Phi_2(\nu) \neq 0$ for some dominant element $\nu \in [\lambda]$. Then $\Phi_1 = \Phi_2$.*

Lemma 2.5. *Let Φ be a basal element in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$. Let $\nu_1, \nu_2 \in [\lambda]$ be two dominant elements. If both $\Phi(\nu_1)$ and $\Phi(\nu_2)$ are nonzero and so are irreducible representations of G. Then*

$$\mathrm{AV}_{\mathbb{C}}(\Phi(\nu_1)) = \mathrm{AV}_{\mathbb{C}}(\Phi(\nu_2)).$$

Proof. This is implied by [Vog81, Part (a) of Proposition 7.2.22 and Part (b) of Proposition 7.3.10]. □

In the setting of Lemma 2.5, we define the complex associated variety of Φ to be $\mathrm{AV}_{\mathbb{C}}(\Phi) := \mathrm{AV}_{\mathbb{C}}(\Phi(\nu))$, where ν is a dominant element in $[\lambda]$ with $\Phi(\nu)$ nonzero.

Recall that $\mathsf{S} \subseteq \mathrm{Nil}(\mathfrak{g}^*)$ is an $\mathrm{Inn}(\mathfrak{g})$-stable Zariski closed set.

Lemma 2.6. *Let Φ be a basal element in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$. Then $\Phi \in \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))$ if and only if $\mathrm{AV}_{\mathbb{C}}(\Phi) \subseteq \mathsf{S}$.*

Proof. The "only if" part is trivial. The "if" part is implied by [Vog81, Part (a) of Proposition 7.2.22] and [Vog81, Part (b) of Proposition 7.2.22]. □

Lemma 2.6 implies that the set

$$\mathcal{B}_{[\lambda],\mathsf{S}}(G) := \{\Phi \in \mathcal{B}_{[\lambda]}(G) \mid \mathrm{AV}_{\mathbb{C}}(\Phi) \subseteq \mathsf{S}\} \quad \text{is a basis of } \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G)). \tag{2.1}$$

Thus $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))$ is a basal subrepresentation of $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$, namely a subrepresentation as well as a basal subspace.

Lemma 2.7. *For all $\nu \in [\lambda]$, the evaluation map (at ν)*

$$\mathrm{ev}_{\nu,\mathsf{S}} \,:\, \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G)) \longrightarrow \mathcal{K}_{\nu,\mathsf{S}}(G)$$

is surjective.

Proof. By using the action of $W_{[\lambda]}$, we assume without loss of generality that ν is dominant. Then the lemma follows by Lemmas 2.2 and 2.6. □

2.2. **Proof of Theorem 1.2.** Theorem 1.2 is an immediate consequence of Lemma 2.7 and the following proposition.

Proposition 2.8. *Suppose $\mathcal{S}$ is a basal $W_{[\lambda]}$-submodule of $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$. For each $\nu \in [\lambda]$, the evaluation map ev_ν at ν descents to an isomorphism*

$$\overline{\mathrm{ev}_\nu|_{\mathcal{S}}} : \mathcal{S}_{W_\nu} \longrightarrow \mathrm{ev}_\nu(\mathcal{S}),$$

where

$$\mathcal{S}_{W_\nu} := \mathcal{S}/\,\mathrm{Span}\,\{\, \Phi - w\cdot\Phi \mid \Phi \in \mathcal{S},\, w \in W_\nu \,\}$$

is the maximal W_ν-invariant quotient of $\mathcal{S}$.

Proof. Without loss of generality we assume that ν is dominant. Let $\mathcal{B}_{\mathcal{S}}$ be the basis of $\mathcal{S}$. We have

$$\begin{aligned}
\ker(\mathrm{ev}_\nu|_{\mathcal{S}}) &= \mathrm{Span}\,\{\, \Phi \in \mathcal{B}_{\mathcal{S}} \mid \Phi(\nu) = 0 \,\} \quad \text{(by Lemma 2.2 and Lemma 2.4)}\\
&\subseteq \mathrm{Span}\left\{ \Phi - s_\alpha\cdot\Phi \;\middle|\; \begin{array}{l} \Phi \in \mathcal{B}_{\mathcal{S}},\\ \alpha \text{ is a simple root of } \Delta^+_{[\lambda]}\\ \text{such that } \langle \lambda, \alpha^\vee\rangle = 0 \end{array} \right\}\\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.3)}\\
&\subseteq \mathrm{Span}\,\{\, \Phi - w\cdot\Phi \mid \Phi \in \mathcal{S},\, w \in W_\nu \,\}\\
&\subseteq \ker(\mathrm{ev}_\lambda|_{\mathcal{S}}). \qquad \text{(by } (\Phi - w\cdot\Phi)(\nu) = \Phi(\nu) - \Phi(w^{-1}\cdot\nu) = 0)
\end{aligned}$$

Therefore

$$\ker(\mathrm{ev}_\lambda|_{\mathcal{S}}) = \mathrm{Span}\,\{\, \Phi - w\cdot\Phi \mid \Phi \in \mathcal{S},\, w \in W_\lambda \,\}\,.$$

and the proposition follows. □

2.3. **Highest weight modules and coherent continuation representations.** Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Let $\mathrm{Rep}(\mathfrak{g},\mathfrak{b})$ denote the category of finitely generated $\mathfrak{g}$-modules that are unions of finite-dimensional $\mathfrak{b}$-submodules, and let $\mathrm{Rep}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b})$ denote its full subcategory of the modules whose complex associated variety is contained in S. Write $\mathcal{K}(\mathfrak{g},\mathfrak{b})$ and $\mathcal{K}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b})$ respectively for the Grothendieck groups of $\mathrm{Rep}(\mathfrak{g},\mathfrak{b})$ and $\mathrm{Rep}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b})$, and form the coherent continuation representations $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$ and $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b}))$.

Let H be a Cartan subgroup of G such that its complexified Lie algebra $\mathfrak{h}$ is contained in $\mathfrak{b}$. Recall that a $(\mathfrak{g},H)$-module is defined to be a $\mathfrak{g}$-module V together with a locally-finite representation of H on it such that

- $h\cdot(X\cdot(h^{-1}\cdot u)) = (\mathrm{Ad}_h(X))\cdot u$, for all $h\in H, X\in\mathcal{U}(\mathfrak{g}), u\in V$ (Ad stands for the Adjoint representation);
- the differential of the representation of H and the restriction of the representation of $\mathfrak{g}$ yields the same representation of $\mathfrak{h}$ on V.

Let $\mathrm{Rep}(\mathfrak{g},H,\mathfrak{b})$ denote the category of finitely generated $(\mathfrak{g},H)$-modules that are unions of finite-dimensional $\mathfrak{b}$-submodules. We define the subcategory $\mathrm{Rep}_{\mathsf{S}}(\mathfrak{g},H,\mathfrak{b})$ of $\mathrm{Rep}(\mathfrak{g},H,\mathfrak{b})$, the Grothendieck groups $\mathcal{K}(\mathfrak{g},H,\mathfrak{b})$ and $\mathcal{K}_{\mathsf{S}}(\mathfrak{g},H,\mathfrak{b})$, and the coherent continuation representations $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(\mathfrak{g},H,\mathfrak{b}))$ and $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g},H,\mathfrak{b}))$, as before.

Proposition 2.9. *The representation* $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g},H,\mathfrak{b}))$ *of* $W_{[\lambda]}$ *is isomorphic to a subrepresentation of* $(\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b})))^k$*, for some* $k\in\mathbb{N}$.

Proof. Write Inn_H for the Zariski closure of the image of H under the adjoint representation $G\to\mathrm{Inn}(\mathfrak{g})$, which is an algebraic torus. Write Q_H for the group of algebraic characters of Inn_H (which is isomorphic to the root lattice). By pulling-back through the homomorphism $H\to\mathrm{Inn}_H$, we view Q_H as a set of characters on H. The tensor product $\beta\otimes\gamma\in\mathrm{Irr}(H)$ is defined for every $\beta\in Q_H$ and $\gamma\in\mathrm{Irr}(H)$. This yields a free action of Q_H on the set $\mathrm{Irr}(H)$.

For each Q_H-orbit $\Gamma\subseteq\mathrm{Irr}(H)$, write $\mathrm{Rep}_{\mathsf{S},\Gamma}(\mathfrak{g},H,\mathfrak{b})$ for the full subcategory of $\mathrm{Rep}_{\mathsf{S}}(\mathfrak{g},H,\mathfrak{b})$ whose objects are the modules V such that every irreducible subquotient of $V|_H$ (V viewed as a representation of H) belongs to Γ. Write $\mathcal{K}_{\mathsf{S},\Gamma}(\mathfrak{g},H,\mathfrak{b})$ for the Grothendieck group of the category $\mathrm{Rep}_{\mathsf{S},\Gamma}(\mathfrak{g},H,\mathfrak{b})$. Then we have a decomposition

$$\mathcal{K}_{\mathsf{S}}(\mathfrak{g},H,\mathfrak{b}) = \bigoplus_{\Gamma\in Q_H\backslash\mathrm{Irr}(H)} \mathcal{K}_{\mathsf{S},\Gamma}(\mathfrak{g},H,\mathfrak{b}),$$

of $\mathcal{R}(\mathfrak{g})$-modules, and

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g},H,\mathfrak{b})) = \bigoplus_{i=1}^{k}\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S},\Gamma_i}(\mathfrak{g},H,\mathfrak{b})),$$

for a finite number of orbits $\Gamma_1,\Gamma_2,\cdots,\Gamma_k\in Q_H\backslash\mathrm{Irr}(H)$ ($k\in\mathbb{N}$). Thus it remains to show that $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S},\Gamma}(\mathfrak{g},H,\mathfrak{b}))$ is isomorphic to a subrepresentation of $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b}))$.

For each $\gamma\in\Gamma$, put

$$M(\gamma) := \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\gamma,$$

which is a module in $\mathrm{Rep}_\Gamma(\mathfrak{g},H,\mathfrak{b})$, where the $\mathcal{U}(\mathfrak{g})$-action is given by the left multiplication, and the H-action is given by

$$h\cdot(X\otimes u) := \mathrm{Ad}_h(X)\otimes h\cdot u, \qquad h\in H,\ X\in\mathcal{U}(\mathfrak{g}),\ u\in\gamma.$$

Note that $\{M(\gamma)\}_{\gamma\in\Gamma}$ is a basis of the space

$$\mathcal{K}_\Gamma(\mathfrak{g},H,\mathfrak{b}) := \mathcal{K}_{\mathrm{Nil}(\mathfrak{g}^*),\Gamma}(\mathfrak{g},H,\mathfrak{b}).$$

Thus the forgetful functor

$$\mathrm{Rep}_\Gamma(\mathfrak{g},H,\mathfrak{b}) := \mathrm{Rep}_{\mathrm{Nil}(\mathfrak{g}^*),\Gamma}(\mathfrak{g},H,\mathfrak{b}) \to \mathrm{Rep}(\mathfrak{g},\mathfrak{b})$$

induces an injective linear map

$$\mathcal{K}_\Gamma(\mathfrak{g}, H, \mathfrak{b}) \to \mathcal{K}(\mathfrak{g}, \mathfrak{b}).$$

This map is a $\mathcal{R}(\mathfrak{g})$-module homomorphism, and induces an injective $\mathcal{R}(\mathfrak{g})$-module homomorphism

$$\mathcal{K}_{\Gamma,\mathsf{S}}(\mathfrak{g}, H, \mathfrak{b}) \to \mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b}).$$

The above homomorphism induces an embedding

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S},\Gamma}(\mathfrak{g}, H, \mathfrak{b})) \to \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b})),$$

and the proposition follows. □

2.4. **A result of Casian.** Similar to the subspace $\mathcal{K}_{\lambda,\mathsf{S}}(G) \subseteq \mathcal{K}(G)$, we define the subspace $\mathcal{K}_{\lambda,\mathsf{S}}(\mathfrak{g}, H, \mathfrak{b}) \subseteq \mathcal{K}(\mathfrak{g}, H, \mathfrak{b})$ in the obvious way. Let $\{H_1, H_2, \cdots, H_r\}$ $(r \in \mathbb{N}^+)$ be a set of representatives of the conjugacy classes of Cartan subgroups of G. For each $i = 1, 2, \cdots, r$, fix a Borel subalgebra $\mathfrak{b}_i$ of $\mathfrak{g}$ that contains the complexified Lie algebra of H_i.

Proposition 2.10. *There is an injective $\mathcal{R}(\mathfrak{g})$-module homomorphism*

$$\gamma_G : \mathcal{K}(G) \to \bigoplus_{i=1}^{r} \mathcal{K}(\mathfrak{g}, H_i, \mathfrak{b}_i)$$

such that

$$\gamma_G(\mathcal{K}_{\lambda,\mathsf{S}}(G)) \subseteq \bigoplus_{i=1}^{r} \mathcal{K}_{\lambda,\mathsf{S}}(\mathfrak{g}, H_i, \mathfrak{b}_i) \tag{2.2}$$

for any $\lambda \in {}^a\mathfrak{h}^*$ and any $\mathrm{Inn}(\mathfrak{g})$*-stable Zariski closed subset* S *of* $\mathrm{Nil}(\mathfrak{g}^*)$.

Proof. This follows from the work of Casian ([Cas86]). See also [McG98]. Since the proposition is not explicitly stated in [Cas86], we briefly recall the argument of Casian for the convenience of the reader.

Let $\mathfrak{n}_i$ denote the nilpotent radical of $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{b}_i$ $(i = 1, 2, \cdots, r)$. For every $q \in \mathbb{Z}$, let $\gamma^q_{\mathfrak{n}_i}$ denote the q-th right derived functor of the following left exact functor from the category of $\mathfrak{g}$-modules to itself:

$$V \to \{u \in V \mid \mathfrak{n}_i^k \cdot v = 0 \text{ for some } k \in \mathbb{N}^+\}.$$

Fix a Cartan involution θ of G and write K for its fixed point group (which is a maximal compact subgroup of G). Without loss of generality we assume that all H_i's are θ-stable.

For every Casselman-Wallach representation V of G, write $V_{[K]}$ for the space of K-finite vectors in V, which is a $(\mathfrak{g}, K)$-module of finite length. Then $\gamma^q_{\mathfrak{n}_i}(V_{[K]})$ is naturally a representation in $\mathrm{Rep}(\mathfrak{g}, H_i, \mathfrak{b}_i)$ ([Cas86, Corollary 4.9]).

We define a linear map

$$\gamma_G : \mathcal{K}(G) \to \bigoplus_{i=1}^{r} \mathcal{K}(\mathfrak{g}, H_i, \mathfrak{b}_i)$$

given by

$$\gamma_G(V) = \left\{ \sum_{q\in\mathbb{Z}} (-1)^q \gamma^q_{\mathfrak{n}_i}(V_{[K]}) \right\}_{i=1,2,\cdots,r}$$

for every Casselman-Wallach representation V of G. The Osborne conjecture (see [Cas86, Theorem 3.1]) and [Cas86, Corollary 4.9]) implies that the map γ_G is injective.

Proposition 4.11 of [Cas86] implies that the functor $\gamma^q_{\mathfrak{n}_i}$ commutes with tensor product with the finite-dimensional representations. Thus γ_G is a $\mathcal{R}(\mathfrak{g})$-homomorphism. Finally, [Cas86, Corollary 4.15] implies that γ_G satisfies the property in (2.2). $\square$

Proposition 2.10 implies that the representation $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))$ of $W_{[\lambda]}$ is isomorphic to a subrepresentation of $\bigoplus_{i=1}^r \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, H_i, \mathfrak{b}_i))$. Together with Proposition 2.9, this implies the following result.

Proposition 2.11. *The representation* $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))$ *of* $W_{[\lambda]}$ *is isomorphic to a subrepresentation of* $(\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b})))^k$*, for some* $k \in \mathbb{N}$.

2.5. **Blocks and coherent continuation representations.** Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is contained in $\mathfrak{b}$. Then $\mathfrak{h}$ is identified with ${}^a\mathfrak{h}$ (since $\mathfrak{b}$ has been fixed) and we view λ as an element of $\mathfrak{h}^*$. Define the Verma module

$$\mathrm{M}(\lambda) := \mathrm{M}(\mathfrak{g}, \mathfrak{b}, \lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho},$$

where $\rho \in \mathfrak{h}^*$ is the half sum of the weights of $\mathfrak{b}$, $\mathbb{C}_{\lambda-\rho}$ is the one-dimensional $\mathfrak{h}$-module corresponds to the character $\lambda - \rho \in \mathfrak{h}^*$, and every $\mathfrak{h}$-module is viewed an a $\mathfrak{b}$-module as usual. Write $\mathrm{L}(\lambda) = \mathrm{L}(\mathfrak{g}, \mathfrak{b}, \lambda)$ for the unique irreducible quotient of $\mathrm{M}(\lambda)$.

Let $\mathrm{Block}_\lambda(\mathfrak{g}, \mathfrak{b})$ denote the full subcategory of $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$ consisting of the modules with generalized infinitesimal character λ whose weights are contained in $\lambda - \rho + Q$, which is called a block in $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$. Let $\mathcal{K}(\mathrm{Block}_\lambda(\mathfrak{g}, \mathfrak{b}))$ denote the Grothendieck group of this category. Then both

$$\{\mathrm{L}(\mathfrak{g}, \mathfrak{b}, w\lambda)\}_{w \in W_{[\lambda]}/W_\lambda} \quad \text{and} \quad \{\mathrm{M}(\mathfrak{g}, \mathfrak{b}, w\lambda)\}_{w \in W_{[\lambda]}/W_\lambda}$$

are bases of $\mathcal{K}(\mathrm{Block}_\lambda(\mathfrak{g}, \mathfrak{b}))$.

The following result is a theorem of Soergel, in a weak form that we require.

Theorem 2.12. *(*[Soe90, Section 2.5, Theorem 11]*) Let* $\mathfrak{g}_i$ *be a reductive complex Lie algebras, with a Borel subalgebra* $\mathfrak{b}_i$ *and a Cartan subalgebra* $\mathfrak{h}_i \subseteq \mathfrak{b}_i$ $(i = 1, 2)$*. Let* $W_i \subseteq \mathrm{GL}(\mathfrak{h}_i)$ *be the Weyl group of* $\mathfrak{g}_i$ *and let* $\lambda_i \in \mathfrak{h}_i^*$ *be dominant. Write* $Q_i \subseteq \mathfrak{h}_i^*$ *for the root lattice of* $\mathfrak{g}_i$*. Put* $[\lambda_i] := \lambda_i + Q_i$*, and let* $W_{\lambda_i} \subseteq W_{[\lambda_i]}$ *denote the stabilizers of* λ_i *and* $[\lambda_i]$ *in* W_i*, respectively. Suppose that there is a group isomorphism* $\varphi : W_{[\lambda_1]} \to W_{[\lambda_2]}$ *that takes the set of simple reflections in* $W_{[\lambda_1]}$ *onto the set of simple reflections in* $W_{[\lambda_2]}$*, and takes* W_{λ_1} *onto* W_{λ_2}*. Then there is a linear isomorphism* $\mathcal{K}(\mathrm{Block}_{\lambda_1}(\mathfrak{g}_1, \mathfrak{b}_1)) \to \mathcal{K}(\mathrm{Block}_{\lambda_2}(\mathfrak{g}_2, \mathfrak{b}_2))$ *that sends* $\mathrm{L}(\mathfrak{g}_1, \mathfrak{b}_1, w_1\lambda_1)$ *to* $\mathrm{L}(\mathfrak{g}_2, \mathfrak{b}_2, \varphi(w_1)\lambda_2)$ *and sends* $\mathrm{M}(\mathfrak{g}_1, \mathfrak{b}_1, w_1\lambda_1)$ *to* $\mathrm{M}(\mathfrak{g}_2, \mathfrak{b}_2, \varphi(w_1)\lambda_2)$*, for all* $w_1 \in W_{[\lambda_1]}$*.*

Recall that $Q \subseteq \mathfrak{h}^*$ is the root lattice. For each Q-coset $\Lambda \subseteq \mathfrak{h}^*$, let $\mathrm{Rep}_\Lambda(\mathfrak{g}, \mathfrak{b})$ denote the full subcategory of $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$ consisting of the modules whose weights are contained in $\Lambda - \rho$. Write $\mathcal{K}_\Lambda(\mathfrak{g}, \mathfrak{b})$ for the Grothendieck group of this subcategory. Then

$$\mathcal{K}(\mathfrak{g}, \mathfrak{b}) = \bigoplus_{\Lambda \in Q \backslash \mathfrak{h}^*} \mathcal{K}_\Lambda(\mathfrak{g}, \mathfrak{b})$$

and hence

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}(\mathfrak{g}, \mathfrak{b})) = \bigoplus_{\Lambda \in Q \backslash \mathfrak{h}^*} \mathrm{Coh}_{[\lambda]}(\mathcal{K}_\Lambda(\mathfrak{g}, \mathfrak{b})).$$

Note that

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}_\Lambda(\mathfrak{g}, \mathfrak{b})) \neq \{0\} \quad \text{only if} \quad \Lambda = w \cdot [\lambda] \text{ for some } w \in W.$$

Thus

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}(\mathfrak{g}, \mathfrak{b})) = \bigoplus_{w \in W/W_{[\lambda]}} \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w[\lambda]}(\mathfrak{g}, \mathfrak{b})). \tag{2.3}$$

Set

$$W'_{[\lambda]} := \{w \in W \mid w\Delta^+_{[\lambda]} = \Delta^+_{w[\lambda]}\}.$$

Then the group multiplications yield a bijective map

$$W'_{[\lambda]} \times W_{[\lambda]} \to W.$$

Let $w' \in W'_{[\lambda]}$. By evaluating at an element $\nu \in [\lambda]$, we get a linear map

$$\mathrm{ev}_\nu : \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b})) \to \mathcal{K}(\mathrm{Block}_{w'\nu}(\mathfrak{g}, \mathfrak{b})). \tag{2.4}$$

The following lemma is an analogue of Lemma 2.1 for highest weight modules.

