Theta Correspondence and The Orbit Method

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Contents

1 Theta Correspondence: Howe's theory

1.1 Dual pairs

- W: a finite-dimensional real symplectic vector space.
  - σ: the anti-involution of $\mathrm{End}_{\mathbb{R}}(W)$ determined by $\langle\, , \,\rangle_W$.
- (A, A'): a pair of σ-stable semisimple $\mathbb{R}$-subalgebras of $\mathrm{End}_{\mathbb{R}}(W)$ that are mutual centralizers of each other.
  - $G := A \cap \mathrm{Sp}(W)$ and $G' := A' \cap \mathrm{Sp}(W)$.
- (G, G'): a reductive dual pair in $\mathrm{Sp}(W)$.
- Construction: $(G, G') = (G(V), G(V'))$, where

$$V : \text{an } \epsilon\text{-Hermitian space over D, } \epsilon = \pm 1;$$

$$V' : \text{an } \epsilon'\text{-Hermitian space over D, } \epsilon\epsilon' = -1;$$

$$(\mathrm{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}).$$

1.2 The oscillator representation

- $\mathrm{H}(W) := W \times \mathbb{R}$, the Heisenberg group with group multiplication

$$(u,t)\cdot(u',t') = (u+u', t+t'+\langle u,u'\rangle_W), \quad u,u' \in W,\ t,t' \in \mathbb{R}.$$

- Fix a nontrivial unitary character $\psi : \mathbb{R} \to \mathbb{C}^\times$.

- **Stone-von Neumann Theorem**: there exists a unique irreducible unitary representation of $\mathrm{H}(W)$ with central character ψ.

- Segal, Shale, Weil: This extends to a unitary representation $\widehat{\omega}$ of

$$J := \widetilde{\mathrm{Sp}}(W) \ltimes \mathrm{H}(W),$$

where $\widetilde{\mathrm{Sp}}(W)$ is the so-called metaplectic cover of $\mathrm{Sp}(W)$.

- ω: the space of smooth vectors of $\widehat{\omega}|_{\mathrm{H}(W)}$.
  - called a smooth oscillator representation.

1.3 Theta lifting

(G, G'): a reductive dual pair in $\mathrm{Sp}(W)$.

- Let $\widetilde{G}$ and $\widetilde{G}'$ be the inverse images of G and G' in $\widetilde{\mathrm{Sp}}(W)$.
- The main object of study is the spectrum of

$$\omega|_{\widetilde{G}\cdot\widetilde{G}'}.$$

Let $\pi \in \mathrm{Rep}(\widetilde{G})$ (a Casselman-Wallach representation).

- The full theta lift of π:

$$\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi) := (\omega \widehat{\otimes} \pi^\vee)_{\widetilde{G}}, \qquad \text{(the Hausdorff coinvariant space).}$$

- The theta lift $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ of π:

$$\text{the largest semisimple quotient of } \Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi).$$

- **Howe duality theorem**: if π is irreducible, then $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ is irreducible or zero.

1.4 Theta lifting via matrix coefficient integrals

- $G = \mathrm{G}(V)$, where V is an ϵ-Hermitian space over D.
- Fix a maximal compact subgroup K_V of $\mathrm{G}(V)$.
- Ψ_V: the function of $\mathrm{G}(V)$ such that
  - it is bi-K_V-invariant; and for all hyperbolic elements $g \in \mathrm{G}(V)$,

$$\Psi_V(g) = \prod_a \left(\frac{1+a}{2}\right)^{-\frac{1}{2}},$$

    where a runs over all eigenvalues of $g \in \mathrm{End}(V \otimes_{\mathbb{R}} \mathbb{C})$.
- Ξ_V: the bi-K_V-invariant Harish-Chandra's Ξ function on $\mathrm{G}(V)$.

