Local limit theorems via Landau-Kolmogorov inequalities and smoothing

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(joint work with Nathan Ross, UC Berkeley)

Overview

- Recap: Local Limit Theorem.
- A simple idea how to prove it
- A quantitative version of the simple idea
- Landau-Kolmogorov inequalities
- Bounds on the smoothness
- Application 1: triangle counts in Erdős-Rényi random graph
- Application 2: isolated vertices in Erdős-Rényi random graph
- Application 3: magnetization in the Currie-Weiss model

Local Limit Theorem

- Let φ_{μ,σ^2} be the density of a Gaussian r.v. with mean μ and variance σ^2.
- An sequence of integer valued r.v.s $W_1, W_2, \ldots$ satisfies the LLT if

$$\boxed{\sup_k \left| \mathbb{P}[W_n = k] - \varphi_{\mu_n,\sigma_n^2}(k) \right| = \mathrm{o}\left(\frac{1}{\sigma_n}\right) \qquad (n \to \infty).}$$

- Equivalently, if F_n is the c.d.f. of W_n and $\Phi_{\mu,\sigma^2} = \int \varphi_{\mu,\sigma^2}$, then

$$\boxed{\|\Delta F_n - \Delta\Phi_{\mu_n,\sigma_n^2}\|_\infty = \mathrm{o}\left(\frac{1}{\sigma_n}\right) \qquad (n \to \infty).}$$

where $\Delta F(k) = F(k+1) - F(k)$.

LLT for sum of i.i.d. r.v.s

Gnedenko (1948), etc.

- Let $X_1, X_2, \ldots$ be integer-valued and i.i.d. with span 1.
- Let F_n be the c.d.f. of $X_1 + \cdots + X_n$
- Then

$$\boxed{\|\Delta F_n - \Delta \Phi_{\mu_n, \sigma_n^2}\|_\infty = \mathrm{O}\left(\frac{1}{\sigma_n^2}\right) \qquad (n \to \infty).}$$

- Note: This is a rate of convergence.
- Our main goal: prove LLTs for sums of dependent r.v.s with rates of convergence

A simple idea

McDonald (1979), Penrose and Peres (2011), R. (2005)

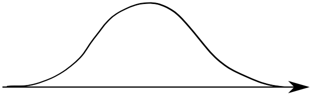
- Let $(W_n)_{n\geqslant 1}$ be integer valued r.v.s with $\mathrm{Var}\, W_n \to \infty$.
- McDonald (1979) proved that

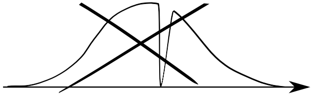
$(W_n)_{n\geqslant 1}$ satisfies CLT and distribution of W_n is "smooth"

$\Rightarrow$

$(W_n)_{n\geqslant 1}$ satisfies LLT

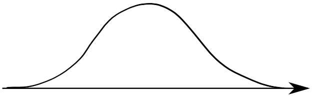
- "Smooth" means that $\|F''\|_\infty$ is small; i.e. the density F' looks like



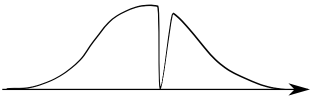


Smoothness

- Let F be the c.d.f. of a r.v. with variance σ^2.
- The following plots show F'



$$\left\|F^{(k)}\right\|_\infty = \mathrm{O}\left(\frac{1}{\sigma^k}\right) \quad \left[\text{e.g. } F(x) = \Phi\left(\frac{x}{\sigma}\right)\right]$$



$$\left\|F^{(1)}\right\|_\infty = \mathrm{O}\left(\frac{1}{\sigma}\right), \quad \left\|F^{(2)}\right\|_\infty \gg \frac{1}{\sigma^2}$$

- Similar for discrete r.v.s: replace $F^{(k)}$ by $\Delta^k F$, where

$$\Delta F(x) = F(x+1) - F(x), \quad \Delta^2 F(x) = \Delta F(x+1) - \Delta F(x), \quad \ldots$$

A quantitative theorem
R.&R. (2012)

- A quantitative version of McDonald (1979):

$$\boxed{\|\Delta F - \Delta G\|_\infty \leqslant C\|F - G\|_\infty^{1/2} \times \left(\|\Delta^2 F\|_\infty + \|\Delta^2 G\|_\infty\right)^{1/2}}$$

- Problem: $\|\Delta^2 F\|_\infty$ is difficult to bound!
- Using $\|\Delta^k F\|_\infty \leqslant \|\Delta^{k+1} F\|_1$, we obtain

$$\boxed{\|\Delta F - \Delta G\|_\infty \leqslant C\|F - G\|_\infty^{1/2} \times \left(\|\Delta^3 F\|_1 + \|\Delta^3 G\|_1\right)^{1/2}}$$

- Note: Often $\|\Delta^k F\|_\infty \approx \|\Delta^{k+1} F\|_1$
- $\|\Delta^{k+1} F\|_1$ is much easier to bound

Landau-Kolmogorov inequalities

Landau (1913), Hardy, Littlewood, and Pólya (1934), Kolmogorov (1962), etc.