Lemma 2.13. *The evaluating map* (2.4) *is surjective for all* $\nu \in [\lambda]$*, and is bijective when* ν *is regular.*

Proof. The proof of Lemma 2.1 works for this case. See also [Mil, Theorem 7.7]. □

For every $w_0 \in W_{[\lambda]}$, define a map

$$\begin{array}{rcl}\Psi_{w'w_0} := \Psi_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_0} : [\lambda] & \to & \mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b}), \\ \nu & \mapsto & \mathrm{M}(\mathfrak{g}, \mathfrak{b}, w'w_0\nu).\end{array}$$

Then $\Psi_{w'w_0} \in \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b}))$, and Lemma 2.13 implies that

$$\{\Psi_{w'w_0}\}_{w_0 \in W_{[\lambda]}} \text{ is a basis of } \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b})). \tag{2.5}$$

Theorem 2.12 and Lemma 2.13 imply that there is a unique element

$$\overline{\Psi}_{w'w_0} := \overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_0} \in \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b}))$$

such that

$$\overline{\Psi}_{w'w_0}(\nu) = \mathrm{L}(\mathfrak{g}, \mathfrak{b}, w'w_0\nu) \quad \text{for every } \nu \in [\lambda] \text{ that is regular and dominant.} \tag{2.6}$$

Then

$$\{\overline{\Psi}_{w'w_0}\}_{w_0 \in W_{[\lambda]}} \text{ is also a basis of } \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b})).$$

We view $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b}))$ as a basal space with this basis. Similar to Lemma 2.2, we know that for all dominant element $\nu \in [\lambda]$, $\overline{\Psi}_{w'w_0}(\nu)$ is either zero or an irreducible object in $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$.

Extend the representation of $W_{[\lambda]}$ on $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b}))$ to the group

$$W_{[\lambda]} \times W_{[\lambda]} \supset \{1\} \times W_{[\lambda]} = W_{[\lambda]}$$

such that

$$(w_1, w_2) \cdot \Psi_{w'w_0} = \Psi_{w'w_1w_0w_2^{-1}} \quad \text{for all } w_0, w_1, w_2 \in W_{[\lambda]}.$$

Write $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b}))$ for the basal space $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g}, \mathfrak{b}))$ equipped with the above representation of $W_{[\lambda]} \times W_{[\lambda]}$.

2.6. **Primitive ideals and Goldie rank polynomials.** Write $\mathrm{J}(\lambda) = \mathrm{J}(\mathfrak{g}, \mathfrak{b}, \lambda)$ for the annihilator ideal of $\mathrm{L}(\lambda)$. Write $\mathfrak{h} = \mathfrak{h}_{\mathrm{s}} \oplus \mathfrak{c}$, where $\mathfrak{h}_{\mathrm{s}} = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{c}$ is the center of $\mathfrak{g}$. Then $\mathfrak{h}^* = \mathfrak{h}_{\mathrm{s}}^* \oplus \mathfrak{c}^*$.

Let $w \in W$. Then there is a unique polynomial function $\tilde{p}_{\mathfrak{g},[\lambda],w}$ on $\mathfrak{h}^*$, called a *Goldie rank polynomial* such that (see [Jos80a, Section 5.12] and [Jos85, Section 2.10])

- it is $\mathfrak{c}^*$-invariant (under the translations);
- for all $\nu \in [\lambda]$ that is regular and dominant,

$$\tilde{p}_{\mathfrak{g},[\lambda],w}(\nu) = \text{Goldie rank of } \mathcal{U}(\mathfrak{g})/\,\mathrm{J}(w \cdot \nu).$$

Let J be a primitive ideal of $\mathcal{U}(\mathfrak{g})$ whose infinitesimal character is represented by an element in $[\lambda]$. Then there is a unique polynomial function $\tilde{p}_{[\lambda],J}$ on $\mathfrak{h}^*$ such that (see [Jos80a, Section 5.12])

$$\tilde{p}_{[\lambda],J} = \tilde{p}_{\mathfrak{g},[\lambda],w_0}$$

for all $w_0 \in W_{[\lambda]}$ with

$$\text{(2.7)} \qquad \operatorname{Ann}(\overline{\Psi}_{w_0}(\nu)) = J \text{ for some dominant element } \nu \in [\lambda].$$

Here Ann indicates the annihilator ideal of a $\mathcal{U}(\mathfrak{g})$-module. Note that such elements w_0 and ν exist by [Duf77, Theorem 1].

For every $\sigma \in \operatorname{Irr}(W)$, its fake degree is defined to be

$$\text{(2.8)} \qquad a(\sigma) := \min\{a \in \mathbb{N} \mid \sigma \text{ occurs in the } a\text{-th symmetric power } \mathrm{S}^a(\mathfrak{h})\}.$$

This is well-defined since every $\sigma \in \operatorname{Irr}(W)$ occurs in the symmetric power $\mathrm{S}(\mathfrak{h}) = \bigoplus_{a\in\mathbb{N}} \mathrm{S}^a(\mathfrak{h})$. The representation σ is said to be univalent if it occurs in $\mathrm{S}^{a(\sigma)}(\mathfrak{h}_{\mathrm{s}})$ with multiplicity one. Note that $\mathfrak{h}_{\mathrm{s}} = \mathfrak{h}/\mathfrak{c} = \mathfrak{h}/\mathfrak{h}^W$, where $\mathfrak{h}^W$ denotes the W-fixed vectors in $\mathfrak{h}$.

Theorem 2.14 ([Jos80b, Theorem 5.4 and Theorem 5.5])**.** *Let J_1 and J_2 be two primitive ideals of $\mathcal{U}(\mathfrak{g})$ whose infinitesimal characters are represented by a common element in $[\lambda]$. Then*

$$J_1 = J_2 \quad \text{if and only if} \quad \tilde{p}_{[\lambda],J_1} = \tilde{p}_{[\lambda],J_2}.$$

Moreover, for each primitive ideal J of $\mathcal{U}(\mathfrak{g})$ whose infinitesimal character is represented by an element in $[\lambda]$,

- *the $W_{[\lambda]}$-subrepresentation $\sigma_{[\lambda],J}$ in $\mathbb{C}[\mathfrak{h}^*]$ generated by $\tilde{p}_{[\lambda],J}$ is a univalent irreducible representation of $W_{[\lambda]}$;*
- *the polynomial $\tilde{p}_{[\lambda],J}$ is homogeneous whose degree equals the fake degree $a(\sigma_J)$.*

The above representation $\sigma_{[\lambda],J}$ is called the *Goldie rank representation* attached to J.

Write $\check{\Delta} := \{\alpha^\vee \mid \alpha \in \Delta\} \subseteq \mathfrak{h}$ for the set of coroots. Recall the set

$$\Delta_{[\lambda]} := \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}$$

and put

$$\check{\Delta}_{[\lambda]} := \{\alpha^\vee \mid \alpha \in \Delta_{[\lambda]}\}.$$

Let $\check{\mathfrak{g}}$ denote the Langlands dual of $\mathfrak{g}$ so that $\mathfrak{h}^*$ is identified with a Cartan subalgebra of $\check{\mathfrak{g}}$ and $\check{\Delta}$ is identified with the root system of $\check{\mathfrak{g}}$. Let $\check{\mathfrak{g}}_{[\lambda]}$ denote the Lie subalgebra of $\check{\mathfrak{g}}$ containing $\mathfrak{h}^*$ whose root system equals $\check{\Delta}_{[\lambda]}$. Let $\mathfrak{g}_{[\lambda]}$ denote the Langlands dual of $\check{\mathfrak{g}}_{[\lambda]}$ so that $\mathfrak{h}$ is identified with a Cartan subalgebra of $\check{\mathfrak{g}}_{[\lambda]}$ and $\Delta_{[\lambda]}$ is identified with the root system of $\mathfrak{g}_{[\lambda]}$. Then the Weyl group of $\check{\mathfrak{g}}_{[\lambda]}$ is identified with $W_{[\lambda]}$. Write $Q_{[\lambda]} \subseteq \mathfrak{h}^*$ for the root lattice for $\mathfrak{g}_{[\lambda]}$.

Lemma 2.15. *If $w \in W_{[\lambda]}$, then the polynomial function $\tilde{p}_{\mathfrak{g},[\lambda],w}$ equals a nonzero scalar multiple of $\tilde{p}_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w}$.*

Proof. Recall the Jantzen matrix $\{a_{\mathfrak{g},[\lambda]}(w_1,w_2) \in \mathbb{Z}\}_{w_1,w_2\in W_{[\lambda]}}$ that is determined by the equality

$$\mathrm{L}(w_1\nu) = \sum_{w_2\in W_{[\lambda]}} a_{\mathfrak{g},[\lambda]}(w_1,w_2)\cdot \mathrm{M}(w_2\nu) \qquad (\text{as elements of } \mathcal{K}(\mathfrak{g},\mathfrak{b})), \quad w_1 \in W_{[\lambda]}.$$

for some (and all) $\nu \in [\lambda]$ that is dominant and regular (see [Jan79, Section 2.15]). By Theorem 2.12 (Soegel's theorem), we have that

$$\text{(2.9)} \qquad a_{\mathfrak{g},[\lambda]}(w_1,w_2) = a_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]}}(w_1,w_2), \qquad w_1,w_2 \in W_{[\lambda]}.$$

Suppose that $w \in W_{[\lambda]}$. Define a polynomial function $p_{\mathfrak{g},[\lambda],w}$ on $\mathfrak{h}^* \times \mathfrak{h}$ by

$$p_{\mathfrak{g},[\lambda],w}(\nu, x) := \sum_{w' \in W_{[\lambda]}} a_{\mathfrak{g},[\lambda]}(w, w') \cdot \langle w'\nu, x\rangle^m, \qquad \nu \in \mathfrak{h}^*,\ x \in \mathfrak{h}, \tag{2.10}$$

where m is the smallest non-negative integer that makes the right-hand side of (2.10) a nonzero polynomial function. Then

$$p_{\mathfrak{g},[\lambda],w}(\nu, x) = \tilde{p}_{\mathfrak{g},[\lambda],w}(\nu) \cdot \tilde{p}'_{\mathfrak{g},[\lambda],w}(x), \qquad \nu \in \mathfrak{h}^*,\ x \in \mathfrak{h}, \tag{2.11}$$

for a unique polynomial function $\tilde{p}'_{\mathfrak{g},[\lambda],w}$ on $\mathfrak{h}$ (see [Kin81] and [Jos84, Section 5.1]).

Applying the above argument to $\mathfrak{g}_{[\lambda]}$, we have that

$$p_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w}(\nu, x) = \tilde{p}_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w}(\nu) \cdot \tilde{p}'_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w}(x), \qquad \nu \in \mathfrak{h}^*,\ x \in \mathfrak{h}.$$

By (2.9), we have that $p_{\mathfrak{g},[\lambda],w} = p_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w}$, and therefore the lemma follows. □

Write $w = w'w_0$, where $w' \in W'_{[\lambda]}$ and $w_0 \in W_{[\lambda]}$.

Lemma 2.16. *The equality*

$$\tilde{p}_{\mathfrak{g},[\lambda],w} = w'^{-1} \cdot \tilde{p}_{\mathfrak{g},w'[\lambda],w'w_0w'^{-1}}$$

holds.

Proof. Suppose that $\nu \in [\lambda]$ is regular and dominant. Then $w'\nu \in w'[\lambda]$ is also regular and dominant. Thus we have that

$$\begin{aligned} \tilde{p}_{\mathfrak{g},[\lambda],w}(\nu) &= \text{Goldie rank of } \mathcal{U}(\mathfrak{g})/\,\mathrm{J}(w\nu) \\ &= \text{Goldie rank of } \mathcal{U}(\mathfrak{g})/\,\mathrm{J}((w'w_0w'^{-1})(w'\nu)) \\ &= \tilde{p}_{\mathfrak{g},w'[\lambda],w'w_0w'^{-1}}(w'\nu) \\ &= (w'^{-1} \cdot \tilde{p}_{\mathfrak{g},w'[\lambda],w'w_0w'^{-1}})(\nu). \end{aligned}$$

This implies the lemma. □

Proposition 2.17. *The polynomial function $\tilde{p}_{\mathfrak{g},[\lambda],w}$ is a nonzero scalar multiple of $\tilde{p}_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0}$.*

Proof. We first note that

$$w'^{-1} \cdot \tilde{p}_{\mathfrak{g}_{w'[\lambda]},Q_{w'[\lambda]},w'w_0w'^{-1}} = \tilde{p}_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0}.$$

From Lemma 2.15, $\tilde{p}_{\mathfrak{g},w'[\lambda],w'w_0w'^{-1}}$ is a nonzero scalar multiple of $\tilde{p}_{\mathfrak{g}_{w'[\lambda]},Q_{w'[\lambda]},w'w_0w'^{-1}}$. Thus

$$w'^{-1}.\tilde{p}_{\mathfrak{g},w'[\lambda],w'w_0w'^{-1}} = \text{a nonzero scalar multiple of } \tilde{p}_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0},$$

Therefore the proposition follows, in view of Lemma 2.16. □

For every univalent irreducible representation σ_0 of $W_{[\lambda]}$, the W-subrepresentation of $\mathrm{S}^{a(\sigma_0)}(\mathfrak{h}_{\mathrm{s}})$ generated by

$$\mathbb{C} \otimes \sigma_0 \subseteq \mathrm{S}^0(\mathfrak{h}_{\mathrm{s}}^{W_{[\lambda]}}) \otimes \mathrm{S}^{a(\sigma_0)}(\mathfrak{h}_{\mathrm{s}} \cap [\mathfrak{g}_{[\lambda]}, \mathfrak{g}_{[\lambda]}]) \subseteq \mathrm{S}^{a(\sigma_0)}(\mathfrak{h}_{\mathrm{s}})$$

is irreducible and univalent, with the same fake degree as that of σ_0 (This result is due to Macdonald, Lusztig, and Spaltenstein. See [Car93, Chapter 11]). This irreducible representation of W is called the j-induction of σ_0, to be denoted by $j_{W_{[\lambda]}}^{W}(\sigma_0)$.

Write $\sigma_{\mathfrak{g},[\lambda],w}$ for the W-subrepresentation of $\mathrm{S}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$ (the space of polynomial functions) generated by $\tilde{p}_{\mathfrak{g},[\lambda],w}$. Similarly, we also have a $W_{[\lambda]}$-subrepresentation $\sigma_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0}$ of $\mathbb{C}[\mathfrak{h}^*]$ generated by $\tilde{p}_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0}$ which is nothing but the Goldie rank representation $\sigma_{J(w_0\nu)}$ for any regular dominant $\nu \in [\lambda]$ (*cf.* Theorem 2.14).

By Proposition 2.17,

$$\sigma_{\mathfrak{g},[\lambda],w} = j_{W_{[\lambda]}}^{W}(\sigma_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0}). \tag{2.12}$$

For every $w \in W$, it is known that $\sigma_{\mathfrak{g},[\lambda],w}$ is irreducible and Springer (see [Hot84] and [Jos85, Section 2.10]). Here an irreducible representation of W is said to be Springer if it corresponds to a trivial local system of a nilpotent orbit in $\mathfrak{g}^*$, under the Springer correspondence. Write $\mathcal{O}_{\mathfrak{g},[\lambda],w} \subseteq \mathfrak{g}^*$ for the corresponding nilpotent orbit:

$$\mathcal{O}_{\mathfrak{g},[\lambda],w} := \mathrm{Springer}^{-1}(\sigma_{\mathfrak{g},[\lambda],w}).$$

Lemma 2.18. *([Jos85, Theorem 3.9]) For every $w \in W$ and every $\nu \in [\lambda]$ that is regular and dominant, the associated variety of* $\mathrm{J}(w\nu)$ *equals the Zariski closure of $\mathcal{O}_{\mathfrak{g},[\lambda],w} \subseteq \mathfrak{g}^*$.*

2.7. **Cell representations in the coherent continuation setting.**

Definition 2.19. *Let E be a finite group. A basal representation V of E is basal vector space carrying a representation of E. A basal subrepresentation of a basal representation V is a subrepresentation of V that is simultaneously a basal subspace.*

For examples, $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ and $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ (recall that $w' \in W'_{[\lambda]}$) are basal representations of $W_{[\lambda]}$, and $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ is a basal representations of $W_{[\lambda]} \times W_{[\lambda]}$. Theorem 2.12 implies that the basal representations $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ and $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ only depend on the set of simple reflections in $W_{[\lambda]}$ (see also see [BV83a, Corollary 2.3]). In particular, they do not depend on $w' \in W'_{[\lambda]}$.

Let V be a basal representation of a finite group E, with basal elements $\mathcal{B} \subseteq V$. For each subset $\mathcal{C} \subseteq \mathcal{B}$, write $\langle \mathcal{C} \rangle$ for the smallest basal subrepresentation of V containing $\mathcal{C}$. For each $\phi \in \mathcal{B}$, write $\langle \phi \rangle := \langle \{\phi\} \rangle$ for simplicity. We define an equivalence relation $\approx$ on $\mathcal{B}$ by

$$\phi_1 \approx \phi_2 \quad \text{if and only if} \quad \langle \phi_1 \rangle = \langle \phi_2 \rangle \qquad (\phi_1, \phi_2 \in \mathcal{B}).$$

An equivalence class of the relation $\approx$ on the set $\mathcal{B}$ is called a cell in V.

We say that a subset $\mathcal{C}$ of $\mathcal{B}$ is order closed if for any $\phi \in \mathcal{B}$, we have $\phi \in \mathcal{C}$ whenever $\langle \phi' \rangle \leq \langle \phi \rangle$ for some $\phi' \in \mathcal{C}$. This is equivalent to saying that $\langle \mathcal{C} \rangle$ is spanned by $\mathcal{C}$. In general, write $\overline{\mathcal{C}}$ for the smallest order closed subset of $\mathcal{B}$ containing $\mathcal{C}$.

When $\mathcal{C}$ is a cell in V, define the cell representation attached to $\mathcal{C}$ by

$$V(\mathcal{C}) := \langle \overline{\mathcal{C}} \rangle / \langle \overline{\mathcal{C}} \setminus \mathcal{C} \rangle.$$

Note that the set $\overline{\mathcal{C}} \setminus \mathcal{C}$ is order closed, and $\{\phi + \langle \overline{\mathcal{C}} \setminus \mathcal{C} \rangle\}_{\phi \in \mathcal{C}}$ is a basis of $V(\mathcal{C})$.

We are particularly interested in the basal representation $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ of $W_{[\lambda]} \times W_{[\lambda]}$. Its set of basal elements is

$$\mathcal{B} := \{\overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_0} \mid w_0 \in W_{[\lambda]}\}.$$

We will write $\langle \mathcal{C} \rangle_{LR} := \langle \mathcal{C} \rangle$, the smallest basal subrepresentation of $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ containing $\mathcal{C}$, where $\mathcal{C}$ is a subset of $\mathcal{B}$. The corresponding order on the basis is denoted by $\underset{LR}{\leq}$.