- A positive function Ψ on $\mathrm{G}(V)$ is said to be ν-bounded ($\nu \in \mathbb{R}$), if

$$\Psi \text{ is bounded by } \Psi_V^{\nu} \cdot \Xi_V \text{ (up to logarithmic growth).}$$

$\rightsquigarrow$ a notion of π to be ν-bounded (using matrix coefficients)

$\rightsquigarrow$ a notion of π to be convergent for $\Theta_{\widetilde{G}}^{\widetilde{G}'}$ (requiring a specific ν-bound)

- In the convergent range, the absolutely convegent integral $\int_G \langle \tilde{g}\phi, \phi' \rangle \cdot \langle \tilde{g}v', v \rangle \, dg$ on $\omega \times \pi^\vee \times \bar{\omega} \times \pi$ yields a continuous bilinear map

$$(\omega \widehat{\otimes} \pi^\vee) \times (\bar{\omega} \widehat{\otimes} \pi) \to \mathbb{C}.$$

- Define

$$\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi) := \frac{\omega \widehat{\otimes} \pi^\vee}{\text{the left kernel of the bilinear map}}.$$

This is a quotient of $\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$, and hence in $\mathrm{Rep}(\widetilde{G}')$.

1.5 Preservation of unitarity

- **Theorem** (Barbasch-Ma-S.-Z.): Suppose that π is convergent (or "a bit better") for $\Theta_{\widetilde{G}}^{\widetilde{G}'}$. If π is unitarizable, so is $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi)$.
- **Remarks**:
  - One needs effective criterion for $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi) \neq \{0\}$.
  - Earlier work by Li, He, and Harris-Li-Sun.

2 The Orbit Method: Kirillov and Kostant

Unitary dual problem: For a Lie group G, describe

$$\widehat{G} := \{\text{irreducible unitary representation of } G\}/\sim .$$

- G acts on $\sqrt{-1}\mathfrak{g}^*$ (the linear dual of $\mathfrak{g}$) by the coadjoint action.
- $\sqrt{-1}\mathfrak{g}^*/G$: coadjoint orbits.
- **The Orbit Method**:

$$\widehat{G} \leftrightsquigarrow \sqrt{-1}\mathfrak{g}^*/G.$$

2.1 The beginning: Kirillov (1962)

- G: connected, simply connected, nilpotent Lie group.
- There is a canonical bijection:

$$\widehat{G} = \sqrt{-1}\mathfrak{g}^*/G.$$

- This gives a method for constructing all irreducible unitary representations of G, from coadjoint orbits.

2.2 Further development

- Auslander and Kostant (1971): for a larger class of Lie groups (connected, simply connected, solvable Lie groups of type I)
- Kostant: Quantization and Unitary Representations (1970)
  - differential geometric constructions
- ......
- The task: coadjoint orbits $\rightsquigarrow$ unitary representations
  - also called quantization

2.3 Reductive Lie groups

- For a reductive Lie group G, we have (for some n)

$$\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R}) = \{n \times n \text{ real matrices}\}.$$

- The trace form $< X, Y >= \mathrm{Tr}(XY)$ identifies $\mathfrak{g}^*$ with $\mathfrak{g}$.
- **Jordan decomposition**: Any element $X \in \mathfrak{g}$ has a unique decomposition

$$X = X_h + X_e + X_n,$$

where X_h is hyperbolic, X_e is elliptic, and X_n is nilpotent, and X_h, X_e and X_n commute with each other.

Vogan (ICM 1986):

- The orbit method should serve as a unifying principle for the unitary dual problem of reductive Lie groups (despite its problems).

2.4 What we know about quantization

- Hyperbolic orbits: parabolic induction (Real analysis, from the 1950's)
- Elliptic orbits: cohomological induction (Complex geometry, from the 1970's)
- Nilpotent orbits: ?
  - Terminology: unipotent representations

Vogan:

- "Three kinds of coadjoint orbit, two kinds of quantization".
- Quantization should involve 3 steps in the following order:
  - the nilpotent step; the elliptic step; the hyperbolic step.

3 Special unipotent representations: Arthur and Barbasch-Vogan

3.1 Langlands and Arthur

- A major insight of Langlands is that representation theory of G is largely controlled by some other group, known as the Langlands dual $G^\vee$.

$$\text{e.g. } G = \mathrm{Sp}(2n, \mathbb{R}), \quad G^\vee = \mathrm{SO}(2n+1, \mathbb{C}).$$

- Langlands classification (1973): Irreducible representations of G, in terms of (local) Langlands parameters
- Arthur classification (2013): Discrete spectrum of L^2 automorphic forms for quasi-split classical groups, in terms of (global) Arthur parameters.