Theorem (Kwong and Zettl (1988))

Let $1 \leqslant p, q, r \leqslant \infty$, let $1 \leqslant k < n$. There is a positive constant C such that

$$\|\Delta^k f\|_q \leqslant C \|f\|_p^{1-\beta} \|\Delta^n f\|_r^{\beta} \qquad \forall f$$

with

$$\beta = \frac{k - 1/q + 1/p}{n - 1/r + 1/p}$$

if and only if

$$\frac{n}{q} \leqslant \frac{n-k}{p} + \frac{k}{r}.$$

General estimates

R&R (2012)

Theorem

Let F and G be c.d.f.s with integer support. Then, for $k \geqslant 2$,

$$\|\Delta F - \Delta G\|_\infty \leqslant C\|F - G\|_\infty^{1-\frac{1}{k-1}} \times \left(\|\Delta^k F\|_1 + \|\Delta^k G\|_1\right)^{\frac{1}{k-1}}$$

$$\|\Delta F - \Delta G\|_\infty \leqslant C\|F - G\|_1^{1-\frac{2}{k}} \times \left(\|\Delta^k F\|_1 + \|\Delta^k G\|_1\right)^{\frac{2}{k}}$$

$$\|\Delta F - \Delta G\|_1 \leqslant C\|F - G\|_1^{1-\frac{1}{k}} \times \left(\|\Delta^k F\|_1 + \|\Delta^k G\|_1\right)^{\frac{1}{k}}$$

Tools to bound $\|\Delta^n F\|_1$

Barbour and Xia (1999), Mattner and Roos (2007), R.&R. (2012)

- Let $F(\cdot|X)$ be the conditional c.d.f. of a random variable given X. Then

$$\|\Delta^k F\|_1 \leqslant \mathbb{E}\|\Delta^k F(\cdot|X)\|_1$$

- Let $F_1, F_2, \ldots, F_k$ be discrete c.d.f. Then

$$\|\Delta^{k+1}(F_1 * \cdots * F_k)\|_1 \leqslant \prod_{i=1}^{k} \|\Delta^2 F_i\|_1$$

- Assume $\|\Delta^2 F\|_1 < 2$. Then

$$\|\Delta^2 F^{n*}\|_1 \leqslant \frac{6.4}{\sqrt{n} \times \sqrt{2 - \|\Delta^2 F\|_1}}$$

A smoothness estimate under dependence

R.&R. (2012)

Theorem

Let (W, W') be an exchangeable pair. Let

$$Q_{+1}(W) = \mathbb{P}[W' = W + 1 | W],$$
$$Q_{-1}(W) = \mathbb{P}[W' = W - 1 | W].$$

Then, with F the c.d.f. of W,

$$\|\Delta^2 F\|_1 \leqslant \frac{\sqrt{\operatorname{Var} Q_{+1}(W)}}{\mathbb{E} Q_{+1}(W)} + \frac{\sqrt{\operatorname{Var} Q_{-1}(W)}}{\mathbb{E} Q_{-1}(W)}.$$

- A similar (but more complicated) estimate holds for $\|\Delta^3 F\|_1$.
- Ideas borrowed from Stein's method.

1: Triangles in Erdős-Rényi random graph

R.&R. (2012), based on L_1 bounds of Barbour, Karoński, and Ruciński (1989)

- Let $G(n, p)$ be an Erdős-Rényi random graph
- Let $F_{n,p}$ denote the c.d.f. of the number of triangles in $G(n, p)$
- Let $\Phi_{n,p}$ denote a Gaussian c.d.f. with the same mean and variance as $F_{n,p}$.

Theorem

If $p \sim c/n^{\alpha}$ *for* $c > 0$ *and* $1/2 \leqslant \alpha < 1$, *then* $\sigma_n^2 \asymp n^{3-3\alpha}$ *and*

$$\|\Delta F_{n,p} - \Delta\Phi_{n,p}\|_{\infty} = \mathrm{O}\left(\frac{1}{\sigma_n} \times \frac{1}{n^{(1-\alpha)/2}}\right).$$

Bounds on the smoothness

Lemma

Under the previous conditions,

$$\|\Delta^2 F_{n,p}\|_1 = \mathrm{O}\left(\frac{1}{\sigma_n}\right), \qquad \|\Delta^3 F_{n,p}\|_1 = \mathrm{O}\left(\frac{1}{\sigma_n^2}\right).$$

Proof.

Exchangeable pair: pick one edge at random and resample.
Then, tedious moment bounding exercise. □

2: Isolated vertices in Erdős-Rényi random graph

R.&R. (2012), based on L_1 bounds of Barbour et al. (1989)

- Let $G(n, p)$ be an Erdős-Rényi random graph
- Let $F_{n,p}$ denote the c.d.f. of the number of isolated vertices in $G(n, p)$
- Let $\Phi_{n,p}$ denote a Gaussian c.d.f. with the same mean and variance as $F_{n,p}$.

Theorem

If $\lim\big(\log(n) - np\big) = \infty$ *and either* $\lim(np) = \infty$ *or* $\lim(np) = c > 0$, *then* $\sigma_n^2 \to \infty$ *and*

$$\|\Delta F_{n,p} - \Delta\Phi_{n,p}\|_\infty = \mathrm{O}\left(\frac{1}{\sigma_n} \times \frac{1}{\sigma_n^{1/2}}\right)$$

3: Magnetization in the Currie-Weiss model

R.&R. (2012), based on uniform bounds of Eichelsbacher and Löwe (2010) and convergence results of Ellis, Newman, and Rosen (1980)

- Consider spin configurations $\sigma \in \{-1, +1\}^n$ with Gibbs measure

$$\mathbb{P}[\sigma] = Z_{\beta,h,n}^{-1} \times \exp\left\{-\frac{\beta}{n}\sum_{i<j}\sigma_i\sigma_j + h\sum_i \sigma_i\right\}.$$

- Let F_n be the c.d.f. of the total magnetization $\sum_i \sigma_i/2$.

Theorem

Let $0 < \beta < 1$*. If* $h = 0$*,*

$$\|\Delta F_n - \Delta\Phi_n\|_\infty = \mathrm{O}\left(\frac{1}{\sigma_n} \times \frac{1}{n^{1/4}}\right)$$

and if $h \neq 0$*,*

$$\|\Delta F_n - \Delta\Phi_n\|_\infty = \mathrm{o}\left(\frac{1}{\sigma_n}\right).$$

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