Note that every irreducible representation of $W_{[\lambda]}$ is self-dual. Hence (2.5) implies that

$$\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b})) \cong \bigoplus_{\sigma \in \mathrm{Irr}(W_{[\lambda]})} \sigma \otimes \sigma \tag{2.13}$$

as representations of $W_{[\lambda]} \times W_{[\lambda]}$. By a double cell in $\mathrm{Irr}(W_{[\lambda]})$, we mean a set of the form

$$\{\sigma \in \mathrm{Irr}(W_{[\lambda]}) \mid \sigma \otimes \sigma \text{ occurs in the cell representation } \mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))(\mathcal{C})\},$$

where $\mathcal{C}$ is a cell in $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$. The isomorphism (2.13) implies that $\mathrm{Irr}(W_{[\lambda]})$ is disjoint union of all its double cells. As noted previously, Theorem 2.12 implies that the notion of double cells in $\mathrm{Irr}(W_{[\lambda]})$ only depends on the set of simple reflections in $W_{[\lambda]}$.

The following conjecture is widely anticipated, although to our knowledge no proof has appeared in the literature. See [Vog82, page 1055].

Conjecture 2.20. *For every cell $\mathcal{C}$ in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$, the set*

$$\{\sigma \in \mathrm{Irr}(W_{[\lambda]}) \mid \sigma \text{ occurs in the cell representation } \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))(\mathcal{C})\}$$

is contained in a single double cell in $\mathrm{Irr}(W_{[\lambda]})$.

Remark 2.21. We call a double cell in $\mathrm{Irr}(W_{[\lambda]})$ a Lusztig double cell. We also call a cell in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ a Harish-Chandra cell, and the associated cell representation a Harish-Chandra cell representation.

Remark 2.22. We note McGovern's observation ([McG98, Page 213]) which amounts to the assertion in Conjecture 2.20. It appears to us that the argument is inadequate as presented.

Recall that an irreducible representation $\sigma_0 \in \mathrm{Irr}(W_{[\lambda]})$ is special (in the sense of Lusztig) if and only if it is isomorphic to $\sigma_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0}$ for some $w_0 \in W_{[\lambda]}$ (see [BV83a, Theorem 1.1]). Let $\mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]}) \subseteq \mathrm{Irr}(W_{[\lambda]})$ denote the subset of special representations.

Lemma 2.23. *For every $\sigma_0 \in \mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]})$, the set*

$$\{\overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_0} \in \mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b})) \mid \sigma_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0} \cong \sigma_0\}$$

is a cell in $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$, and σ_0 is the unique special representation in the double cell in $\mathrm{Irr}(W_{[\lambda]})$ attached to this cell.

Proof. See [BV83a, Theorem 2.6 and Corollary 2.16] and [Vog82, Corollary 14.11]. Note that by Theorem 2.12, we may assume without loss of generality that λ is integral so that $\mathfrak{g} = \mathfrak{g}_{[\lambda]}$. □

For every cell $\mathcal{C}$ in $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$, write $\mathcal{O}_{\mathcal{C}} := \mathcal{O}_{\mathfrak{g},[\lambda],w'w_0}$, where w_0 is an element of $W_{[\lambda]}$ such that $\overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_0} \in \mathcal{C}$. By Lemma 2.23, this is independent of the choice of w_0.

Lemma 2.24. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two cells in $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ such that $\langle \mathcal{C}_1\rangle_{LR} \subseteq \langle \mathcal{C}_2\rangle_{LR}$. Then*

$$\overline{\mathcal{O}_{\mathcal{C}_1}} \subseteq \overline{\mathcal{O}_{\mathcal{C}_2}},$$

where $\overline{}$ indicates the Zariski closure.

Proof. For every basal element ϕ in $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$, write $\langle\phi\rangle_L$ for the smallest $W_{[\lambda]} \times \{1\}$-stable basal subspace of $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ containing ϕ, and likewise write $\langle\phi\rangle_R$ for the smallest $\{1\} \times W_{[\lambda]}$-stable basal subspace of $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ containing ϕ.

By the argument in [BV85, Section 3], it suffices to prove the following statement: if there exists $\phi_1 \in \mathcal{C}_1$ and $\phi_2 \in \mathcal{C}_2$ such that $\langle\phi_1\rangle_L \subseteq \langle\phi_2\rangle_L$ or $\langle\phi_1\rangle_R \subseteq \langle\phi_2\rangle_R$, then $\overline{\mathcal{O}_{\mathcal{C}_1}} \subseteq \overline{\mathcal{O}_{\mathcal{C}_2}}$.

Write $\phi_i = \overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_i}$, where $w_i \in W_{[\lambda]}$ $(i = 1, 2)$. Let ν be a regular dominant element in $[\lambda]$. We first assume that $\langle\phi_1\rangle_R \subseteq \langle\phi_2\rangle_R$. Note that the analogue of Lemma 2.6 in the setting of highest weight modules still holds (with the same proof). Thus Lemma 2.18 implies that

$$\phi_2 \in \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\overline{\mathcal{O}_{\mathcal{C}_2}},w'[\lambda]}(\mathfrak{g},\mathfrak{b})),$$

which further implies that

$$\phi_1 \in \langle\phi_1\rangle_R \subseteq \langle\phi_2\rangle_R \subseteq \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\overline{\mathcal{O}_{\mathcal{C}_2}},w'[\lambda]}(\mathfrak{g},\mathfrak{b})).$$

Thus $\mathrm{L}(\mathfrak{g},\mathfrak{b},w'w_1\nu) \in \mathcal{K}_{\overline{\mathcal{O}_{\mathcal{C}_2}},w'[\lambda]}(\mathfrak{g},\mathfrak{b})$, and Lemma 2.18 implies that $\overline{\mathcal{O}_{\mathcal{C}_1}} \subseteq \overline{\mathcal{O}_{\mathcal{C}_2}}$.

Now we assume that $\langle\phi_1\rangle_L \subseteq \langle\phi_2\rangle_L$. Put $\phi_i' := \overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_i^{-1}}$ $(i = 1, 2)$. By [Lus84, Lemma 5.2], we know that $\phi_i' \in \mathcal{C}_i$, and by the argument in [BV85, Section 3], $\langle\phi_1'\rangle_R \subseteq \langle\phi_2'\rangle_R$. Therefore $\overline{\mathcal{O}_{\mathcal{C}_1}} \subseteq \overline{\mathcal{O}_{\mathcal{C}_2}}$ by the earlier argument. This finishes the proof of the lemma. $\square$

We define a preorder $\underset{LR}{\leq}$ on $\mathrm{Irr}(W_{[\lambda]})$ such that $\sigma_1 \underset{LR}{\leq} \sigma_2$ $(\sigma_1, \sigma_2 \in \mathrm{Irr}(W_{[\lambda]}))$ if and only if for some cell $\mathcal{C}$ in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'\cdot[\lambda]}(\mathfrak{g},\mathfrak{b}))$,

$$\sigma_1 \otimes \sigma_1 \text{ occurs in } \langle\overline{\mathcal{C}}\rangle \,/\, \langle\overline{\mathcal{C}} \setminus \mathcal{C}\rangle \qquad \text{and} \qquad \sigma_2 \otimes \sigma_2 \text{ occurs in } \langle\overline{\mathcal{C}}\rangle\,.$$

We say $\sigma_1 \underset{LR}{\approx} \sigma_2$ if and only $\sigma_1 \underset{LR}{\leq} \sigma_2 \underset{LR}{\leq} \sigma_1$. Clearly double cells in $\mathrm{Irr}(W_{[\lambda]})$ are nothing but the equivalent classes of $\underset{LR}{\approx}$. As before, the preorder $\underset{LR}{\leq}$ on $\mathrm{Irr}(W_{[\lambda]})$ only depends on the set of simple reflections in $W_{[\lambda]}$.

Lemma 2.25. *Let J_1 and J_2 be two primitive ideals of $\mathcal{U}(\mathfrak{g})$ whose infinitesimal characters are represented by a common element in $[\lambda]$. Suppose that $J_1 \subsetneq J_2$. Then*

- *the associated variety of J_1 properly contains that of J_2; and*
- *$\sigma_{[\lambda],J_1} \underset{LR}{\leq} \sigma_{[\lambda],J_2}$ and $\sigma_{[\lambda],J_1} \neq \sigma_{[\lambda],J_2}$.*

Proof. The first statement is [BK76, Korollar 3.6]. For the second one, by using the coherent continuation representation as in (2.7), we assume without loss of generality that J_1 and J_2 have the same regular infinitesimal character in $[\lambda]$. Now the statement is proved in [BV83a, Proof of Proposition 3.18 (c)]. $\square$

Lemma 2.26. *(*[BV83a, Lemma 3.22]*) Let $\sigma_1, \sigma_2 \in \mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]})$. If $\sigma_1 \underset{LR}{\leq} \sigma_2$ and $\sigma_1 \neq \sigma_2$, then $a(\sigma_1) < a(\sigma_2)$.*

2.8. **Proof of Theorem 1.3 via $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b}))$.** Similar to (2.3), we have a decomposition (as $W_{[\lambda]}$-modules)

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b})) = \bigoplus_{w'\in W/W_{[\lambda]}} \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b})), \tag{2.14}$$

where $\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b})$ is the Grothendieck group of $\mathrm{Rep}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b})$, and the latter is the full subcategory of $\mathrm{Rep}(\mathfrak{g},\mathfrak{b})$ whose objects are modules that belong to both $\mathrm{Rep}_{\mathsf{S}}(\mathfrak{g},\mathfrak{b})$ and $\mathrm{Rep}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b})$.

Using Lemma 2.18, the same argument as the proof of (2.1) shows that $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ is a basal subrepresentation of $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ spanned by

$$\{\overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_0} \mid w_0 \in W_{[\lambda]},\, \mathcal{O}_{\mathfrak{g},[\lambda],w'w_0} \subseteq \mathsf{S}\}.$$

Put

$$\mathrm{Irr}^{\mathrm{sp}}_{\mathsf{S}}(W_{[\lambda]}) := \{\sigma_0 \in \mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]}) \mid \mathrm{Springer}^{-1}(j^W_{W_{[\lambda]}}\sigma_0) \subseteq \mathsf{S}\},$$

and (2.12) implies that $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ is spanned by

$$\{\overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_0} \mid w_0 \in W_{[\lambda]},\, \sigma_{\mathfrak{g}_{[\lambda]},Q_{[\lambda]},w_0} \in \mathrm{Irr}^{\mathrm{sp}}_{\mathsf{S}}(W_{[\lambda]})\}. \tag{2.15}$$

Recall the set $\mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})$ defined in (1.1), which equals

$$\{\sigma \in \mathrm{Irr}(W_{[\lambda]}) \mid \sigma \text{ lies in the same double cell with an element of } \mathrm{Irr}^{\mathrm{sp}}_{\mathsf{S}}(W_{[\lambda]})\}.$$

Lemma 2.24 implies that $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ is a subrepresentation of $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$.

Proposition 2.27. *As a representation of $W_{[\lambda]} \times W_{[\lambda]}$, we have*

$$\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b})) \cong \bigoplus_{\sigma\in\mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})} \sigma\otimes\sigma.$$

Proof. Recall that

$$\mathcal{B} := \{\overline{\Psi}_{\mathfrak{g},\mathfrak{b},[\lambda],w'w_0} \mid w_0 \in W_{[\lambda]}\}.$$

Denote by $\mathcal{B}_0$ the set (2.15), which is a basis of $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$. By Lemma 2.23, it is a union of cells in $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$. Choose a filtration

$$\mathcal{B}_0 \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_k = \emptyset \qquad (k \in \mathbb{N})$$

of $\mathcal{B}_0$ such that

- $\mathcal{C}_i := \mathcal{B}_i \setminus \mathcal{B}_{i+1}$ is a cell in $\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))$ for all $i = 0, 1, \ldots, k-1$, and
- for all $i_1, i_2 \in \{0, 1, \ldots, k-1\}$, $\langle \mathcal{C}_{i_1}\rangle_{LR} \subseteq \langle \mathcal{C}_{i_2}\rangle_{LR}$ implies that $i_1 \geq i_2$.

Then it is elementary to see that $\langle \mathcal{B}_i\rangle_{LR}$ is spanned by $\mathcal{B}_i$ ($i = 0, 1, 2, \cdots, k$). Let $\sigma_i \in \mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]})$ denote the unique special representation such that $\sigma_i \otimes \sigma_i$ occurs in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{w'[\lambda]}(\mathfrak{g},\mathfrak{b}))(\mathcal{C}_i)$. Then Lemmas 2.18 and 2.23 imply that

$$\{\sigma_0, \sigma_1, \ldots, \sigma_{k-1}\} = \mathrm{Irr}^{\mathrm{sp}}_{\mathsf{S}}(W_{[\lambda]}).$$

Now we have that

$$\begin{aligned}
&\mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b}))\\
\cong\ & \bigoplus_{i=0}^{k-1} \langle \mathcal{B}_i\rangle_{LR}/\langle \mathcal{B}_{i+1}\rangle_{LR}\\
\cong\ & \bigoplus_{i=0}^{k-1} \mathrm{Coh}^{LR}_{[\lambda]}(\mathcal{K}_{\mathsf{S},w'[\lambda]}(\mathfrak{g},\mathfrak{b}))(\mathcal{C}_i)\\
\cong\ & \bigoplus_{\sigma \in \mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})} \sigma \otimes \sigma.
\end{aligned}$$

□

Finally, Theorem 1.3 follows from Proposition 2.11, (2.14), and Proposition 2.27.

2.9. **Proof of Theorem 1.4.** Recall that for every basal element $\Phi \in \mathcal{B}_{[\lambda]}(G)$,

$$\mathrm{AV}_{\mathbb{C}}(\Phi) = \mathrm{AV}_{\mathbb{C}}(\Phi(\nu)) \quad \text{for all } \nu \in [\lambda] \text{ that is regular and dominant.}$$

Recall the definition of fake degree in (2.8). The following lemma is due to King [Kin81]. See also [Vog82, Corollary 14.11] and [Cas86, Theorem 3.4].

Lemma 2.28. *For every Harish-Chandra cell $\mathcal{C}$ in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$, there is a unique representation $\sigma_{\mathcal{C}} \in \mathrm{Irr}(W_{[\lambda]})$ whose fake degree is minimal among all representations occurring in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))(\mathcal{C})$. Moreover, for every $\Phi \in \mathcal{C}$, $\sigma_{\mathcal{C}}$ is the Goldie rank representation attached to the primitive ideal $\mathrm{Ann}(\Phi(\nu))$ for any regular dominant $\nu \in [\lambda]$. In particular, $\sigma_{\mathcal{C}}$ is special and*

$$\mathrm{AV}_{\mathbb{C}}(\Phi) = \mathrm{Springer}^{-1}(j^W_{W_{[\lambda]}}\sigma_{\mathcal{C}})$$

for every $\Phi \in \mathcal{C}$. □

Lemma 2.29. *Assume that for every Harish-Chandra cell $\mathcal{C}$ in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$, the set*

$$\{\sigma \in \mathrm{Irr}(W_{[\lambda]}) \mid \sigma \text{ occurs in } \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))(\mathcal{C})\}$$

is contained in a single double cell in $\mathrm{Irr}(W_{[\lambda]})$. Then for every $\sigma \in \mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})$,

$$[\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))] = [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))]. \tag{2.16}$$

Proof. Note that

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G)) \cong \bigoplus_{\mathcal{C} \text{ is a cell in } \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))} \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))(\mathcal{C}),$$

and Lemma 2.28 implies that

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G)) \cong \bigoplus_{\mathcal{C} \text{ is a cell in } \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G)),\ \sigma_{\mathcal{C}} \in \mathrm{Irr}^{\mathrm{sp}}_{\mathsf{S}}(W_{[\lambda]})} \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))(\mathcal{C}).$$

Then Lemma 2.28 and the assumption of the lemma imply (2.16). □

Under the assumption of Lemma 2.29, we have that

$$\begin{aligned} & \sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) \\ = & \sum_{\sigma \in \mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})} [1_{W_\lambda} : \sigma] \cdot [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G))] \quad \text{(by (1.2))} \\ = & \sum_{\sigma \in \mathrm{Irr}_{\mathsf{S}}(W_{[\lambda]})} [1_{W_\lambda} : \sigma] \cdot [\sigma : \mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))] \quad \text{(by Lemma 2.29)}. \end{aligned}$$

This completes the proof of Theorem 1.4.

2.10. **A formula of $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ à la Barbasch and Vogan.** Let H be a Cartan subgroup of G with complexified Lie algebra $\mathfrak{h}$. Write $\mathfrak{t}$ for the complexified Lie algebra of the unique maximal compact subgroup of H. As before, denote by $\Delta_{\mathfrak{h}} \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{g}$. A root $\alpha \in \Delta_{\mathfrak{h}}$ is called imaginary if $\alpha^\vee \in \mathfrak{t}$. An imaginary root $\alpha \in \Delta_{\mathfrak{h}}$ is said to be compact if the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are contained in the complexified Lie algebra of a common compact subgroup of G.

Note that every Casselman-Wallach representation of H is finite dimensional, and for every $\Gamma \in \mathrm{Irr}(H)$, there is a unique element $\mathrm{d}\Gamma \in \mathfrak{h}^*$ such that the differential of Γ is isomorphic to a direct sum of one-dimensional representations of $\mathfrak{h}$ attached to $\mathrm{d}\Gamma$.

For every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$, write

$$\xi_{\mathfrak{b}} : {}^a\mathfrak{h} \to \mathfrak{h}$$

for the linear isomorphism attached to $\mathfrak{b}$, whose transpose inverse is still denoted by $\xi_{\mathfrak{b}} : {}^a\mathfrak{h}^* \to \mathfrak{h}^*$. Recall that $\Delta^+ \subseteq {}^a\mathfrak{h}^*$ denotes the set of positive roots. For every element w in the abstract Weyl group W, put

$$\delta(w, \mathfrak{b}) := \frac{1}{2} \cdot \sum_{\alpha \text{ is an imaginary root in } \xi_{\mathfrak{b}} w \Delta^+} \alpha - \sum_{\beta \text{ is a compact imaginary root in } \xi_{\mathfrak{b}} w \Delta^+} \beta \in \mathfrak{h}^*.$$

Denote by $Q_{\mathfrak{h}} \subseteq \mathfrak{h}^*$ the root lattice of $\mathfrak{g}$. For every $\nu \in \mathfrak{h}^*$, put $[\nu] := \nu + Q_{\mathfrak{h}}$. Write $\widetilde{\mathrm{Irr}}_{[\lambda]}(H)$ for the set of all triples $\gamma = (\mathfrak{b}, \Gamma, \nu)$ with the following properties:

- $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, $\Gamma \in \mathrm{Irr}(H)$, and $\nu \in \mathfrak{h}^*$;
- $\mathrm{d}\Gamma - \nu = \delta(1, \mathfrak{b})$;
- $\xi_{\mathfrak{b}}([\lambda]) = [\nu]$.

Define the real Weyl group

$$W_H := (\text{the normalizer of } \mathfrak{h} \text{ in } G)/H.$$

As usual, $Q_{\mathfrak{h}}$ is identified with a group of characters on H. Define an action of the group $W_H \ltimes Q_{\mathfrak{h}}$ on the set $\widetilde{\mathrm{Irr}}_{[\lambda]}(H)$ by requiring that

$$\beta \cdot (\mathfrak{b}, \Gamma, \nu) := (\mathfrak{b}, \beta \otimes \Gamma, \beta + \nu)$$

and

$$g\cdot(\mathfrak{b},\Gamma,\nu):=(g\cdot\mathfrak{b},g\cdot\Gamma,g\cdot\nu)$$

for all $\beta\in Q_{\mathfrak{h}}$, $g\in W_H$ and $(\mathfrak{b},\gamma,\nu)\in\widetilde{\mathrm{Irr}}_{[\lambda]}(H)$.