3.2 Arthur's unipotent representations

- Of special interest are the so-called unipotent Arthur parameters.

  Roughly, such a parameter is given by a nilpotent orbit $\check{\mathcal{O}}$ of $\check{\mathfrak{g}}$.

- Arthur's conjecture (1983): $\exists$ unipotent Arthur packets (corresponding to unipotent Arthur parameters)
  - Basic unitary representations that are of interest for global applications.
  - Arthur (in 2013, and followed by Mok) proved the classification theorem for the so-called quasi-split classical groups.
  - There was no (local) definition/characterization of representations in such unipotent packets.

3.3 Barbasch-Vogan

- Special unipotent attached to a nilpotent orbit $\check{\mathcal{O}}$ in $\check{\mathfrak{g}}$.
  - Approximately it requires the representation to have the smallest size, among all those with certain fixed Laplace-Beltrami eigenvalues.
- Notation:

  $\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{$ special unipotent repns. attached to $\check{\mathcal{O}}$ $\}$.
- Barbasch and Vogan reserved the name "unipotent" for a wider class of unitary representations, as in the orbit philosophy.
  - We do not know how to define/characterize "unipotent" representations.

3.4 The definition of special unipotent representations

G: a real reductive group, and $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$.

$\rightsquigarrow$ infinitesimal character $\chi_{\check{\mathcal{O}}}$ (by way of an $\mathfrak{sl}_2$-triple)

$\rightsquigarrow$ the maximal ideal $I_{\check{\mathcal{O}}} \subset \mathcal{U}(\mathfrak{g})$, by a theorem of Dixmier.

- **Definition** (Barbasch-Vogan, 1985):

  $\pi \in \mathrm{Irr}(G)$ is called **special unipotent** attached to $\check{\mathcal{O}}$ if

  $$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) = I_{\check{\mathcal{O}}}.$$

  $$\Longleftrightarrow \pi \text{ has infinitesimal character} \chi_{\check{\mathcal{O}}} \text{ and } \mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}}.$$

  – Here $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$ is the Barbasch-Vogan dual of $\check{\mathcal{O}}$, which is special in the sense of Lusztig.

3.5 Arthur-Barbasch-Vogan conjecture

- **Conjecture** (1980s): $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ consist of unitary representations.
  - **Barbasch** (1989): proved the conjecture for complex classical groups.

Other questions:

- The size of $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$?
- Construction of special unipotent representations?
  - **Barbasch-Vogan** (1985): answered the above questions (and more) for complex semisimple groups.

4 Real classical groups and special unipotent representations

	G	$\mathbf{G}$	$\mathbf{G}^\vee$	
D_n	$\mathrm{O}(p, 2n-p)$	$\mathrm{O}(2n, \mathbb{C})$	$\mathrm{O}(2n, \mathbb{C})$	D_n
C_n	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$	B_n
B_n	$\mathrm{O}(p, 2n+1-p)$	$\mathrm{O}(2n+1, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$	C_n
$\tilde{C}_n$	$\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$	C_n
D_n	$\mathrm{O}^*(2n)$	$\mathrm{SO}(2n, \mathbb{C})$	$\mathrm{SO}(2n, \mathbb{C})$	D_n
C_n	$\mathrm{Sp}(p, n-p)$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$	B_n
A_n	$\mathrm{U}(p, n-p)$	$\mathrm{GL}(n, \mathbb{C})$	$\mathrm{GL}(n, \mathbb{C})$	A_n

Barbasch-Ma-S.-Z.:

- constructs and classifies all special unipotent repns. of G. As a consequence, the Arthur-Barbasch-Vogan's conjecture holds for G:

All special unipotent repns. of G are unitarizable.

References:

1. D. Barbasch, J.-J. Ma and B. Sun and C.-B. Zhu, Special unipotent representations of real classical groups: counting and reduction to good parity, arXiv:2205.05266.
2. ——, Special unipotent representations of real classical groups: construction and unitarity, arXiv:1712.05552.

Remarks:

- The unitarizability of special unipotent representations for quasisplit classical groups is independently due to Adams, Arancibia Robert and Mezo, as a consequence of their result:

$$\text{Arthur packet} = \text{ABV packet}.$$

- Numerous earlier work on unipotent representations in the framework of theta correspondence: Sahi, Huang-Zhu, Huang-Li, Brylinsky, He, Trapa, Paul-Trapa, Moeglin, Barbasch, ...

4.1 The main tool: theta lifting

- Relevance of theta correspondence for unitary representation theory: Howe, Kashiwara-Vergne, Adams, Li, ...
- Theta lifting: constructive theory (analysis)
  - Non-vanishing: variant of doubling method; degenerate principal series (Earlier work by Rallis, Kudla-Rallis, Lee-Zhu, He, ...)
  - How to distinguish representations constructed by theta lifting

4.2 Counting the representations: algebra and combinatorics

- Count the size of $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ via combinatorial parameters;
  - coherent continuation representation, primitive ideals and double cells (Joseph, Lusztig, Barbasch-Vogan, Vogan);
  - The combinatorial parameters arise from branching laws of Weyl group representations.
- **Theorem** (Barbasch-Ma-S-Z): If $\check{\mathcal{O}}$ has good parity, then

$$\sharp\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \begin{cases} 2^{\sharp\mathrm{PP}_\star(\check{\mathcal{O}})}\cdot\sharp\mathrm{PBP}_G(\check{\mathcal{O}}), & \text{if } \star = C, \tilde{C}; \\ 2\cdot 2^{\sharp\mathrm{PP}_\star(\check{\mathcal{O}})}\cdot\sharp\mathrm{PBP}_G(\check{\mathcal{O}}), & \text{if } \star = B, D. \end{cases}$$

  - PBP stands for painted bipartitions (painting rules depending on the type $\star$ of the group G);
  - $2^{\sharp\mathrm{PP}_\star(\check{\mathcal{O}})}$: the size of Lusztig's canonical quotient.

4.3 Constructing the representations from the parameters

- Motivated by the counting, we define the parameter set:

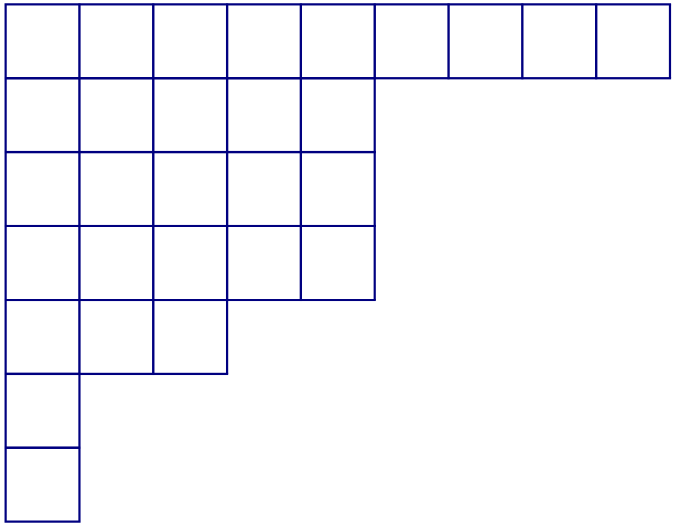
$$\mathrm{PBP}_G^{\mathrm{ext}}(\check{\mathcal{O}}) := \mathrm{PBP}_G(\check{\mathcal{O}}) \times \{\wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}})\}, \quad \star \in \{B, C, D, \widetilde{C}\}.$$