On the other hand, we have the cross action of $W_{[\lambda]}$ on the set $\widetilde{\mathrm{Irr}}_{[\lambda]}(H)$ (see [Vog81, Definition 8.3.1]):

$$\begin{aligned}&w\times(\mathfrak{b},\Gamma,\nu)\\:=\ &((\xi_{\mathfrak{b}}w^{-1}\xi_{\mathfrak{b}}^{-1})\cdot\mathfrak{b},\ ((\xi_{\mathfrak{b}}w^{-1}\xi_{\mathfrak{b}}^{-1})\cdot\nu-\nu)\otimes(\delta(w^{-1},\mathfrak{b})-\delta(1,\mathfrak{b}))\otimes\Gamma,\ (\xi_{\mathfrak{b}}w^{-1}\xi_{\mathfrak{b}}^{-1})\cdot\nu),\end{aligned}$$

for all $w\in W_{[\lambda]}$ and $(\mathfrak{b},\Gamma,\nu)\in\widetilde{\mathrm{Irr}}_{[\lambda]}(H)$. It is routine to check that these two actions are well-defined. Moreover, the actions of $W_{[\lambda]}$ and W_H commute with each other, and

$$w\cdot(\beta\cdot(w^{-1}\cdot(\mathfrak{b},\Gamma,\nu)))=((\xi_{\mathfrak{b}}w\xi_{\mathfrak{b}}^{-1})\cdot\beta)\cdot(\mathfrak{b},\Gamma,\nu)$$

for all $w\in W_{[\lambda]}$, $\beta\in Q_{\mathfrak{h}}$ and $(\mathfrak{b},\Gamma,\nu)\in\widetilde{\mathrm{Irr}}_{[\lambda]}(H)$. Thus the cross action descends to an action of $W_{[\lambda]}$ on the quotient set

$$\overline{\mathrm{Irr}}_{[\lambda]}(H):=(W_H\ltimes Q_{\mathfrak{h}})\backslash(\widetilde{\mathrm{Irr}}_{[\lambda]}(H)).$$

We define the set of parameters for the $\mathcal{K}(G)$-valued coherent families on $[\lambda]$ to be

$$\mathcal{P}_{[\lambda]}(G):=\bigsqcup_H\overline{\mathrm{Irr}}_{[\lambda]}(H),\tag{2.17}$$

where H runs over a representative set of conjugacy classes of Cartan subgroups of G. For each $\overline{\gamma}\in\mathcal{P}_{[\lambda]}(G)$, we have two $\mathcal{K}(G)$-valued coherent families $\Psi_{\overline{\gamma}}$ and $\overline{\Psi}_{\overline{\gamma}}$ on $[\lambda]$ such that

$$\Psi_{\overline{\gamma}}(\mu)=\mathrm{X}((\Gamma,\nu))\quad\text{and}\quad\overline{\Psi}_{\overline{\gamma}}(\mu)=\overline{\mathrm{X}}((\Gamma,\nu))\tag{2.18}$$

for all regular dominant element $\mu\in[\lambda]$ and all $(\mathfrak{b},\Gamma,\nu)\in\overline{\gamma}$ such that $\xi_{\mathfrak{b}}(\mu)=\nu$. Here $\mathrm{X}((\Gamma,\nu))$ is the standard representation defined in [Vog81, Notational Convention 6.6.3] and $\overline{\mathrm{X}}((\Gamma,\nu)))$ is its unique irreducible subrepresentation. See [Vog81, Theorem 6.5.10 and Theorem 7.2.10].

By Langlands classification, $\{\overline{\Psi}_{\overline{\gamma}}\}_{\overline{\gamma}\in\mathcal{P}_{[\lambda]}(G)}$ is the set of basal elements in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G))$ (see [Vog81, Theorem 6.6.14]). The set $\{\Psi_{\overline{\gamma}}\}_{\overline{\gamma}\in\mathcal{P}_{[\lambda]}(G)}$ is also a basis ([Vog81, Proposition 6.6.7]).

Since G is in the Harish-Chandra's class, the real Weyl group W_H is identified with a subgroup of the Weyl group $W_{\mathfrak{h}}$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Write $\mathfrak{t}_{\mathrm{im}}$ for the subspace of $\mathfrak{t}$ spanned by the set $\{\alpha^\vee\mid\alpha\text{ is an imaginary root in }\Delta_{\mathfrak{h}}\}$. Then W_H stabilizers $\mathfrak{t}_{\mathrm{im}}$, and we define a character

$$\mathrm{sgn}_{\mathrm{im}}:W_H\to\mathbb{C}^\times,\quad w\mapsto(\text{the determinant of the map }w:\mathfrak{t}_{\mathrm{im}}\to\mathfrak{t}_{\mathrm{im}}).$$

This is a quadratic character.

Let $\overline{\gamma}\in\overline{\mathrm{Irr}}_{[\lambda]}(H)$. Write $W_{\overline{\gamma}}$ for the stabilizer of $\overline{\gamma}$ in $W_{[\lambda]}$. Pick a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$ that extends to a triple $(\mathfrak{b},\gamma,\nu)\in\overline{\gamma}$, which is unique up to the action of W_H. It is clear that $\xi_{\mathfrak{b}}w\xi_{\mathfrak{b}}^{-1}\in W_H$ for all $w\in W_{\overline{\gamma}}$. We define a quadratic character

$$\mathrm{sgn}_{\overline{\gamma}}:W_{\overline{\gamma}}\to\mathbb{C}^\times,\qquad w\mapsto\mathrm{sgn}_{\mathrm{im}}(\xi_{\mathfrak{b}}w\xi_{\mathfrak{b}}^{-1}).$$

This is independent of the choice of $\mathfrak{b}$.

The coherent continuation representation may be computed by the use of the basis $\{\Psi_{\overline{\gamma}}\}_{\overline{\gamma}\in\mathcal{P}_{[\lambda]}(G)}$ (see [Vog82, Section 14]). The following result is due to Barbasch-Vogan, in a suitably modified form from [BV83b, Proposition 2.4]. As its proof follows the same line as that of [BV83b, Proposition 2.4], we will be content to state the precise result.

Theorem 2.30 (*cf.* [BV83b, Proposition 2.4])**.** *As a representation of $W_{[\lambda]}$,*

$$\mathrm{Coh}_{[\lambda]}(\mathcal{K}(G)) \cong \bigoplus_{\overline{\gamma}} \mathrm{Ind}_{W_{\overline{\gamma}}}^{W_{[\lambda]}} \mathrm{sgn}_{\overline{\gamma}},$$

where $\overline{\gamma}$ runs over a representative set of the $W_{[\lambda]}$-orbits in $\mathcal{P}_{[\lambda]}(G)$ under the cross action.

3. Explicit calculation of the coherent continuation representation

In the rest of the paper we will freely use the notation of Sections 1.4-1.7. Recall the element $\lambda_{\check{\mathcal{O}}} \in W\backslash{}^a\mathfrak{h}^*$. We pick an arbitrary representative in ${}^a\mathfrak{h}^*$, which we still write as $\lambda_{\check{\mathcal{O}}}$. If $\star \in \{\widetilde{A}, \widetilde{C}\}$, we let $\mathcal{K}'(G)$ denote the Grothendieck group of the category of genuine Casselman-Wallach representations of G. Otherwise, put $\mathcal{K}'(G) := \mathcal{K}(G)$. Similar notations such as $\mathcal{K}'_{\mathsf{S}}(G)$ and $\mathcal{K}'_{\lambda,\mathsf{S}}(G)$ will be used without further explanation.

Recall the Barbasch-Vogan duality map ([BV85, Appendix] and [BMSZ20, Section 1])

$$\begin{array}{rcl} d_{\mathrm{BV}} : \overline{\mathrm{Nil}}(\check{\mathfrak{g}}) & \rightarrow & \overline{\mathrm{Nil}}(\mathfrak{g}^*), \\ \check{\mathcal{O}} & \mapsto & \text{the unique open } \mathrm{Inn}(\mathfrak{g})\text{-orbit in the associated variety of } I_{\star,\check{\mathcal{O}}}, \end{array}$$

where $\overline{\mathrm{Nil}}(\check{\mathfrak{g}})$ denotes the set of $\check{G}$-orbits in $\mathrm{Nil}(\check{\mathfrak{g}})$, and $\overline{\mathrm{Nil}}(\mathfrak{g}^*)$ denotes the set of $\mathrm{Inn}(\mathfrak{g})$-orbits in $\mathrm{Nil}(\mathfrak{g}^*)$.

As an obvious variant of Corollary 1.6 for $\lambda = \lambda_{\check{\mathcal{O}}}$, we have that

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sum_{\sigma \in {}^L\mathscr{C}_{\lambda_{\check{\mathcal{O}}}}} [\sigma : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G))] \leq \sum_{\sigma \in {}^L\mathscr{C}_{\lambda_{\check{\mathcal{O}}}}} [\sigma : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))], \tag{3.1}$$

and the equality holds if the analogous condition for a Harish-Chandra cell in $\mathrm{Coh}_{[\lambda]}(\mathcal{K}'(G))$ holds. Here $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}})$ and $\overline{\mathcal{O}}$ denotes its Zariski closure.

3.1. **Some Weyl groups and quadratic characters.**

Let S_n denote the permutation group of the set $\{1, 2, \ldots, n\}$ $(n \geq 0)$, which is identified with the Weyl group of $\mathfrak{gl}_n(\mathbb{C})$. Put $\mathsf{W}_n := \mathsf{S}_n \ltimes \{\pm 1\}^n$, which is identified with the Weyl group of type B_n or C_n.

Define a quadratic character

$$\varepsilon : \mathsf{W}_n \to \{\pm 1\}, \quad (s, (x_1, x_2, \cdots, x_n)) \mapsto x_1 x_2 \cdots x_n. \tag{3.2}$$

Let W'_n denote the kernel of this character, which is identified with the Weyl group of type D_n. As always, sgn denotes the sign character (of an appropriate Weyl group). Since S_n is a quotient of W_n, we may inflate the sign character of S_n to obtain a character of W_n, to be denoted by $\overline{\mathrm{sgn}}$. Then we have that

$$\varepsilon = \mathrm{sgn} \otimes \overline{\mathrm{sgn}}.$$

The group W_n is naturally embedded in S_{2n} via the homomorphism determined by

$$(i, i+1) \in \mathsf{S}_n \mapsto (2i-1, 2i+1)(2i, 2i+2), \qquad (1 \leq i \leq n-1), \tag{3.3}$$

and

$$(1, \cdots, 1, \underbrace{-1}_{j\text{-th term}}, 1, \cdots, 1) \in \{\pm 1\}^n \mapsto (2j-1, 2j), \qquad (1 \leq j \leq n). \tag{3.4}$$

Here $(i, i+1)$, $(2i-1, 2i+1)$, etc., indicate the involutions in the permutation groups. Note that ε is also the restriction of sgn of S_{2n} to W_n.

We view W_n and S_n as reflection groups acting on $\mathbb{C}^n$ as usual. Let $\mathsf{H}_n := \mathsf{W}_n \ltimes \{\pm 1\}^n$, to be viewed as a subgroup in W_{2n} such that

- the first factor W_n sits in $\mathsf{S}_{2n} \subseteq \mathsf{W}_{2n}$ as in (3.3) and (3.4),

- the element $(1,\cdots,1,\underbrace{-1}_{i\text{-th term}},1,\cdots,1)\in\{\pm1\}^n$ acts on $\mathbb{C}^{2n}$ by

$$(x_1,x_2,\cdots,x_{2n})\mapsto(x_1,\cdots,x_{2i-2},-x_{2i},-x_{2i-1},x_{2i+1},\cdots,x_{2n}).$$

Note that H_n is also a subgroup of W'_{2n}. Define a quadratic character

$$\begin{array}{rcl}\tilde{\varepsilon}:\mathsf{H}_n=\mathsf{W}_t\ltimes\{\pm1\}^t & \to & \{\pm1\},\\ (g,(a_1,a_2,\cdots,a_t)) & \mapsto & a_1a_2\cdots a_n.\end{array}$$

3.2. **Coherent continuation representations in type A.** Suppose that $\star\in\{A^{\mathbb{R}},A^{\mathbb{H}},A,\widetilde{A}\}$ for the moment. We call an integer to have the good parity (depends on $\star$ and $\check{G}$), if it has the same parity as

$$\begin{cases}\text{the parity of the rank of } \check{G}, & \text{if } \star\in\{A^{\mathbb{R}},A^{\mathbb{H}},A\};\\ 1+\text{ the parity of the rank of } \check{G}, & \text{if } \star=\widetilde{A}.\end{cases}$$

The other parity is called the bad parity. We have a decomposition

$$\check{\mathcal{O}}=\check{\mathcal{O}}_{\mathrm{g}}\overset{r}{\sqcup}\check{\mathcal{O}}_{\mathrm{b}}$$

of Young diagram, where every nonzero row length of $\check{\mathcal{O}}_{\mathrm{g}}$ and $\check{\mathcal{O}}_{\mathrm{b}}$ has respectively good and bad parity. Then

$$W_{[\lambda_{\check{\mathcal{O}}}]}=\mathsf{S}_{|\check{\mathcal{O}}_{\mathrm{g}}|}\times\mathsf{S}_{|\check{\mathcal{O}}_{\mathrm{b}}|}.$$

The proof of the following Propositions 3.1-3.4 follows from Theorem 2.30 by direct computation. We omit the details.

Proposition 3.1. *Suppose that $\star=A^{\mathbb{R}}$. Let*

$$\mathcal{C}_l:=\bigoplus_{\substack{t,c,d\in\mathbb{N}\\2t+c+d=l}}\mathrm{Ind}_{\mathsf{W}_t\times\mathsf{S}_c\times\mathsf{S}_d}^{\mathsf{S}_l}\varepsilon\otimes1\otimes1.$$

Then as $W_{[\lambda_{\check{\mathcal{O}}}]}$-modules,

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G))\cong\mathcal{C}_{|\check{\mathcal{O}}_{\mathrm{g}}|}\otimes\mathcal{C}_{|\check{\mathcal{O}}_{\mathrm{b}}|}.$$

Proposition 3.2. *Suppose that $\star=A^{\mathbb{H}}$. Let*

$$\mathcal{C}_l:=\begin{cases}\mathrm{Ind}_{\mathsf{W}_{\frac{l}{2}}}^{\mathsf{S}_l}\epsilon, & \textit{if } l \textit{ is even;}\\ 0, & \textit{otherwise.}\end{cases}$$

Then as $W_{[\lambda_{\check{\mathcal{O}}}]}$-modules,

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G))\cong\mathcal{C}_{|\check{\mathcal{O}}_e|}\otimes\mathcal{C}_{|\check{\mathcal{O}}_o|}.$$

Proposition 3.3. *Suppose that $\star\in\{A,\widetilde{A}\}$ so that $G=\mathrm{U}(p,q)$ or $\widetilde{\mathrm{U}}(p,q)$. Let*

$$\mathcal{C}_{p,q}:=\bigoplus_{\substack{t,s,r\in\mathbb{N}\\t+r=p,t+s=q}}\mathrm{Ind}_{\mathsf{W}_t\times\mathsf{S}_s\times\mathsf{S}_r}^{\mathsf{S}_{p+q}}1\otimes\mathrm{sgn}\otimes\mathrm{sgn},$$

$$\mathcal{C}_{\mathrm{b}}:=\begin{cases}\mathrm{Ind}_{\mathsf{W}_{\frac{|\check{\mathcal{O}}_{\mathrm{b}}|}{2}}}^{\mathsf{S}_{|\check{\mathcal{O}}_{\mathrm{b}}|}}1, & \textit{if } |\check{\mathcal{O}}_{\mathrm{b}}| \textit{ is even;}\\ 0, & \textit{otherwise.}\end{cases}$$

Then as $W_{[\lambda_{\check{\mathcal{O}}}]}$-modules,

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))\cong\begin{cases}\mathcal{C}_{p-\frac{1}{2}|\check{\mathcal{O}}_{\mathrm{b}}|,q-\frac{1}{2}|\check{\mathcal{O}}_{\mathrm{b}}|}\otimes\mathcal{C}_{\mathrm{b}}, & \textit{if } |\check{\mathcal{O}}_{\mathrm{b}}| \textit{ is even and } p,q\geq\frac{1}{2}|\check{\mathcal{O}}_{\mathrm{b}}|;\\ 0, & \textit{otherwise.}\end{cases}$$

3.3. **Coherent continuation representations in type BCD.** In this subsection, we assume that $\star \in \{ B, \widetilde{C}, C, D, C^*, D^* \}$. Recall the notion of good parity (depends on $\star$):

$$\text{the good parity} = \begin{cases} \text{odd}, & \text{when } \star \in \{ C, C^*, D, D^* \}; \\ \text{even}, & \text{when } \star \in \{ B, \widetilde{C} \}. \end{cases}$$

The other parity is called the bad parity.

Recall the nilpotent $\check{G}$-orbit $\check{\mathcal{O}} \subseteq \mathrm{Nil}(\check{\mathfrak{g}})$. Recall the semisimple element $\lambda_{\check{\mathcal{O}}} \in \check{\mathfrak{g}}$ which is uniquely determined by $\check{\mathcal{O}}$ up to $\check{G}$-conjugation. Put $e_{\check{\mathcal{O}}} := \exp(2\pi\sqrt{-1}\lambda_{\check{\mathcal{O}}})$, whose square equals the identity element of $\check{G}$ (here π is the the ratio of the circumference, and $\exp : \check{\mathfrak{g}} \to \check{G}$ is the exponential map). Let $\check{V}$ denote the standard representation of $\check{G}$, which is equipped with a $\check{G}$-invariant bilinear form that is symmetric or skew-symmetric. We have an orthogonal decomposition

$$\check{V} = \check{V}_{\mathrm{g}} \oplus \check{V}_{\mathrm{b}},$$

where

$$\check{V}_{\mathrm{g}} := \begin{cases} \text{the eighenspace of } e_{\check{\mathcal{O}}} \text{ with eigenvalue } 1, & \text{if } \check{G} \text{ is special orthogonal ;} \\ \text{the eighenspace of } e_{\check{\mathcal{O}}} \text{ with eigenvalue } -1, & \text{if } \check{G} \text{ is symplectic} \end{cases}$$

and

$$\check{V}_{\mathrm{b}} := \begin{cases} \text{the eighenspace of } e_{\check{\mathcal{O}}} \text{ with eigenvalue } -1, & \text{if } \check{G} \text{ is special orthogonal;} \\ \text{the eighenspace of } e_{\check{\mathcal{O}}} \text{ with eigenvalue } 1, & \text{if } \check{G} \text{ is symplectic.} \end{cases}$$

Write $\check{G}_{\mathrm{g}}$ and $\check{G}_{\mathrm{b}}$ for the isometry groups of $\check{V}_{\mathrm{g}}$ and $\check{V}_{\mathrm{b}}$ respectively. Then we have identification

$$(\check{G}_{\mathrm{g}}, \check{G}_{\mathrm{b}}) = \begin{cases} (\mathrm{Sp}_{2n_{\mathrm{g}}}(\mathbb{C}), \mathrm{Sp}_{2n_{\mathrm{b}}}(\mathbb{C})), & \text{if } \star \in \{ B, \widetilde{C} \}; \\ (\mathrm{SO}_{2n_{\mathrm{g}}+1}(\mathbb{C}), \mathrm{SO}_{2n_{\mathrm{b}}}(\mathbb{C})), & \text{if } \star \in \{ C, C^* \}; \\ (\mathrm{SO}_{2n_{\mathrm{g}}}(\mathbb{C}), \mathrm{SO}_{2n_{\mathrm{b}}}(\mathbb{C})), & \text{if } \star \in \{ D, D^* \}, \end{cases}$$

where $2n_{\mathrm{b}}$ equals the sum of the bad row lengths of $\check{\mathcal{O}}$, and $n_{\mathrm{b}} + n_{\mathrm{g}}$ equals the rank of $\check{\mathfrak{g}}$.

Note that $\lambda_{\check{\mathcal{O}}} \in \check{\mathfrak{g}}_{\mathrm{g}} \times \check{\mathfrak{g}}_{\mathrm{b}} \subseteq \check{\mathfrak{g}}$, where $\check{\mathfrak{g}}_{\mathrm{g}}$ and $\check{\mathfrak{g}}_{\mathrm{b}}$ denote the Lie algebras of $\check{G}_{\mathrm{g}}$ and $\check{G}_{\mathrm{b}}$, respectively. We have a unique $\check{G}_{\mathrm{g}}$-orbit $\check{\mathcal{O}}_{\mathrm{g}} \subseteq \mathrm{Nil}(\check{\mathfrak{g}}_{\mathrm{g}})$ and a unique $\check{G}_{\mathrm{b}}$-orbit in $\check{\mathcal{O}}_{\mathrm{b}} \subseteq \mathrm{Nil}(\check{\mathfrak{g}}_{\mathrm{b}})$ such that $\lambda_{\check{\mathcal{O}}_{\mathrm{g}}} + \lambda_{\check{\mathcal{O}}_{\mathrm{b}}}$ is $(\check{G}_{\mathrm{g}} \times \check{G}_{\mathrm{b}})$-conjugate to $\lambda_{\check{\mathcal{O}}}$. Moreover,

$$\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}} \overset{r}{\sqcup} \check{\mathcal{O}}_{\mathrm{b}}$$

as Young diagrams, and $\check{\mathcal{O}}_{\mathrm{b}} = 2\check{\mathcal{O}}'_{\mathrm{b}}$ for a unique Young diagram $\check{\mathcal{O}}'_{\mathrm{b}}$. The orbit $\check{\mathcal{O}}_{\mathrm{g}}$ has good parity in the sense that all its nonzero row lengths has good parity, and likewise the orbit $\check{\mathcal{O}}_{\mathrm{b}}$ has bad parity. The integral Weyl group $W_{[\lambda_{\check{\mathcal{O}}}]} \subseteq \mathrm{GL}({}^a\mathfrak{h})$ is a direct product:

$$W_{[\lambda_{\check{\mathcal{O}}}]} = W_{[\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}]} \times W_{[\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}]} = W_{\mathrm{g}} \times W_{\mathrm{b}}, \tag{3.5}$$

where

$$\begin{aligned} W_{\mathrm{g}} &:= \begin{cases} \mathsf{W}_{n_{\mathrm{g}}}, & \text{if } \star \in \{ B, C, C^* \}; \\ \mathsf{W}'_{n_{\mathrm{g}}}, & \text{if } \star \in \{ \widetilde{C}, D, D^* \}, \end{cases} \\ W_{\mathrm{b}} &:= \begin{cases} \mathsf{W}_{n_{\mathrm{b}}}, & \text{if } \star \in \{ B, \widetilde{C} \}; \\ \mathsf{W}'_{n_{\mathrm{b}}}, & \text{if } \star \in \{ C, C^*, D, D^* \}. \end{cases} \end{aligned} \tag{3.6}$$

We now define various Weyl group representations case by case, which will be used to state the formulas of coherent continuation representations. Let $n', p', q' \in \mathbb{N}$.

- Suppose that $\star = B$ and $p'+q'$ is odd. Define

$$\mathcal{C}_{\mathrm{b}}^{n'} := \bigoplus_{2t+c+d=n'} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{W}_c\times\mathsf{W}_d}^{\mathsf{W}_{n'}} \tilde{\varepsilon}\otimes 1\otimes 1,$$

$$\mathcal{C}_{\mathrm{g}}^{p',q'} := \bigoplus_{\substack{0\leq p'-(2t+a+2r)\leq 1,\\ 0\leq q'-(2t+a+2s)\leq 1}} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a\times\mathsf{W}_s\times\mathsf{W}_r}^{\mathsf{W}_{\frac{p'+q'-1}{2}}} \tilde{\varepsilon}\otimes 1\otimes \mathrm{sgn}\otimes\mathrm{sgn}\,.$$

- Suppose that $\star = C^*$. Define

$$\mathcal{C}_{\mathrm{b}}^{n'} := \begin{cases}\mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}_{n'}}\tilde{\varepsilon}, & \text{if } n'=2t \text{ is even;}\\ 0, & \text{otherwise,}\end{cases}$$

$$\mathcal{C}_{\mathrm{g}}^{2p',2q'} := \bigoplus_{(t+s,t+r)=(p',q')} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{W}_s\times\mathsf{W}_t}^{\mathsf{W}_{p'+q'}} \tilde{\varepsilon}\otimes\mathrm{sgn}\otimes\mathrm{sgn}\,.$$

- Suppose that $\star = C$. Define

$$\mathcal{C}_{\mathrm{b}}^{n'} := \bigoplus_{2t+a=n'} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a}^{\mathsf{W}_{n'}} \tilde{\varepsilon}\otimes 1,$$

$$\mathcal{C}_{\mathrm{g}}^{n',n'} := \bigoplus_{2t+a+c+d=n'} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a\times\mathsf{W}_c\times\mathsf{W}_d}^{W_{n'}} \tilde{\varepsilon}\otimes\mathrm{sgn}\otimes 1\otimes 1.$$

- Suppose that $\star = \widetilde{C}$. Define

$$\mathcal{C}_{\mathrm{b}}^{n'} := \bigoplus_{2t+c+d=n'} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{W}_c\times\mathsf{W}_d}^{\mathsf{W}_{n'}} \tilde{\varepsilon}\otimes 1\otimes 1,$$

$$\mathcal{C}_{\mathrm{g}}^{n',n'} := \bigoplus_{2t+a+a'=n'} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a\times\mathsf{S}_{a'}}^{\mathsf{W}_{n'}} \tilde{\varepsilon}\otimes\mathrm{sgn}\otimes 1.$$

- Suppose that $\star = D$ and $p'+q'$ is even. Define

$$\mathcal{C}_{\mathrm{b}}^{n'} := \bigoplus_{2t+a=n'} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a}^{\mathsf{W}_{n'}} \tilde{\varepsilon}\otimes 1,$$

$$\mathcal{C}_{\mathrm{g}}^{p',q'} := \bigoplus_{\substack{2t+c+d+2r=p'\\ 2t+c+d+2s=q'}} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{W}_s\times\mathsf{W}_r\times\mathsf{W}'_c\times\mathsf{W}_d}^{\mathsf{W}_{\frac{p'+q'}{2}}} \tilde{\varepsilon}\otimes\overline{\mathrm{sgn}}\otimes\overline{\mathrm{sgn}}\otimes 1\otimes 1.$$

- Suppose that $\star = D^*$. Define

$$\mathcal{C}_{\mathrm{b}}^{n'} := \begin{cases}\mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}'_{n'}}\tilde{\varepsilon}, & \text{if } n'=2t \text{ is even;}\\ 0, & \text{otherwise,}\end{cases}$$

$$\mathcal{C}_{\mathrm{g}}^{n',n'} := \bigoplus_{2t+a=n'} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a}^{\mathsf{W}'_{n'}} \tilde{\varepsilon}\otimes\mathrm{sgn}\,.$$

Recall that

$$G := \begin{cases}\mathrm{SO}(p,q), & \text{if } \star\in\{B,D\};\\ \mathrm{O}^*(2n), & \text{if } \star=D^*;\\ \mathrm{Sp}_{2n}(\mathbb{R}), & \text{if } \star=C;\\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}), & \text{if } \star=\widetilde{C};\\ \mathrm{Sp}(p,q), & \text{if } \star=C^*.\end{cases}$$

Recall from (1.8) the group

$$G'_{\mathrm{b}} := \begin{cases} \mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star \in \{ B, C, D \}; \\ \widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star = \widetilde{C}; \\ \mathrm{GL}_{\frac{n_{\mathrm{b}}}{2}}(\mathbb{H}), & \text{if } \star \in \{ C^*, D^* \}. \end{cases}$$

Define

$$(p_{\mathrm{g}}, q_{\mathrm{g}}) := \begin{cases} (p - n_{\mathrm{b}}, q - n_{\mathrm{b}}), & \text{if } \star \in \{ B, D \}; \\ (n - n_{\mathrm{b}}, n - n_{\mathrm{b}}), & \text{if } \star \in \{ C, \widetilde{C}, D^* \}; \\ (2p - n_{\mathrm{b}}, 2q - n_{\mathrm{b}}), & \text{if } \star = C^*. \end{cases}$$

Then G has a closed subgroup isomorphic to G'_{b} if and only if $p_{\mathrm{g}}, q_{\mathrm{g}} \geq 0$.

Proposition 3.4. *If* $p_{\mathrm{g}}, q_{\mathrm{g}} \geq 0$, *then* $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))$ *is isomorphic to the restriction to* $W_{[\lambda_{\check{\mathcal{O}}}]}$ *of*

$$\begin{cases} \mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}} \otimes \mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}}, q_{\mathrm{g}}}, & \text{if } \star \in \{ B, C, \widetilde{C}, C^*, D \}; \\ \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}'_{n_{\mathrm{b}}, n_{\mathrm{g}}}} (\mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}} \otimes \mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}}, q_{\mathrm{g}}}), & \text{if } \star = D^*, \end{cases}$$

where

$$\mathsf{W}'_{n_{\mathrm{b}}, n_{\mathrm{g}}} := \left(\mathsf{W}_{n_{\mathrm{b}}} \times \mathsf{W}_{n_{\mathrm{g}}}\right) \cap \mathsf{W}'_{n_{\mathrm{g}} + n_{\mathrm{b}}}.$$

Otherwise, $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G)) = \{0\}$.

4. Special unipotent representations in type A

Let YD_n be the set of Young diagrams of total size n. We identify YD_n with the set $\overline{\mathrm{Nil}}(\mathfrak{g}) := \mathrm{GL}_n(\mathbb{C}) \backslash \mathrm{Nil}(\mathfrak{gl}_n(\mathbb{C}))$ of complex nilpotent orbits and also with the set $\mathrm{Irr}(\mathsf{S}_n)$ via the Springer correspondence (see [Car93, 11.4]). More specifically, for $\mathcal{O}' \in \overline{\mathrm{Nil}}(\mathfrak{g}) = \mathsf{YD}_n$, the Springer correspondence is given by Macdonald's construction for S_n via j-induction:

$$\mathrm{Springer}(\mathcal{O}') = j_{\prod_j \mathsf{S}_{\mathbf{c}_j(\mathcal{O}')}}^{\mathsf{S}_n} \mathrm{sgn}.$$

Suppose that $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}$ in this section. Recall that $W_{[\lambda_{\check{\mathcal{O}}}]} = \mathsf{S}_{|\check{\mathcal{O}}_{\mathrm{g}}|} \times \mathsf{S}_{|\check{\mathcal{O}}_{\mathrm{b}}|}$. It is easy to verify that

$$(4.1)\qquad \begin{aligned} W_{\lambda_{\check{\mathcal{O}}}} &= \prod_{i \in \mathbb{N}^+} \mathsf{S}_{\mathbf{c}_i(\check{\mathcal{O}}_{\mathrm{g}})} \times \prod_{i \in \mathbb{N}^+} \mathsf{S}_{\mathbf{c}_i(\check{\mathcal{O}}_{\mathrm{b}})}, \\ {}^L\mathscr{C}_{\lambda_{\check{\mathcal{O}}}} &= \{ \tau_{\lambda_{\check{\mathcal{O}}}} \}, \quad \text{where } \tau_{\lambda_{\check{\mathcal{O}}}} := (j_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}) \otimes \mathrm{sgn} = \check{\mathcal{O}}_{\mathrm{g}}^t \otimes \check{\mathcal{O}}_{\mathrm{b}}^t. \end{aligned}$$

Here and as before, a superscript "t" indicates the transpose of a Young diagram.

4.1. Special unipotent representations of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{H})$.

Special unipotent representations of general linear groups $\mathrm{GL}_n(F)$, where $F = \mathbb{R}$ or $\mathbb{H}$, are well-known and comprise representations of the form

$$\mathrm{Ind}_P^{\mathrm{GL}_n(F)} \chi,$$

where P is a parabolic subgroup of $\mathrm{GL}_n(F)$, and χ is a character of P trivial on the connected component of P. See [Vog86, Page 450]. We will review their classifications in the framework of this article.

Recall from Definition 1.8 the set $\mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}})$, which is the set of paintings on $\check{\mathcal{O}}^t$ that has type $A^{\mathbb{R}}$. It is easy to see that

$$(4.2)\qquad \#(\mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}})) = \prod_{r \in \mathbb{N}^+} (\#\{ i \in \mathbb{N}^+ \mid \mathbf{r}_i(\check{\mathcal{O}}) = r \} + 1).$$

Proposition 4.1. *Suppose $\star = A^{\mathbb{R}}$ so that $G = \mathrm{GL}_n(\mathbb{R})$. Then*

$$[\tau_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G))] = \#(\mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}_{\mathrm{g}})) \times \#(\mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}_{\mathrm{b}})) = \#(\mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}})).$$

Proof. In view of Proposition 3.1 and the description of the left cell representation $\tau_{\lambda_{\check{\mathcal{O}}}}$ in (4.1), the first equality follows from Pieri's rule ([GW09, Corollary 9.2.4]) and the following branching formula (see [BV83b, Lemma 4.1 (b)]):

$$\mathrm{Ind}_{\mathsf{W}_n}^{\mathsf{S}_{2n}} \varepsilon = \bigoplus_{\substack{\sigma \in \mathsf{YD}_{2n} \\ \mathbf{c}_i(\sigma) \text{ is even for all } i \in \mathbb{N}^+}} \sigma. \tag{4.3}$$

The last equality follows from (4.2). □

Let $\mathrm{sgn}_a \colon \mathrm{GL}_a(\mathbb{R}) \to \{\pm 1\}$ be the character given by sign of determinant and 1_a be the trivial character of $\mathrm{GL}_a(\mathbb{R})$. For $\tau \in \mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}})$, we attache the representation

$$\pi_\tau := \bigtimes_j \underbrace{1_j \times \cdots \times 1_j}_{c_j\text{-terms}} \times \underbrace{\mathrm{sgn}_j \times \cdots \times \mathrm{sgn}_j}_{d_j\text{-terms}}. \tag{4.4}$$

where

- j runs over all nonzero column lengths in $\check{\mathcal{O}}^t$,
- d_j is the number of columns of length j ending with the symbol "d",
- c_j is the number of columns of length j ending with the symbol "•" or "c", and
- "×" denotes the parabolic induction.

Then π_τ is irreducible and belongs to $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ (see [Vog86, Theorem 3.8] and [ABV91, Example 27.5]).

Theorem 4.2 (*cf.* [ABV91, Example 27.5])**.** *Let $\star = A^{\mathbb{R}}$ so that $G = \mathrm{GL}_n(\mathbb{R})$. Then the map*

$$\begin{array}{ccc} \mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G), \\ \tau & \mapsto & \pi_\tau \end{array}$$

is bijective.

Proof. The injectivity is proved in [Vog86, Theorem 3.8]. Then the map is bijective by Corollary 1.6 and Proposition 4.1. □

Theorem 4.3. *Assume that $\star = A^{\mathbb{H}}$ so that $G = \mathrm{GL}_n(\mathbb{H})$. Then*

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \begin{cases} \{\pi_{\check{\mathcal{O}}}\}, & \text{if } \check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}; \\ \emptyset, & \text{otherwise,} \end{cases}$$

where

$$\pi_{\check{\mathcal{O}}} := 1_{\mathbf{r}_1(\check{\mathcal{O}})/2} \times 1_{\mathbf{r}_2(\check{\mathcal{O}})/2} \times \cdots \times 1_{\mathbf{r}_k(\check{\mathcal{O}})/2},$$

k is the number of nonempty rows of $\check{\mathcal{O}}$ and 1_a denotes the trivial representation of $\mathrm{GL}_a(\mathbb{H})$ $(a \geq 0)$.

Proof. For the irreducibility of $\pi_{\check{\mathcal{O}}}$, see [Vog86, Theorem 3.8]. As in the case of $\mathrm{GL}_n(\mathbb{R})$, we know that $\pi_{\check{\mathcal{O}}} \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$ (*cf.* [BV85, Lemma 8.3]). In view of the description of left cell representation $\tau_{\lambda_{\check{\mathcal{O}}}}$ in (4.1), it follows from (4.3) that

$$[\tau_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G))] = \begin{cases} 1, & \text{if } \check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}; \\ 0, & \text{otherwise.} \end{cases}$$

This proves the theorem. □

4.2. **Special unipotent representations of** $\mathrm{U}(p,q)$**.** In this subsection, we suppose $\star \in \{A, \widetilde{A}\}$ so that

$$G = \begin{cases} \mathrm{U}(p,q), & \text{if } \star = A; \\ \widetilde{\mathrm{U}}(p,q), & \text{if } \star = \widetilde{A}. \end{cases}$$

Recall from Definition 1.8 the set $\mathsf{PAP}_\star(\check{\mathcal{O}})$, which is the set of paintings on $\check{\mathcal{O}}^t$ that has type $\star$. For $\mathcal{P} \in \mathsf{PAP}_\star(\check{\mathcal{O}})$, we have defined its signature $(p_\mathcal{P}, q_\mathcal{P})$ in (1.7). Recall that $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) = \check{\mathcal{O}}^t \subseteq \mathrm{Nil}(\mathfrak{g}^*)$. Let $\overline{\mathrm{Nil}}_G(\mathcal{O})$ denote the set of G-orbits in $(\sqrt{-1}\mathfrak{g}_0^*) \cap \mathcal{O}$, where $\mathfrak{g}_0$ denotes the Lie algebra of G which equals $\mathfrak{u}(p,q)$.

We first consider the case when $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$. In this good parity setting, we will state a counting result on $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$. The elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ can be constructed by cohomological induction explicitly and they are irreducible and unitary due to [Mat96, Tra01], see also [Tra04, Section 2] and [MR19, Section 4]. We refer the reader to [BMSZ21] for the construction of all elements of $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ by the method of theta lifting.

Theorem 4.4 (*cf.* [BV83b, Theorem 4.2] and [Tra04, Theorem 2.1])**.** *Suppose that* $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$. *Then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp\{\mathcal{P} \in \mathrm{PAP}_{A^*}(\check{\mathcal{O}}) \mid (p_\mathcal{P}, q_\mathcal{P}) = (p,q)\} = \sharp(\overline{\mathrm{Nil}}_G(\mathcal{O})). \tag{4.5}$$

Moreover, for every $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$*, its wavefront set* $\mathrm{WF}(\pi)$ *is the closure in* $\sqrt{-1}\mathfrak{g}_0^*$ *of a unique orbit* $\mathscr{O}_\pi \in \overline{\mathrm{Nil}}_G(\mathcal{O})$*, and the map*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}}(G) & \longrightarrow & \overline{\mathrm{Nil}}_G(\mathcal{O}), \\ \pi & \mapsto & \mathscr{O}_\pi. \end{array} \tag{4.6}$$

is bijective.

Proof. The Harish-Chandra cell representations in $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))$ are all irreducible and explicitly described in terms of the Springer correspondence by [Bož02, Lemma 4]. Thus the equality holds in (3.1), and the first equality in (4.5) follows from Proposition 3.3, (4.1), (4.3), and Pieri's rule ([GW09, Corollary 9.2.4]). The second equality in (4.5) will follow directly from the bijectivity of (4.6).

The assignment of wavefront set yields a bijection

$$\{\text{cell in the basal representation } \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G))/\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}\setminus\mathcal{O}}(G))\} \to \overline{\mathrm{Nil}}_G(\mathcal{O}),$$

and every cell representation in $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G))/\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}\setminus\mathcal{O}}(G))$ is irreducible and isomorphic to $\tau_{\lambda_{\check{\mathcal{O}}}}$. See [BV83b, Theorem 4.2] and [Bož02, Theorem 5] (we use [SV00, Theorem 1.4] to rephrase the result in terms of real nilpotent orbits).

Recall from Proposition 1.5 that

$$[1_{W_{\lambda_{\check{\mathcal{O}}}}} : \tau_{\lambda_{\check{\mathcal{O}}}}] = 1.$$

Assume without loss of generality that $\lambda_{\check{\mathcal{O}}} \in {}^a\mathfrak{h}^*$ is dominant. As in the proof of Theorem 1.3, for each cell $\mathcal{C}$ in $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G))$ that is not a cell in $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}\setminus\mathcal{O}}(G))$, there is a unique element $\Phi_\mathcal{C} \in \mathcal{C}$ such that $\Phi_\mathcal{C}(\lambda_{\check{\mathcal{O}}}) \neq 0$. Then

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{\Phi_\mathcal{C}(\lambda_{\check{\mathcal{O}}})\}_{\mathcal{C} \text{ is a cell in } \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G)) \text{ that is not a cell in } \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}\setminus\mathcal{O}}(G))}.$$

This implies the bijectivity assertion of the theorem. □

Now we consider the general case.