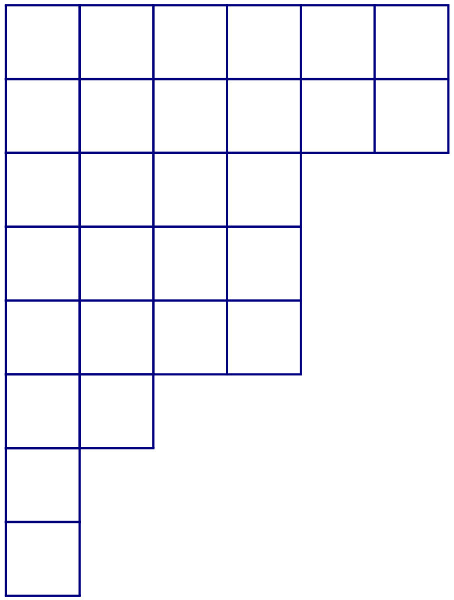
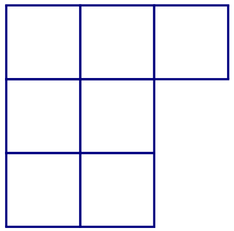
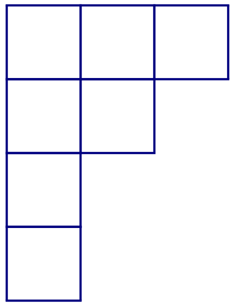
- For each $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}_G^{\mathrm{ext}}(\check{\mathcal{O}})$, we construct $\pi_{\boldsymbol{\tau}} \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$ inductively:

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \check{\Theta}_{\tau'}^{\tau}(\pi_{\boldsymbol{\tau}'}) \otimes (1_{p_\tau, q_\tau}^{+,-})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\Theta}_{\tau'}^{\tau}(\pi_{\boldsymbol{\tau}'} \otimes \det{}^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C};. \end{cases}$$

- Basic idea: descent the combinatorial parameter $\boldsymbol{\tau}$ to $\boldsymbol{\tau}'$ of a smaller group, and then theta-lift the representation.

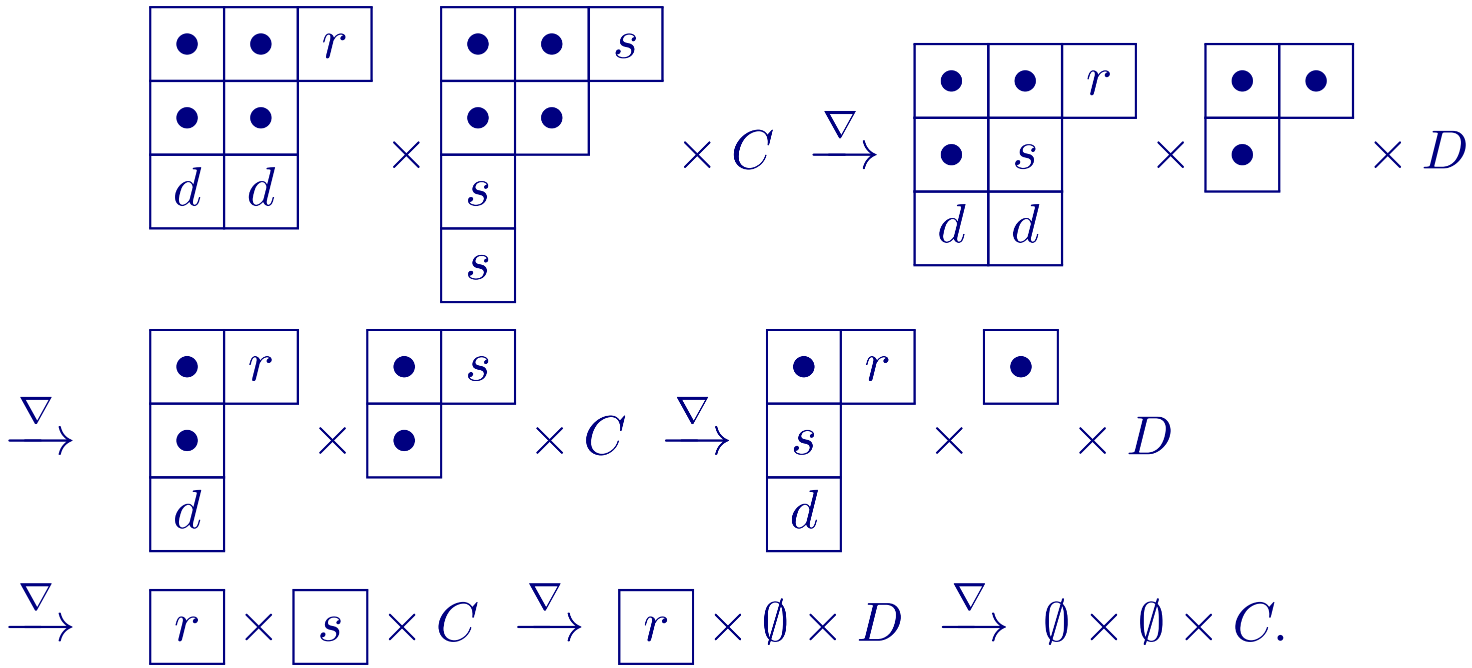
Example: $G = \mathrm{Sp}(28, \mathbb{R})$, $\check{G} = \mathrm{O}(29, \mathbb{C})$.

$\check{\mathcal{O}} =$ 

$\mathcal{O} =$  $\xrightarrow{\text{Springer}} (\imath, \jmath) =$  $\times$ 

$$\mathrm{PP}_\star(\check{\mathcal{O}}) = \{(1, 2), (5, 6)\}$$

Painted bipartition (with symbols $\bullet, s, r, c, d$) and descent:



Corresponding Lie groups

$$
\begin{aligned}
& \mathrm{Sp}(28,\mathbb{R}) \to \mathrm{O}(10,10) \\
\to \quad & \mathrm{Sp}(14,\mathbb{R}) \to \mathrm{O}(5,5) \\
\to \quad & \mathrm{Sp}(4,\mathbb{R}) \to \mathrm{O}(2,0) \to \mathrm{Sp}(0,\mathbb{R}).
\end{aligned}
$$

4.4 Distinguishing the representations

- Moment maps: symmetric space setting

$$\begin{array}{ccccc} \mathfrak{p}_{\mathsf{s}} & \xleftarrow{M_{\mathsf{s}}} & \mathcal{X}_{\mathsf{s},\mathsf{s}'} & \xrightarrow{M_{\mathsf{s}'}} & \mathfrak{p}_{\mathsf{s}'}\,, \\ \phi^*\phi & \longleftarrow\!\dashv & \phi & \longmapsto & \phi\phi^* \end{array}$$

$\rightsquigarrow$ notion of the descent of a nilpotent $K_{\mathsf{s},\mathbb{C}}$-orbit:

$$\mathscr{O} \mapsto \mathscr{O}' =: \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}).$$

$\rightsquigarrow$ notion of the geometric theta lift:

$$\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'} : \mathcal{K}_{\mathsf{s}'}(\mathscr{O}') \to \mathcal{K}_{\mathsf{s}}(\mathscr{O}).$$

– Earlier work: Kraft-Procesi, Przebinda, Nishiyama-Ochiai-Zhu

- **Associated cycles**:
  - The associated cycle in the theta lifting setting has an upper bound via geometric theta lift.

    Theorem (Barbasch-Ma-S-Z): If $\mathcal{O}' = \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O})$ is regular descent, then

    $$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')) \preceq \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\pi')).$$

    * Earlier work: Nishiyama-Ochiai-Taniguchi-Yamashita-Kato, Nishiyama-Zhu, Loke-Ma.
    * Equality is achieved in a good situation, by using the doubling method.
  - In the construction of special unipotent representations by iterated theta lifting, we thus know the associated cycles every step of the way.
    * Enough to tell two representations π_{τ_1} and π_{τ_2} apart.

Finally we may

- conclude the **exhaustion** of special unipotent representations by their counting;
- establish the **unitarity** by using our concrete models.

4.5 A themetic diagram: analysis, combinatorics, and geometry

$$\begin{array}{ccccc}
 & \text{Classical} & & \text{geometry} & \\
\nearrow & & \downarrow & & \nwarrow \\
\text{Comb. parameter} & \underset{\color{blue}{descent}}{\longrightarrow} & \odot & \overset{\color{blue}{lift}}{\longrightarrow} & \text{Representation} \\
\searrow & & \downarrow & & \swarrow \\
 & \text{Associated} & & \text{cycle} &
\end{array}$$

Theta Correspondence and The Orbit Method

are compatible

(for special unipotent representations)!

Thank you!