Lemma 4.5. *The set* $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ *is empty unless*

- $|\check{\mathcal{O}}_{\mathrm{b}}|$ *is even and* $p, q \geq \frac{1}{2}|\check{\mathcal{O}}_{\mathrm{b}}|$*; and*
- *each nonzero row length occurs in* $\check{\mathcal{O}}_{\mathrm{b}}$ *with even multiplicity.*

Proof. Suppose that $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is nonempty. By (3.1), we have that

$$[\tau_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))] \neq 0.$$

Then by Proposition 3.3, (4.1), and (4.3), the two conditions of the lemma are satisfied. □

Assume that the two conditions in Lemma 4.5 are satisfied. Then G has a Levi subgroup that is identified with $G'_{\mathrm{b}} \times G_{\mathrm{g}}$, where $G'_{\mathrm{b}} := \mathrm{GL}_{\frac{|\check{\mathcal{O}}_{\mathrm{b}}|}{2}}(\mathbb{C})$ and

$$G_{\mathrm{g}} = \begin{cases} \mathrm{U}(p - \frac{|\check{\mathcal{O}}_{\mathrm{b}}|}{2}, q - \frac{|\check{\mathcal{O}}_{\mathrm{b}}|}{2}), & \text{if } \star = A; \\ \widetilde{\mathrm{U}}(p - \frac{|\check{\mathcal{O}}_{\mathrm{b}}|}{2}, q - \frac{|\check{\mathcal{O}}_{\mathrm{b}}|}{2}), & \text{if } \star = \widetilde{A}. \end{cases}$$

Write $\check{\mathcal{O}}_{\mathrm{b}} = 2\check{\mathcal{O}}'_{\mathrm{b}}$ and let $\pi_{\check{\mathcal{O}}'_{\mathrm{b}}}$ denote the unique element in $\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\mathrm{GL}_{\frac{|\check{\mathcal{O}}_{\mathrm{b}}|}{2}})$. Then for every $\pi_0 \in \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$, the normalized parabolic induction $\pi_{\check{\mathcal{O}}'_{\mathrm{b}}} \rtimes \pi_0$ is irreducible by [Mat96, Theorem 3.2.2] and is an element of $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ (*cf.* [MR19, Theorem 5.3]).

Theorem 4.6. *Assume that the two conditions in Lemma 4.5 are satisfied. Then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})), \tag{4.7}$$

and the map

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G), \\ \pi_0 & \mapsto & \pi_{\check{\mathcal{O}}'_{\mathrm{b}}} \rtimes \pi_0 \end{array} \tag{4.8}$$

is bijective.

Proof. As in the good parity case, the structure of cells in the basal representation $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))$ (see [Bož02, Theorem 5]) implies the equality in (3.1). Hence

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = [\tau_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))].$$

Similarly,

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) = [\tau_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}]}(\mathcal{K}'(G_{\mathrm{g}}))].$$

Recall the representation $\mathcal{C}_{\mathrm{b}}$ from Proposition 3.3. Note that $[\check{\mathcal{O}}^t_{\mathrm{b}} : \mathcal{C}_{\mathrm{b}}] = 1$, in view of (4.3). The above three equalities clearly imply (4.7).

By the calculation of the wavefront set of the induced representations ([Bar00, Corollary 5.0.10]), Theorem 4.4 implies that all the representations $\pi_{\check{\mathcal{O}}'_{\mathrm{b}}} \rtimes \pi_0$, where π_0 varies in $\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$, have pairwise distinct wavefront sets. Thus the map (4.8) is injective. Hence it is bijection by the counting assertion (4.7). □

Remark 4.7. When $\check{\mathcal{O}} \neq \check{\mathcal{O}}_{\mathrm{g}}$, the wavefront set $\mathrm{WF}(\pi)$, where $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, may not be the closure of a single orbit in $\overline{\mathrm{Nil}}_G(\mathcal{O})$.

5. Counting of special unipotent representations in type BCD

In this section, we assume that $\star \in \{B, C, \widetilde{C}, C^*, D, D^*\}$. Proposition 3.4 and (3.1) and imply that the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is empty unless $p_{\mathrm{g}}, q_{\mathrm{g}} \geq 0$. For this reason we assume that $p_{\mathrm{g}}, q_{\mathrm{g}} \geq 0$ throughout this section.

To ease the notation, for every sequence $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ ($k \geq 0$) of integers, we let $[a_1, a_2, \cdots, a_k]_{\mathrm{col}}$ denote the Young diagram whose i-th column has length a_i if $1 \leq i \leq k$ and length 0 otherwise. Likewise, we let $[a_1, a_2, \cdots, a_k]_{\mathrm{row}}$ denote the Young diagram whose i-th row has length a_i if $1 \leq i \leq k$ and length 0 otherwise.

As usual, we identify $\mathrm{Irr}(\mathsf{W}_n)$ ($n \geq 0$) with the set of bipartitions $\tau = (\tau_L, \tau_R)$ of total size n ([Car93, Section 11.4]). Here the total size refers to $|\tau_L| + |\tau_R|$. We also let $(\tau_L, \tau_R)_I \in \mathrm{Irr}(\mathsf{W}'_n)$ denote the irreducible representation given by

- the restriction of $(\tau_L,\tau_R)\in \mathrm{Irr}(\mathsf{W}_n)$ if $\tau_L\neq\tau_R$, and
- the induced representation $\mathrm{Ind}_{\mathsf{S}_n}^{\mathsf{W}'_n}\tau_L$ if $\tau_L=\tau_R$.

Note that

$$(\tau_L,\tau_R)_I=(\tau_R,\tau_L)_I\in \mathrm{Irr}(\mathsf{W}'_n).$$

5.1. **The left cells.** In this subsection, we describe the Lusztig left cell ${}^L\mathscr{C}_{\lambda_{\check{\mathcal{O}}}}$ attached to $\lambda_{\check{\mathcal{O}}}$.

Define two Young diagrams

(5.1)
$$\tau_{L,\mathrm{b}}:=\begin{cases}\left[\frac12(\mathbf{r}_1(\check{\mathcal{O}}'_\mathrm{b})+1),\frac12(\mathbf{r}_2(\check{\mathcal{O}}'_\mathrm{b})+1),\cdots,\frac12(\mathbf{r}_c(\check{\mathcal{O}}'_\mathrm{b})+1)\right]_{\mathrm{col}}, & \text{if } \star\in\{B,\widetilde{C}\};\\ \left[\frac12\mathbf{r}_1(\check{\mathcal{O}}'_\mathrm{b}),\frac12\mathbf{r}_2(\check{\mathcal{O}}'_\mathrm{b}),\cdots,\frac12\mathbf{r}_c(\check{\mathcal{O}}'_\mathrm{b})\right]_{\mathrm{col}}, & \text{if } \star\in\{C,C^*,D,D^*\},\end{cases}$$

and

(5.2)
$$\tau_{R,\mathrm{b}}:=\begin{cases}\left(\frac12(\mathbf{r}_1(\check{\mathcal{O}}'_\mathrm{b})-1),\frac12(\mathbf{r}_2(\check{\mathcal{O}}'_\mathrm{b})-1),\cdots,\frac12(\mathbf{r}_c(\check{\mathcal{O}}'_\mathrm{b})-1)\right)_{\mathrm{col}}, & \text{if } \star\in\{B,\widetilde{C}\};\\ \left(\frac12\mathbf{r}_1(\check{\mathcal{O}}'_\mathrm{b}),\frac12\mathbf{r}_2(\check{\mathcal{O}}'_\mathrm{b}),\cdots,\frac12\mathbf{r}_c(\check{\mathcal{O}}'_\mathrm{b})\right)_{\mathrm{col}}, & \text{if } \star\in\{C,C^*,D,D^*\},\end{cases}$$

where $c:=\mathbf{c}_1(\check{\mathcal{O}}'_\mathrm{b})$.

Recall from (3.5) that $W_{[\lambda_{\check{\mathcal{O}}}]}=W_{[\lambda_{\check{\mathcal{O}}_\mathrm{g}}]}\times W_{[\lambda_{\check{\mathcal{O}}_\mathrm{b}}]}=W_\mathrm{g}\times W_\mathrm{b}$. Define an irreducible representation $\tau_\mathrm{b}\in\mathrm{Irr}(W_\mathrm{b})$ attached to $\check{\mathcal{O}}_\mathrm{b}$ by

(5.3)
$$\tau_\mathrm{b}:=\begin{cases}(\tau_{L,\mathrm{b}},\tau_{R,\mathrm{b}}), & \text{if } \star\in\{B,\widetilde{C}\};\\ (\tau_{L,\mathrm{b}},\tau_{R,\mathrm{b}})_I, & \text{if } \star\in\{C,C^*,D,D^*\}.\end{cases}$$

Recall the set $\mathrm{PP}_\star(\check{\mathcal{O}}_\mathrm{g})$ from Definition 1.14. Put

$$\mathrm{A}(\check{\mathcal{O}}):=\mathrm{A}(\check{\mathcal{O}}_\mathrm{g}):=\text{the power set of }\mathrm{PP}_\star(\check{\mathcal{O}}_\mathrm{g}),$$

which is identified with the free $\mathbb{F}_2$-vector space with free basis $\mathrm{PP}(\check{\mathcal{O}}_\mathrm{g})$. Here $\mathbb{F}_2:=\mathbb{Z}/2\mathbb{Z}$ is the field with two elements only. Note that $\{\emptyset,\mathrm{PP}(\check{\mathcal{O}}_\mathrm{g})\}$ is a subgroup of $\mathrm{A}(\check{\mathcal{O}})$. Define

$$\bar{\mathrm{A}}(\check{\mathcal{O}}):=\bar{\mathrm{A}}(\check{\mathcal{O}}_\mathrm{g}):=\begin{cases}\mathrm{A}(\check{\mathcal{O}})/\{\emptyset,\mathrm{PP}(\check{\mathcal{O}}_\mathrm{g})\}, & \text{if } \star=\widetilde{C};\\ \mathrm{A}(\check{\mathcal{O}}), & \text{otherwise.}\end{cases}$$

Generalizing (1.11), for each $\wp\in\mathrm{A}(\check{\mathcal{O}})$, we define a pair

$$(\imath_\wp,\jmath_\wp):=(\imath_\star(\check{\mathcal{O}},\wp),\jmath_\star(\check{\mathcal{O}},\wp))$$

of Young diagrams as in what follows.

If $\star=B$, then

$$\mathbf{c}_1(\jmath_\wp)=\frac{\mathbf{r}_1(\check{\mathcal{O}}_\mathrm{g})}{2},$$

and for all $i\geq 1$,

$$(\mathbf{c}_i(\imath_\wp),\mathbf{c}_{i+1}(\jmath_\wp))=\begin{cases}(\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}}_\mathrm{g})}{2},\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_\mathrm{g})}{2}), & \text{if } (2i,2i+1)\in\wp;\\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_\mathrm{g})}{2},\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}}_\mathrm{g})}{2}), & \text{otherwise.}\end{cases}$$

If $\star=\widetilde{C}$, then for all $i\geq 1$,

$$(\mathbf{c}_i(\imath_\wp),\mathbf{c}_i(\jmath_\wp))=\begin{cases}(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_\mathrm{g})}{2},\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_\mathrm{g})}{2}), & \text{if } (2i-1,2i)\in\wp;\\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_\mathrm{g})}{2},\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_\mathrm{g})}{2}), & \text{otherwise.}\end{cases}$$

If $\star \in \{D, D^*\}$, then

$$\mathbf{c}_1(\imath_\wp) = \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}_{\mathrm{g}}) > 0; \\ 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}_{\mathrm{g}}) = 0, \end{cases}$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_\wp), \mathbf{c}_{i+1}(\imath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}), & \text{if } (2i, 2i+1) \in \wp; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, 0), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}_{\mathrm{g}}; \\ (0, 0), & \text{if } (2i, 2i+1) \text{ is empty in } \check{\mathcal{O}}_{\mathrm{g}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}), & \text{otherwise.} \end{cases}$$

If $\star \in \{C, C^*\}$, then for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_\wp), \mathbf{c}_i(\imath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}), & \text{if } (2i-1, 2i) \in \wp; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}_{\mathrm{g}}; \\ (0, 0), & \text{if } (2i-1, 2i) \text{ is empty in } \check{\mathcal{O}}_{\mathrm{g}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}), & \text{otherwise.} \end{cases}$$

We define an element $\tau_\wp \in \mathrm{Irr}(W_{\mathrm{g}})$ by

$$\tau_\wp := \begin{cases} (\imath_\wp, \jmath_\wp), & \text{if } \star \in \{ B, C, C^* \}; \\ (\imath_\wp, \jmath_\wp)_I, & \text{if } \star \in \{ \widetilde{C}, D, D^* \}. \end{cases} \tag{5.4}$$

Note that if $\star = \widetilde{C}$, then $\tau_\wp = \tau_{\wp^c}$, where $\wp^c$ is the complement of $\wp$ in $\mathrm{PP}_\star(\check{\mathcal{O}}_{\mathrm{g}})$. Therefore in all cases, $\tau_{\bar{\wp}} \in \mathrm{Irr}(W_{\mathrm{g}})$ is obviously defined for every $\bar{\wp} \in \bar{\mathrm{A}}(\check{\mathcal{O}})$.

Remark 5.1. When $\star \neq \widetilde{C}$, $\mathrm{A}(\check{\mathcal{O}})$ gives another description of Lusztig's canonical quotient attached to $\check{\mathcal{O}}$. The set $\mathrm{PP}_\star(\check{\mathcal{O}})$ appears implicitly in [Som01, Section 5].

To simplify the notation, we write ${}^L\mathscr{C}_{\check{\mathcal{O}}} := {}^L\mathscr{C}_{\lambda_{\check{\mathcal{O}}}}$. Recall that ${}^L\mathscr{C}_{\check{\mathcal{O}}}$ is the set of all $\sigma \in \mathrm{Irr}(W_{[\lambda_{\check{\mathcal{O}}}]})$ that occurs in

$${}^L\mathscr{V}_{\check{\mathcal{O}}} := \left(J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}\right) \otimes \mathrm{sgn}\,.$$

Lemma 5.2 (*cf.* Barbasch-Vogan [BV85, Proposition 5.28])**.** *The representation ${}^L\mathscr{V}_{\check{\mathcal{O}}}$ of $W_{[\lambda_{\check{\mathcal{O}}}]}$ is multiplicity free, and the map*

$$\begin{array}{rcl} \bar{\mathrm{A}}(\check{\mathcal{O}}) & \rightarrow & {}^L\mathscr{C}_{\check{\mathcal{O}}}, \\ \bar{\wp} & \mapsto & \tau_{\bar{\wp}} \otimes \tau_{\mathrm{b}} \end{array}$$

is well-defined and bijective. Moreover,

$$\tau_{\check{\mathcal{O}}} := \tau_\emptyset \otimes \tau_{\mathrm{b}}$$

is the unique special representation in ${}^L\mathscr{C}_{\check{\mathcal{O}}}$,

$$\mathrm{Springer}^{-1}(j_{W_{[\lambda_{\check{\mathcal{O}}}]}}^{W}(\tau_{\check{\mathcal{O}}})) = d_{\mathrm{BV}}(\check{\mathcal{O}}), \tag{5.5}$$

and

$$d_{\mathrm{BV}}(\check{\mathcal{O}}) = d_{\mathrm{BV}}(\check{\mathcal{O}}_{\mathrm{g}}) \overset{c}{\sqcup} (\check{\mathcal{O}}'_{\mathrm{b}})^t \overset{c}{\sqcup} (\check{\mathcal{O}}'_{\mathrm{b}})^t \tag{5.6}$$

as Young diagrams.

Proof. When $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$ and $\star \neq \widetilde{C}$, the lemma is proved in [BV85, Proposition 5.28]. When $\star \neq \widetilde{C}$, the equalities (5.5) and (5.6) are proved in [BV85, Proposition A2]. In general, the lemma follows from Lusztig's formula of J-induction in [Lus84, §4.4-4.6]. □

Recall from (3.6) that $W_{\mathrm{g}} = \mathsf{W}'_{n_{\mathrm{g}}}$, when $\star \in \{\widetilde{C}, D, D^*\}$. Since the representation theory of W_n is more elementary than that of W'_n, we prefer to work with W_n instead of W'_n in some situations. For this reason, in all cases we also define

$$\widetilde{\tau}_\wp = (\imath_\wp, \jmath_\wp) \in \mathrm{Irr}(\mathsf{W}_{n_{\mathrm{g}}}), \qquad \wp \in \mathrm{A}(\check{\mathcal{O}}). \tag{5.7}$$

See (5.4) for the description of $\imath_\wp$ and $\jmath_\wp$.

For later use, we record the following lemma, which follows immediately from our explicit descriptions of $\tau_\wp$ and τ_{b}.

Lemma 5.3. *Let $\wp \in \mathrm{A}(\check{\mathcal{O}})$. If $\star = \widetilde{C}$, then*

$$\mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{g}}} \times \mathsf{W}_{n_{\mathrm{b}}}}^{\mathsf{W}_{n_{\mathrm{g}}} \times \mathsf{W}_{n_{\mathrm{b}}}} \tau_\wp \otimes \tau_{\mathrm{b}} = \begin{cases} \widetilde{\tau}_\emptyset \otimes \tau_{\mathrm{b}}, & \text{if } n_{\mathrm{g}} = 0; \\ (\widetilde{\tau}_\wp \otimes \tau_{\mathrm{b}}) \oplus (\widetilde{\tau}_{\wp^c} \otimes \tau_{\mathrm{b}}), & \text{otherwise.} \end{cases}$$

If $\star \in \{D, D^*\}$, then

$$\mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{g}}} \times \mathsf{W}'_{n_{\mathrm{b}}}}^{\mathsf{W}_{n_{\mathrm{g}}} \times \mathsf{W}_{n_{\mathrm{b}}}} \tau_\wp \otimes \tau_{\mathrm{b}} \cong \begin{cases} \widetilde{\tau}_{\mathrm{b}}, & \text{if } n_{\mathrm{g}} = 0; \\ (\widetilde{\tau}_\wp \otimes \widetilde{\tau}_{\mathrm{b}}) \oplus (\widetilde{\tau}^s_\wp \otimes \widetilde{\tau}_{\mathrm{b}}), & \text{otherwise.} \end{cases}$$

Here $\widetilde{\tau}_{\mathrm{b}} = \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}}^{\mathsf{W}_{n_{\mathrm{b}}}} \tau_{\mathrm{b}}$ and $\widetilde{\tau}^s_\wp := \widetilde{\tau}_\wp \otimes \varepsilon$ (recall the quadratic character ε from (3.2)).

5.2. **From coherent continuation representation to counting.** We have defined in (1.12) the set $\mathsf{PBP}_\star(\check{\mathcal{O}})$ when $\check{\mathcal{O}}$ has good parity. Similarly, we make the following definition in the bad parity case.

Definition 5.4. *Let $\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}})$ be the set of all triples $\uptau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \star$ where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted partitions such that*

- *$(\imath, \jmath) = (\tau_{L,\mathrm{b}}, \tau_{R,\mathrm{b}})$ (see (5.3));*
- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c, d\}, & \text{if } \star \in \{B, \widetilde{C}\}; \\ \{\bullet, d\}, & \text{if } \star \in \{C, D\}; \\ \{\bullet\}, & \text{if } \star \in \{C^*, D^*\}; \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \star \in \{C, D\}; \\ \{\bullet\}, & \text{if } \star \in \{B, \widetilde{C}, C^*, D^*\}. \end{cases}$$

We introduce some additional notation. For each bipartition τ, let

$$\mathsf{PBP}_\star(\tau) := \{\, \uptau \text{ is a painted bipartition } \mid \star_\uptau = \star, (\imath_\uptau, \jmath_\uptau) = \tau \,\}$$

and

$$\mathsf{PBP}_G(\tau) := \{\, \uptau \text{ is a painted bipartition } \mid G_\uptau = G, (\imath_\uptau, \jmath_\uptau) = \tau \,\}.$$

Put

$$\widetilde{\mathsf{PBP}}_\star(\check{\mathcal{O}}) := \bigsqcup_{\wp \subseteq \mathrm{PP}(\check{\mathcal{O}})} \mathsf{PBP}_\star(\widetilde{\tau}_\wp)$$

and

$$\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}}) := \bigsqcup_{\wp \subseteq \mathrm{PP}(\check{\mathcal{O}})} \mathsf{PBP}_G(\widetilde{\tau}_\wp),$$

where $\widetilde{\tau}_\wp := (\imath_\wp, \jmath_\wp)$ (see (5.7)).

Recall the group G_{g} from (1.10) (with $l = n_{\mathrm{b}}$). Its Langlands dual group is identified with $\check{G}_{\mathrm{g}}$.

Proposition 5.5. *The equality*

$$\sum_{\bar{\wp} \in \bar{\mathrm{A}}(\check{\mathcal{O}})} [\tau_{\mathrm{b}} \otimes \tau_{\bar{\wp}} : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))] = \sharp(\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}})) \cdot \sharp(\widetilde{\mathsf{PBP}}_{G_{\mathrm{g}}}(\check{\mathcal{O}}_{\mathrm{g}}))$$

holds.

Proof. We use the following formulas ([McG98, p220 (6)]) to compute the multiplicities:

$$
\begin{aligned}
&\mathrm{Ind}_{\mathsf{H}_n}^{\mathsf{W}_{2n}} \tilde{\varepsilon} \cong \bigoplus_{\sigma \in \mathrm{Irr}(\mathsf{S}_n)} (\sigma, \sigma), \\
&\mathrm{Ind}_{\mathsf{H}_n}^{\mathsf{W}'_{2n}} \tilde{\varepsilon} \cong \bigoplus_{\sigma \in \mathrm{Irr}(\mathsf{S}_n)} (\sigma, \sigma)_I, \\
&\mathrm{Ind}_{\mathsf{S}_n}^{\mathsf{W}_n} \mathrm{sgn} \cong \bigoplus_{a+b=n} \mathrm{Ind}_{\mathsf{W}_a \times \mathsf{W}_b}^{\mathsf{W}_n} \overline{\mathrm{sgn}} \otimes \mathrm{sgn} \cong \bigoplus_{a+b=n} ([a]_{\mathrm{col}}, [b]_{\mathrm{col}}), \\
&\mathrm{Ind}_{\mathsf{S}_n}^{\mathsf{W}_n} 1 \cong \bigoplus_{a+b=n} \mathrm{Ind}_{\mathsf{W}_a \times \mathsf{W}_b}^{\mathsf{W}_n} 1 \otimes \epsilon \cong \bigoplus_{a+b=n} ([a]_{\mathrm{row}}, [b]_{\mathrm{row}}).
\end{aligned}
\tag{5.8}
$$

We skip the details when $\star \in \{ B, \widetilde{C}, C, D, C^* \}$, and present the computation for $\star = D^*$, which is the most complicated case. Suppose that $\star = D^*$ so that $G = \mathrm{O}^*(2n)$. Recall that $(W_{\mathrm{b}}, W_{\mathrm{g}}) = (\mathsf{W}'_{n_{\mathrm{b}}}, \mathsf{W}'_{n_{\mathrm{g}}})$, where $n_{\mathrm{b}} = \frac{1}{2}\left|\check{\mathcal{O}}_{\mathrm{b}}\right|$ and $n_{\mathrm{g}} = \frac{1}{2}\left|\check{\mathcal{O}}_{\mathrm{g}}\right|$.

First suppose that $n_{\mathrm{g}} = 0$. Then $\tau_{\mathrm{b}} = (\mathcal{O}'_{\mathrm{b}}, \mathcal{O}'_{\mathrm{b}})_I \in \mathrm{Irr}(\mathsf{W}'_{n_{\mathrm{b}}})$ and

$$
\begin{aligned}
\sum_{\bar{\wp} \in \bar{\mathrm{A}}(\check{\mathcal{O}})} [\tau_{\mathrm{b}} \otimes \tau_{\bar{\wp}} : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))] &= [\tau_{\mathrm{b}} : \mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}}] \\
&= [\tau_{\mathrm{b}} : \mathrm{Ind}_{\mathsf{H}_{\frac{n_{\mathrm{b}}}{2}}}^{\mathsf{W}'_{n_{\mathrm{b}}}} \tilde{\varepsilon}] \\
&= [(\mathcal{O}'_{\mathrm{b}}, \mathcal{O}'_{\mathrm{b}})_I : \bigoplus_{\sigma \in \mathrm{Irr}(\mathsf{S}_n)} (\sigma, \sigma)_I] \\
&= 1 \\
&= \sharp(\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}})) \cdot \sharp(\widetilde{\mathsf{PBP}}_{G_{\mathrm{g}}}(\check{\mathcal{O}}_{\mathrm{g}})).
\end{aligned}
$$

This proves the proposition in the case of $n_{\mathrm{g}} = 0$.

Now we suppose that $n_{\mathrm{g}} > 0$. Recall that

$$\widetilde{\tau}_{\mathrm{b}} := \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}}^{\mathsf{W}_{n_{\mathrm{b}}}} \tau_{\mathrm{b}} = (\mathcal{O}'_{\mathrm{b}}, \mathcal{O}'_{\mathrm{b}}) \in \mathrm{Irr}(\mathsf{W}_{n_{\mathrm{b}}})$$

and

$$\widetilde{\tau}_\wp := (\imath_\wp, \jmath_\wp) \in \mathrm{Irr}(\mathsf{W}_{n_{\mathrm{g}}}) \quad \text{for all } \wp \subseteq \mathrm{PP}(\check{\mathcal{O}}_{\mathrm{g}}).$$

Note that $\imath_\wp \neq \jmath_\wp$ since $\mathbf{c}_1(\imath_\wp) > \mathbf{c}_1(\jmath_\wp)$, which implies that

$$\mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}'_{n_{\mathrm{b}}, n_{\mathrm{g}}}} \tau_{\mathrm{b}} \otimes \tau_\wp \cong (\widetilde{\tau}_{\mathrm{b}} \otimes \widetilde{\tau}_\wp)|_{\mathsf{W}'_{n_{\mathrm{b}}, n_{\mathrm{g}}}}.$$

For ease of notation, write $\mathsf{W}'' := \mathsf{W}'_{n_{\mathrm{b}}, n_{\mathrm{g}}}$. For every finite group E and any two finite-dimensional representations V_1 and V_2 of E, put

$$[V_1, V_2]_E := \dim \mathrm{Hom}_E(V_1, V_2).$$

For each $\wp \in \mathrm{A}(\check{\mathcal{O}}_{\mathrm{g}})$, we have that

$$\begin{aligned}[\tau_{\mathrm{b}} \otimes \tau_{\wp} : \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G))] =&[\tau_{\mathrm{b}} \otimes \tau_{\wp} : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}\times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}''} \mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}} \otimes \mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}},q_{\mathrm{g}}}]_{\mathsf{W}'_{n_{\mathrm{b}}}\times \mathsf{W}'_{n_{\mathrm{g}}}} \\
=&[\mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}\times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}''} \tau_{\mathrm{b}} \otimes \tau_{\wp} : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}\times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}''} \mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}} \otimes \mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}},q_{\mathrm{g}}}]_{\mathsf{W}''} \\
=&[(\widetilde{\tau}_{\mathrm{b}} \otimes \widetilde{\tau}_{\wp})|_{\mathsf{W}''} : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}\times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}''} \mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}} \otimes \mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}},q_{\mathrm{g}}}]_{\mathsf{W}''} \\
=&[\widetilde{\tau}_{\mathrm{b}} \otimes \widetilde{\tau}_{\wp} : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}\times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}_{n_{\mathrm{b}}}\times \mathsf{W}_{n_{\mathrm{g}}}} \mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}} \otimes \mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}},q_{\mathrm{g}}}]_{\mathsf{W}_{n_{\mathrm{b}}}\times \mathsf{W}_{n_{\mathrm{g}}}} \\
=&[\widetilde{\tau}_{\mathrm{b}} : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}}^{\mathsf{W}_{n_{\mathrm{b}}}} \mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}}]_{\mathsf{W}_{n_{\mathrm{b}}}} \cdot [\widetilde{\tau}_{\wp} : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}_{n_{\mathrm{g}}}} \mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}},q_{\mathrm{g}}}]_{\mathsf{W}_{n_{\mathrm{g}}}} \\
=&\sharp(\mathsf{PBP}_{\star}(\check{\mathcal{O}}_{\mathrm{b}}))) \cdot \sharp(\mathsf{PBP}_{G_{\mathrm{g}}}(\widetilde{\tau}_{\wp})).\end{aligned}$$

The last equality follows from the explicit description of $\mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}}$ and $\mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}},q_{\mathrm{g}}}$ in Section 3.3, the branching rules (5.8) and Pieri's rule ([GW09, Corollary 9.2.4]). □

Now Proposition 5.5, Lemma 5.2 and (3.1) imply the following corollary.

Corollary 5.6. *The inequality*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) \leq \sharp(\mathsf{PBP}_{\star}(\check{\mathcal{O}}_{\mathrm{b}})) \cdot \sharp(\widetilde{\mathsf{PBP}}_{G_{\mathrm{g}}}(\check{\mathcal{O}}_{\mathrm{g}}))$$

holds.

When $\check{\mathcal{O}}$ has good parity, we will see from Proposition 6.1 and Proposition 6.2 that

$$\sharp(\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}})) = \begin{cases} \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})), & \text{if } \star \in \{C^*, D^*\}; \\ 2^{\sharp(\mathrm{PP}_{\star}(\check{\mathcal{O}}))} \cdot \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})), & \text{if } \star \in \{B, C, D, \widetilde{C}\}. \end{cases}$$

Corollary 5.6 will thus imply Theorem 1.18 in the introductory section.

5.3. **The case of bad parity.** Recall from (1.8) the group

$$G'_{\mathrm{b}} := \begin{cases} \mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star \in \{\, B, C, D \,\}; \\ \widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star = \widetilde{C}; \\ \mathrm{GL}_{\frac{n_{\mathrm{b}}}{2}}(\mathbb{H}), & \text{if } \star \in \{\, C^*, D^* \,\}. \end{cases}$$

Proposition 5.7. *The equalities*

$$\sharp(\mathsf{PBP}_{\star}(\check{\mathcal{O}}_{\mathrm{b}})) = \sharp(\mathsf{PAP}_{\star'}(\check{\mathcal{O}}'_{\mathrm{b}})) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}))$$

hold, where

$$\star' := \begin{cases} A^{\mathbb{R}}, & \textit{if } \star \in \{\, B, C, \widetilde{C}, D \,\}; \\ A^{\mathbb{H}}, & \textit{if } \star \in \{\, C^*, D^* \,\}. \end{cases} \tag{5.9}$$

Proof. Suppose that $\star \in \{\, C^*, D^* \,\}$. Then

$$\sharp(\mathsf{PBP}_{\star}(\check{\mathcal{O}}_{\mathrm{b}}; \tau_{\mathrm{b}})) = \sharp(\mathsf{PAP}_{A^{\mathbb{H}}}(\check{\mathcal{O}}'_{\mathrm{b}})) = 1.$$

Suppose that $\star \in \left\{\, B, C, \widetilde{C}, D \,\right\}$. It is easy to see that we have a bijection

$$\begin{array}{rcl} \mathsf{PBP}_{\star}(\check{\mathcal{O}}_{\mathrm{b}}) & \rightarrow & \mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_{\mathrm{b}}), \\ (\tau_{L,b}, \mathcal{P}) \times (\tau_{R,b}, \mathcal{Q}) \times & \mapsto & ((\check{\mathcal{O}}'_{\mathrm{b}})^t, \mathcal{P}'), \end{array}$$

where $\mathcal{P}'$ is defined by the condition that

$$\mathcal{P}(\mathbf{c}_j(\tau_{L,b}), j) = d \Longleftrightarrow \mathcal{P}'(\mathbf{r}_j(\check{\mathcal{O}}'_{\mathrm{b}}), j) = d, \quad \text{for all } j = 1, 2, \cdots, \mathbf{c}_1(\check{\mathcal{O}}'_{\mathrm{b}}).$$

The last equality is in Theorems 4.2 and 4.3. □

Set

$$G_{\rm b} = \begin{cases} \mathrm{SO}(n_{\rm b}, n_{\rm b}+1), & \text{if } \star = B; \\ \mathrm{SO}(n_{\rm b}, n_{\rm b}), & \text{if } \star \in \{C, D\}; \\ \widetilde{\mathrm{Sp}}_{2n_{\rm b}}(\mathbb{R}), & \text{if } \star = \widetilde{C}; \\ \mathrm{O}^*(2n_{\rm b}), & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

Note that $\check{G}_{\rm b}$ is the Langlands dual of $G_{\rm b}$ and $\check{\mathcal{O}}_{\rm b}$ is a nilpotent orbit in $\mathrm{Nil}(\check{\mathfrak{g}}_{\rm b})$.

Let $P_{\rm b}$ be a parabolic subgroup of $G_{\rm b}$ containing $G'_{\rm b}$ as a Levi component.

Proposition 5.8. *For every $\pi' \in \mathrm{Unip}_{G'_{\rm b}}(\check{\mathcal{O}}'_{\rm b})$, the normalized induced representation $\mathrm{Ind}_{P_{\rm b}}^{G_{\rm b}} \pi'$ is irreducible and belongs to $\mathrm{Unip}_{G'_{\rm b}}(\check{\mathcal{O}}'_{\rm b})$. Moreover, the map*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}'_{\rm b}}(G'_{\rm b}) & \rightarrow & \mathrm{Unip}_{\check{\mathcal{O}}_{\rm b}}(G_{\rm b}), \\ \pi' & \mapsto & \mathrm{Ind}_{P_{\rm b}}^{G_{\rm b}} \pi' \end{array} \tag{5.10}$$

is bijective.

Proof. By the construction of π' (Theorem 4.2 and 4.3) and a result of Barbasch [Bar00, Corollary 5.0.10], the wavefront cycle of $\mathrm{Ind}_{P_{\rm b}}^{G_{\rm b}} \pi'$ is a single orbit $\mathscr{O}$ with multiplicity one and its complexification is $d_{\rm BV}(\check{\mathcal{O}}_{\rm b})$. Note that every irreducible summand of $\mathrm{Ind}_{P_{\rm b}}^{G_{\rm b}} \pi'$ belongs to $\mathrm{Unip}_{\check{\mathcal{O}}_{\rm b}}(G_{\rm b})$. Hence $\mathrm{Ind}_{P_{\rm b}}^{G_{\rm b}} \pi'$ has to be irreducible.

We now suppose that $\star = D$ so that $G_{\rm b} = \mathrm{SO}(n_{\rm b}, n_{\rm b})$. We will prove the injectivity of the map (5.10). Fix a split Cartan subgroup

$$H = (\mathbb{R}^\times)^{n_{\rm b}} \subseteq G'_{\rm b} \subseteq G_{\rm b}$$

and write $H = MA$ where $M = \{\pm 1\}^{n_{\rm b}}$ is the compact part of H and $A = (\mathbb{R}_+^\times)^{n_{\rm b}}$ is the split part of H. We identify a character of A with a vector in $\mathbb{C}^{n_{\rm b}}$ as usual. Let

$$K := \{(g_1, g_2) \in \mathrm{O}(n_{\rm b}) \times \mathrm{O}(n_{\rm b}) \mid \det(g_1) \cdot \det(g_2) = 1\}$$

be a maximal compact subgroup of $G_{\rm b}$, and let B be a Borel subgroup of $G_{\rm b}$ containing H and the unipotent radical of $P_{\rm b}$.

For each integer l such that $0 \leq l \leq n_{\rm b}$, put

$$\delta_l = \underbrace{1_1 \otimes \cdots \otimes 1_1}_{l} \otimes \underbrace{\mathrm{sgn}_1 \otimes \cdots \otimes \mathrm{sgn}_1}_{n_{\rm b}-l} \in \mathrm{Irr}(M),$$

where 1_1 denotes the trivial character of $\{\pm 1\}$ and sgn_1 denotes the non-trivial character of $\{\pm 1\}$. It is a fine M-type ([Vog81, Definition 4.3.8]). Let μ_l be the restriction to K of the irreducible $\mathrm{O}(n_{\rm b}) \times \mathrm{O}(n_{\rm b})$-representation $\wedge^l \mathbb{C}^{n_{\rm b}} \otimes \wedge^0 \mathbb{C}^{n_{\rm b}}$. Then μ_l is a fine K-type, see [BGG75, §6].

Vogan proves that there is a well-defined injective map

$$\begin{array}{rccc} \mathfrak{X}\colon & W_H \backslash \mathrm{Irr}(H) & \rightarrow & \mathrm{Irr}(G_{\rm b}), \\ & \text{the } W_H\text{-orbit of } \chi \in \mathrm{Irr}(H) & \mapsto & (\mathrm{Ind}_B^{G_{\rm b}} \chi)(\mu_{l_\chi}), \end{array}$$

where W_H denotes the real Weyl group of $G_{\rm b}$ with respect to H, $l_\chi \in \{0, 1, \ldots, n_{\rm b}\}$ is the integer such that $\chi|_M$ is conjugate to δ_{l_χ} by W_H, and $(\mathrm{Ind}_B^{G_{\rm b}} \chi)(\mu_{l_\chi})$ denotes the unique irreducible subquotient in the normalized induced representation $\mathrm{Ind}_B^{G_{\rm b}} \chi$ containing the K-type μ_l. See [Vog81, Theorem 4.4.8].

Now take

$$\pi' = 1_{n_1} \times \cdots \times 1_{n_r} \times \mathrm{sgn}_{n_{r+1}} \times \cdots \times \mathrm{sgn}_{n_k} \in \mathrm{Unip}_{G'_{\rm b}}(\check{\mathcal{O}}'_{\rm b}),$$

where $k \geq r \geq 0$, and $n_1, n_2, \ldots, n_k$ are positive integers with $n_1 + n_2 + \cdots + n_k = n_{\rm b}$.

Write

$$\begin{array}{rcl}\delta_{\pi'} & := & \delta_{n_1+n_2+\cdots+n_r}\in \mathrm{Irr}(M),\\ \nu_{\pi'} & := & \left(\frac{n_1-1}{2},\frac{n_1-3}{2},\dots,\frac{1-n_1}{2},\dots,\frac{n_k-1}{2},\frac{n_k-3}{2},\dots,\frac{1-n_k}{2}\right)\in \mathrm{Irr}(A).\end{array}$$

Clearly the map

$$\begin{array}{rcl}\mathfrak{P}:\mathrm{Unip}_{G'_{\mathrm{b}}}(\check{\mathcal{O}}'_{\mathrm{b}}) & \rightarrow & W_H\backslash \mathrm{Irr}(H),\\ \pi' & \mapsto & \text{the } W_H\text{-orbit of } \delta_{\pi'}\otimes\nu_{\pi'}\end{array}$$

is well-defined and injective.

Note that the map (5.10) equals the composition $\mathfrak{X}\circ\mathfrak{P}$, and hence it is also injective. This proves the injectivity of (5.10) in the case when $\star = D$. The same proof works in the case when $\star\in\{B,C,\widetilde{C}\}$ and we omit the details. When $\star\in\{C^*,D^*\}$, the map (5.10) has to be injective since its domain is a singleton. Thus (5.10) is injective in all cases.

The bijection of (5.10) follows from the injectivity and the counting inequalities below:

$$\left|\mathsf{PAP}_{\star'}(\check{\mathcal{O}}'_{\mathrm{b}})\right| = \left|\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}})\right| \leq \left|\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(G_{\mathrm{b}})\right| \leq \left|\mathsf{PBP}_{\star}(\check{\mathcal{O}}_{\mathrm{b}})\right| = \left|\mathsf{PAP}_{\star'}(\check{\mathcal{O}}'_{\mathrm{b}})\right|.$$

Here $\star'\in\{A^{\mathbb{R}},A^{\mathbb{H}}\}$ is as in (5.9), the first inequality follows from the injectivity of (5.10), the second inequality follows from Corollary 5.6, and the last equality is in Proposition 5.7.

□

5.4. **Matching coherent continuation representations.** Let $\check{G}'_{\mathrm{b}}$ be a Levi factor of a Siegel parabolic subgroup of $\check{G}_{\mathrm{b}}$ such that its Lie algebra $\check{\mathfrak{g}}'_{\mathrm{b}}$ has nonempty intersection with $\check{\mathcal{O}}_{\mathrm{b}}$. Such a Levi factor is unique up to conjugation by $\check{G}_{\mathrm{b}}$. Note that $\check{G}'_{\mathrm{b}}\cong \mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{C})$. We suppose without loss of generality that there is an $\mathfrak{sl}_2$-triple in $\check{\mathfrak{g}}'_{\mathrm{b}}$ whose neutral element equals $2\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}$.

Put

$$\check{G}'_2 := \check{G}'_{\mathrm{b}}\times\check{G}_{\mathrm{g}}\quad\text{and}\quad \check{G}_2 := \check{G}_{\mathrm{b}}\times\check{G}_{\mathrm{g}},$$

whose complexified Lie algebras are respectively denoted by $\check{\mathfrak{g}}'_2$ and $\check{\mathfrak{g}}_2$. Fix a Cartan subalgebra $\check{\mathfrak{h}}'_2$ of $\check{\mathfrak{g}}'_2$ that contains $\lambda_{\check{\mathcal{O}}}$. Let $\check{\Delta}\supseteq\check{\Delta}_2\supseteq\check{\Delta}'_2$ denotes the root systems with respect to $\check{\mathfrak{h}}'_2$ of $\check{\mathfrak{g}}$, $\check{\mathfrak{g}}_2$, and $\check{\mathfrak{g}}'_2$, respectively. Fix a positive system of $\check{\Delta}^+$ of $\check{\Delta}$ such that

$$\langle\lambda_{\check{\mathcal{O}}},\check{\alpha}\rangle\geq 0\quad\text{for all } \check{\alpha}\in\check{\Delta}^+.$$

Put $\check{\Delta}_2^+ := \check{\Delta}^+\cap\check{\Delta}_2$ and $\check{\Delta}_2'^+ := \check{\Delta}^+\cap\check{\Delta}'_2$.

Following [GI19], we set

$$G'_2 := G'_{\mathrm{b}}\times G_{\mathrm{g}}\quad\text{and}\quad G_2 := G_{\mathrm{b}}\times G_{\mathrm{g}},$$

whose complexified Lie algebras are respectively denoted by $\mathfrak{g}'_2$ and $\mathfrak{g}_2$. We identify the universal Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_2$ with the dual of ${}^a\check{\mathfrak{h}}'_2$ such that

- the positive coroot systems are respectively identified with $\check{\Delta}_2$ and $\check{\Delta}_2^+$, if $\star\neq\widetilde{C}$;
- the positive root systems are respectively identified with $\check{\Delta}_2$ and $\check{\Delta}_2^+$, if $\star=\widetilde{C}$, where we use the trace form on $\check{\mathfrak{g}}$ to identify ${}^a\check{\mathfrak{h}}'_2$ with its dual space.

Recall that we have chosen a Siegel parabolic subgroup P_{b} of G_{b} and G'_{b} is identified with a Levi component of it. Using P_{b}, we obviously identify the universal Cartan subalgebra of $\mathfrak{g}'_{\mathrm{b}}$ with that of $\mathfrak{g}_{\mathrm{b}}$. Thus the universal Cartan subalgebra of $\mathfrak{g}'_2$ is also identified with that of $\mathfrak{g}_2$.

Let $G'_{\mathrm{b},\mathbb{C}}$, $G_{\mathrm{b},\mathbb{C}}$, $G_{\mathrm{g},\mathbb{C}}$, and $G_{\mathbb{C}}$ respectively denote the standard complexifications of G'_{b}, G_{b}, G_{g}, and G. Thus $G'_{\mathrm{b},\mathbb{C}}$ is a complex general linear group, and $G_{\mathrm{b},\mathbb{C}}$, $G_{\mathrm{g},\mathbb{C}}$, and $G_{\mathbb{C}}$ are complex special orthogonal groups or complex symplectic groups.

The coherent family and coherent continuation representation may be extended to these lattices, *cf.* [Vog81, Lemma 7.2.6]. By abuse of notation, we have $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G'_2))$ and

$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'(G_2))$ where $\mathcal{K}'(G_2')$ and $\mathcal{K}'(G_2)$ denote the Grothendieck groups of genuine representations for the respective groups, when G is a metaplectic group.

In all cases, one may construct natural maps

$$\mathcal{P}_{[\lambda_{\check{\mathcal{O}}}]}(G_2') \xrightarrow{f'} \mathcal{P}_{[\lambda_{\check{\mathcal{O}}}]}(G_2) \xrightarrow{f} \mathcal{P}_{[\lambda_{\check{\mathcal{O}}}]}(G)$$

by matching regular characters, and furthermore f is injective. See [GI19, p 97] for the odd orthogonal group case, and [RT03, p26] for the metaplectic group case. All other cases are similar (the existence of such maps is a consequence of the Vogan duality [Vog82, Section 11]). Recall the definition of coherent families of standard modules in (2.18). We define

$$\begin{array}{rccll} \varphi := \varphi^{f}\colon & \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G_2)) & \longrightarrow & \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G)) & \\ & \Psi_{\overline{\gamma}_2} & \mapsto & \Psi_{f(\overline{\gamma}_2)}, & \text{for } \overline{\gamma}_2 \in \mathcal{P}_{[\lambda_{\check{\mathcal{O}}}]}(G_2). \end{array} \tag{5.11}$$

Similarly, we define $\varphi^{f'}$ and $\varphi^{f\circ f'}$.

Lemma 5.9. *(i) The integral Weyl group of $[\lambda_{\check{\mathcal{O}}}]$ with respect to G_2 is naturally identified with $W_{[\lambda_{\check{\mathcal{O}}}]}$;*

(ii) The map φ is a basal $W_{[\lambda]}$-module injection;

(iii) Evaluating at $\nu \in [\lambda_{\check{\mathcal{O}}}]$, the map φ induces an injection

$$\varphi_\nu\colon \mathcal{K}_\nu(G_2) \to \mathcal{K}_\nu(G)$$

which preserves the irreducibility.

(iv) The following diagram is commutative:

$$\begin{array}{ccccc} & & \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G_2')) & & \\ & \swarrow^{\varphi^{f'}} & & \searrow^{\varphi^{f\circ f'}} & \\ \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G_2) & & \xrightarrow{\quad\quad\varphi\quad\quad} & & \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G)). \end{array}$$

(v) The maps $\varphi^{f'}$ and $\varphi^{f\circ f'}$ are parabolic induction functors: for any $\Psi \in \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G_2'))$ and $\nu \in [\lambda_{\check{\mathcal{O}}}]$,

$$(\varphi^{f'}(\Psi))(\nu) = \mathrm{Ind}_{P_{\mathrm{b}}\times G_{\mathrm{g}}}^{G_{\mathrm{b}}\times G_{\mathrm{g}}} \Psi(\nu) \quad \textit{and} \quad (\varphi^{f\circ f'}(\Psi))(\nu) = \mathrm{Ind}_P^G \Psi(\nu).$$

Consequently,

$$\varphi_\nu\left((\mathrm{Ind}_{P_{\mathrm{b}}}^{G_{\mathrm{b}}} \pi')\otimes \pi_0\right) = \mathrm{Ind}_P^G(\pi'\otimes\pi_0) \quad \textit{for any } \pi'\otimes\pi_0 \in \mathcal{K}_\nu(G_{\mathrm{b}}'\times G_{\mathrm{g}}).$$

Proof. (i) This is clear by construction.

(ii) That φ is $W_{[\lambda]}$-equivariant can be seen directly from the explicit formula of the coherent continuation action on standard modules. Now the Kazhdan-Lusztig-Vogan algorithm ensures that $\varphi(\overline{\Psi}_{\overline{\gamma}_2}) = \overline{\Psi}_{f(\overline{\gamma}_2)}$ at any regular dominant element $\nu \in [\lambda_{\check{\mathcal{O}}}]$, i.e. φ is basal, see [RT03, Theorem 1.1] for the metaplectic group, [GI19, p91] and [ABV91, Theorem 16.22] for other cases.

(iii) The claim on the injectivity of φ_ν is clear from Proposition 2.8 and the injectivity of (5.11). *cf.* [GI19, Lemma 3.3].

For the second claim, one may easily reduce it to the case when ν is dominant. Let $\pi \in \mathcal{K}_\nu(G_2)$ and $\Psi \in \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G_2))$ be the basal element such that $\Psi(\nu) = \pi$. Now $\varphi(\Psi)$ is also basal by (ii), and so $\varphi_\nu(\pi) := \varphi(\Psi)(\nu)$ must be either irreducible or zero. By the injectivity of φ_ν, it is non-zero and so the claim follows.

(iv) Since standard modules can be constructed via parabolic induction [Vog81, Theorem 6.6.14], the claim follows from induction by stages.

(v) Note that coherent continuation is compatible with parabolic induction [Vog81, Corollary 7.2.10]. In our case, only the 0-th term $\mathscr{R}^0_{\mathfrak{q}}$ in $X_{\mathfrak{q}}$ loc. cit. does not vanish since P_{b} and P are real parabolic subgroups. The claim follows.

□

Proof of Theorem 5.10. In view of Lemma 5.9 (iii) and (v), the injectivity of $\mathfrak{I}$ and the irreducibility of $\pi' \rtimes \pi_0$ follow immediately from the corresponding claims in the bad parity case, see Proposition 5.8.

That the representation $\pi' \rtimes \pi_0$ is in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ now follows directly from the formula on wavefront cycle for parabolic induction [Bar00, Corollary 5.0.10].

The counting result Lemma 5.11 below finally implies the bijectivity of $\mathfrak{I}$. □

5.5. **Reduction to the good parity.** Recall that G is a classical Lie group of type $\star \in \{ B, \widetilde{C}, C, D, C^*, D^* \}$.

Let P be a parabolic subgroup of G whose Levi component is identified with $G'_{\mathrm{b}} \times G_{\mathrm{g}}$ (or $(G'_{\mathrm{b}} \times G_{\mathrm{g}})/\{\pm 1\}$ when $\star = \widetilde{C}$).

Theorem 5.10. *There is a bijection*

$$\begin{array}{rccc} \mathfrak{I}\colon & \mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G), \\ & (\pi', \pi_0) & \mapsto & \pi' \rtimes \pi_0. \end{array}$$

Here $\pi' \rtimes \pi_0$ refers to the normalized parabolically induced representation from P.

The rest of this section will be devoted to the proof of the above theorem.

We first prove the injectivity of $\mathfrak{I}$ by making use of the "endoscopic group" $G_{\mathrm{b}} \times G_{\mathrm{g}}$ and translating the problem to a regular infinitesimal character. The method is used and carefully explained in [Mat04]. See also [GI19, Section 3] for a comprehensive account for the odd orthogonal group case.

Lemma 5.11. *We have*

$$|\mathrm{Unip}_{\check{\mathcal{O}}}(G)| = \left|\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(G_{\mathrm{b}})\right| \cdot \left|\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})\right|.$$

Proof. It clearly suffices to consider the case where n_{b} and n_{g} are both non-zero. Recall that for each primitive ideal J, we have attached an irreducible $W_{[\lambda_{\check{\mathcal{O}}}]}$-representation σ_J (Theorem 2.14).

Without loss of generality we assume that $\lambda_{\check{\mathcal{O}}}$ is dominant. Let Ψ_2 be a basal element in $\mathrm{Coh}_{[\lambda_{\mathrm{b}}]}(\mathcal{K}(G_{\mathrm{b}})) \otimes \mathrm{Coh}_{[\lambda_{\mathrm{g}}]}(\mathcal{K}(G_{\mathrm{g}}))$ such that $\pi_2 := \Psi_2(\lambda_{\check{\mathcal{O}}}) \neq 0$. Then $\pi_2 \in \mathrm{Irr}(G'_{\mathrm{b}} \times G_{\mathrm{g}})$. Put $\Psi := \varphi(\Psi_2)$. Then $\pi := \Psi(\lambda_{\check{\mathcal{O}}}) \in \mathrm{Irr}(G)$.

Via the injective basal homomorphism (5.11), we have $\langle \Psi_2 \rangle \overset{\varphi}{\cong} \langle \Psi \rangle$. By Lemma 2.28, $\sigma_{\mathrm{Ann}(\pi_2)} = \sigma_{\mathrm{Ann}(\pi)}$ is the irreducible representation of $W_{[\lambda_{\check{\mathcal{O}}}]}$ having minimal fake degree occurring in $\langle \Psi_2 \rangle \overset{\varphi}{\cong} \langle \Psi \rangle$.

Let $I_{\lambda_{\check{\mathcal{O}}}}$ and $I'_{\lambda_{\check{\mathcal{O}}}}$ be the maximal primitive ideal in $\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g}_2)$ with infinitesimal character $\lambda_{\check{\mathcal{O}}}$ respectively. By (1.4), $\sigma_{I_{\lambda_{\check{\mathcal{O}}}}} = j_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} = \sigma_{I'_{\lambda_{\check{\mathcal{O}}}}}$ is also the Goldie rank representation attached to $I'_{\lambda_{\check{\mathcal{O}}}}$.

Since $\pi \neq 0$ and $\pi_2 \neq 0$ have infinitesimal character $\lambda_{\check{\mathcal{O}}}$, we have $\mathrm{Ann}(\pi) \subseteq I_{\lambda_{\check{\mathcal{O}}}}$ and $\mathrm{Ann}(\pi_2) \subseteq I'_{\lambda_{\check{\mathcal{O}}}}$. By Lemma 2.25,

$$\begin{aligned} \mathrm{AV}_{\mathbb{C}}(\pi) \subseteq \overline{\mathcal{O}} &\iff \mathrm{Ann}(\pi) = I_{\lambda_{\check{\mathcal{O}}}} \\ \iff \sigma_{\mathrm{Ann}(\pi)} = \sigma_{I_{\lambda_{\check{\mathcal{O}}}}} &\iff \sigma_{\mathrm{Ann}(\pi_2)} = \sigma_{I'_{\lambda_{\check{\mathcal{O}}}}} \\ \iff \mathrm{Ann}(\pi_2) = I'_{\lambda_{\check{\mathcal{O}}}} &\iff \mathrm{AV}_{\mathbb{C}}(\pi_2) \subseteq \overline{\mathcal{O}_{\mathrm{b}}} \times \overline{\mathcal{O}_{\mathrm{g}}} \end{aligned}$$

Consequently, evaluating at $\lambda_{\check{\mathcal{O}}}$ yields a linear bijection

$$\mathcal{K}_{\lambda_{\check{\mathcal{O}}_{\mathrm{b}}},\overline{\mathcal{O}_{\mathrm{b}}}}(G_{\mathrm{b}}) \otimes \mathcal{K}_{\lambda_{\check{\mathcal{O}}_{\mathrm{g}}},\overline{\mathcal{O}_{\mathrm{g}}}}(G_{\mathrm{g}}) \longrightarrow \mathrm{ev}_{\lambda_{\check{\mathcal{O}}}} \left(\mathrm{Im}(\varphi) \cap \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}_{\overline{\mathcal{O}}}(G))\right).$$

We claim that

$$\mathrm{ev}_{\lambda_{\check{\mathcal{O}}}} \left(\mathrm{Im}(\varphi) \cap \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}_{\overline{\mathcal{O}}}(G))\right) = \mathcal{K}_{\lambda_{\check{\mathcal{O}}},\overline{\mathcal{O}}}(G). \tag{5.12}$$

Then the lemma follows by taking the dimension on the two sides.

When $\star \neq D^*$, equation (5.12) holds since φ is an isomorphism (*cf.* **??**).

Now we consider the case where $\star = D^*$. For any positive integer k and a W'_k-module τ, let τ^s denote the twist of τ by any non-trivial element s in $\mathsf{W}_k/\mathsf{W}'_k$. Retain the notation in **??**. As basal $W_{[\lambda]}$-modules, we have

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}(G)) = \mathrm{Coh}_{B_1} \oplus \mathrm{Coh}_{B_2}$$

where

- $\mathrm{Coh}_{B_1} := \mathrm{Im}(\varphi) \cong \mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}} \otimes \mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}},q_{\mathrm{g}}}$ and
- Coh_{B_2} is isomorphic to $(\mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}})^s \otimes (\mathcal{C}_{\mathrm{g}}^{p_{\mathrm{g}},q_{\mathrm{g}}})^s$.

Observe that $[\tau_{\mathrm{b}} : (\mathcal{C}_{\mathrm{b}}^{n_{\mathrm{b}}})^s] = 0$. Hence the Goldie rank representation $\sigma_{\lambda_{\check{\mathcal{O}}}} = \tau_{\mathrm{b}} \otimes \tau$ attached to $I_{\lambda_{\check{\mathcal{O}}}}$ does not occur in Coh_{B_2} and $\mathrm{ev}_{\lambda_{\check{\mathcal{O}}}}(\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}_{\overline{\mathcal{O}}}(G)) \cap \mathrm{Coh}_{B_2}) = 0$ by Lemma 2.28. This implies the validity of (5.12). □

6. Combinatorics of painted bipartitions

In this section, we assume that $\star \in \{B, C, \widetilde{C}, C^*, D, D^*\}$, and $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$, namely $\check{\mathcal{O}}$ has $\star$-good parity. Recall the set $\mathrm{PP}_\star(\check{\mathcal{O}})$ of primitive $\star$-pairs in $\check{\mathcal{O}}$. For each subset $\wp$ of $\mathrm{PP}_\star(\check{\mathcal{O}})$, we have defined a bipartition $\tau_\wp = (\imath_\wp, \jmath_\wp)$ in Section 5.1.

The following two combinatorial results follow by induction on $\mathbf{c}_1(\check{\mathcal{O}})$. As the proof is quite tedious, we omit the details.

Proposition 6.1. *Suppose that* $\star \in \{ C^*, D^* \}$. *Then*

$$\mathsf{PBP}_G(\tau_\wp) = \emptyset, \quad \textit{for all nonempty } \wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}}).$$

Consequently,

$$\sharp(\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}})) = \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})).$$

Proof. Suppose that $\emptyset \neq \wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}})$, and there was an element $\boldsymbol{\tau} = (\imath_\wp, \mathcal{P}) \times (\jmath_\wp, \mathcal{Q}) \times \star \in \mathsf{PBP}_G(\tau_\wp)$.

First assume that $\star = C^*$. Pick an element $(2i-1, 2i) \in \wp$. Then we have that

$$\mathbf{c}_i(\imath_\wp) = \tfrac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}) + 1) > \tfrac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}) - 1) = \mathbf{c}_i(\jmath_\wp). \tag{6.1}$$

By the requirements of a painted bipartition, we also have that

$$\begin{aligned} \mathbf{c}_i(\imath_\wp) &= \sharp\{ j \in \mathbb{N}^+ \mid (i,j) \in \mathrm{Box}(\imath_\wp),\ \mathcal{P}(i,j) = \bullet \} \\ &= \sharp\{ j \in \mathbb{N}^+ \mid (i,j) \in \mathrm{Box}(\jmath_\wp),\ \mathcal{Q}(i,j) = \bullet \} \\ &\leq \mathbf{c}_i(\jmath_\wp). \end{aligned}$$

This contradicts (6.1) and therefore $\mathsf{PBP}_G(\tau_\wp) = \emptyset$.

Now we assume that $\star = D^*$. Pick an element $(2i, 2i+1) \in \wp$. Then we have that

$$\mathbf{c}_{i+1}(\imath_\wp) = \tfrac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}) + 1) > \tfrac{1}{2}(\mathbf{r}_{2i+1}(\check{\mathcal{O}}) - 1) = \mathbf{c}_i(\jmath_\wp). \tag{6.2}$$

By the requirements of a painted bipartition, we also have that

$$\begin{aligned} \mathbf{c}_{i+1}(\imath_\wp) &\leq \sharp\{\, j \in \mathbb{N}^+ \mid (i,j) \in \mathrm{Box}(\imath_\wp),\ \mathcal{P}(i,j) = \bullet \,\} \\ &= \sharp\{\, j \in \mathbb{N}^+ \mid (i,j) \in \mathrm{Box}(\jmath_\wp),\ \mathcal{Q}(i,j) = \bullet \,\} \\ &\leq \mathbf{c}_i(\jmath_\wp). \end{aligned}$$

This contradicts (6.2) and therefore $\mathsf{PBP}_G(\tau_\wp) = \emptyset$. □

The following combinatorial result follows by induction on $\mathbf{c}_1(\check{\mathcal{O}})$. We omit the details.

Proposition 6.2. *Suppose that* $\star \in \{\, B, C, \widetilde{C}, D \,\}$. *Then*

$$\sharp(\mathsf{PBP}_G(\tau_\wp)) = \sharp(\mathsf{PBP}_G(\tau_\emptyset)), \quad \textit{for all } \wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}}).$$

Consequently,

$$\sharp(\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}})) = 2^{\sharp(\mathrm{PP}_\star(\check{\mathcal{O}}))} \cdot \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})).$$

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