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Anomaly cancellation and modularity, II: The $E_8 \times E_8$ case

In memory of Professor LU QiKeng (1927–2015)

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Abstract In this paper we show that both of the Green-Schwarz anomaly factorization formula for the gauge group $E_8 \times E_8$ and the Hořava-Witten anomaly factorization formula for the gauge group E_8 can be derived through modular forms of weight 14. This answers a question of Schwarz. We also establish generalizations of these factorization formulas and obtain a new Hořava-Witten type factorization formula.

Keywords E_8 bundle, anomaly cancellation, Eisenstein series, modular form

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1 Introduction

In [8, 9, 16], it has been shown that both of the Alvarez-Gaumé-Witten miraculous anomaly cancellation formula [1] and the Green-Schwarz anomaly factorization formula [7] for the gauge group $SO(32)$ can be derived (and extended) through a pair of modularly related modular forms, which are over the modular subgroups $\Gamma_0(2)$ and $\Gamma^0(2)$, respectively. In answering a question of Schwarz [19], we deal with the remaining case of gauge group $E_8 \times E_8$ in this article.

Let $Z \to X \to B$ be a fiber bundle with fiber Z being 10-dimensional. Let TZ be the vertical tangent bundle equipped with a metric g^{TZ} and an associated Levi-Civita connection ∇^{TZ} (see [3, Proposition 10.2]). Let $R^{TZ} = (\nabla^{TZ})^2$ be the curvature of ∇^{TZ}, which we also for simplicity denote by R. Let $T_{\boldsymbol{C}}Z$ be the complexification of TZ with the induced Hermitian connection $\nabla^{T_{\boldsymbol{C}}Z}$.

Let (P_1, ϑ_1) and (P_2, ϑ_2) be two principal E_8 bundles with connections over X. Let ρ be the adjoint representation of E_8. Let $W_i = P_i \times_\rho \boldsymbol{C}^{248}$, $i = 1, 2$ be the associated vector bundles, which are of rank 248. We equip both W_1 and W_2 with Hermitian metrics and Hermitian connections, respectively. Let F_i denote the curvature of the bundle W_i. Let "Tr" denote the trace in the adjoint representation. The elementary facts about E_8 tells us that $\mathrm{Tr}F_i^{2n+1} = 0$, $\mathrm{Tr}F_i^4 = \frac{1}{100}(\mathrm{Tr}F_i^2)^2$ and $\mathrm{Tr}F_i^6 = \frac{1}{7200}(\mathrm{Tr}F_i^2)^3$ (see [2]). It is easy to see that $c_2(W_i) = -\frac{1}{2} \cdot 2\pi \cdot \mathrm{Tr}F_i^2$. Simply denote $\mathrm{Tr}F_1^n + \mathrm{Tr}F_2^n$ by $\mathrm{Tr}F^n$.

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The Green-Schwarz anomaly formula [7] asserts that the following factorization for the 12 forms holds[1)]:

$$\begin{aligned}
I_{12} &= \{\widehat{A}(TZ)\mathrm{ch}(W_1+W_2)+\widehat{A}(TZ)\mathrm{ch}(T_{\boldsymbol{C}}Z)-2\widehat{A}(TZ)\}^{(12)}\\
&= \frac{-1}{64\pi^6}\frac{1}{720}\left(-\frac{15}{8}\mathrm{tr}R^2\mathrm{tr}R^4-\frac{15}{32}(\mathrm{tr}R^2)^3+\mathrm{Tr}F^6+\mathrm{Tr}F^2\left(\frac{1}{16}\mathrm{tr}R^4+\frac{5}{64}(\mathrm{tr}R^2)^2\right)-\frac{5}{8}\mathrm{Tr}F^4\mathrm{tr}R^2\right)\\
&= \frac{-1}{4\pi^2}\frac{1}{2}\left(\mathrm{tr}R^2-\frac{1}{30}\mathrm{Tr}F^2\right)\cdot\frac{1}{16\pi^4}\frac{1}{180}\left(\frac{1}{960}(\mathrm{Tr}F^2)^2-\frac{5}{16}\mathrm{Tr}F^4+\frac{1}{32}\mathrm{tr}R^2\mathrm{Tr}F^2-\frac{15}{16}\mathrm{tr}R^4-\frac{15}{64}(\mathrm{tr}R^2)^2\right)\\
&\ \ \vdots\\
&= \left(p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2))\right)\cdot I_8. \qquad (1.1)
\end{aligned}$$

In [11, 12], Hořava and Witten observed, on the other hand, that the following anomaly factorization formula holds for each $i=1,2$:

$$\begin{aligned}
\widehat{I}^i_{12} &= \left\{\widehat{A}(TZ)\mathrm{ch}(W_i)+\frac{1}{2}\widehat{A}(TZ)\mathrm{ch}(T_{\boldsymbol{C}}Z)-\widehat{A}(TZ)\right\}^{(12)}\\
&= \frac{-1}{64\pi^6}\frac{1}{1440}\left(-\frac{15}{8}\mathrm{tr}R^2\mathrm{tr}R^4-\frac{15}{32}(\mathrm{tr}R^2)^3+2\mathrm{Tr}F_i^6+\mathrm{Tr}F_i^2\left(\frac{1}{8}\mathrm{tr}R^4+\frac{5}{32}(\mathrm{tr}R^2)^2\right)-\frac{5}{4}\mathrm{Tr}F_i^4\mathrm{tr}R^2\right)\\
&= \frac{-1}{4\pi^2}\frac{1}{4}\left(\mathrm{tr}R^2-\frac{1}{15}\mathrm{Tr}F_i^2\right)\cdot\widehat{I}^i_8\\
&= \left(\frac{1}{2}p_1(TZ)+\frac{1}{30}c_2(W_i)\right)\cdot\widehat{I}^i_8, \qquad (1.2)
\end{aligned}$$

where $\widehat{I}^i_8$ can be written explicitly as

$$\widehat{I}^i_8=\frac{1}{16\pi^4}\frac{1}{24}\left(-\frac{1}{4}\left(\frac{1}{2}\mathrm{tr}R^2-\frac{1}{30}\mathrm{Tr}F_i^2\right)^2-\frac{1}{8}\mathrm{tr}R^4+\frac{1}{32}(\mathrm{tr}R^2)^2\right),$$

and therefore

$$I_{12}=\widehat{I}^1_{12}+\widehat{I}^2_{12}=\left(\frac{1}{2}p_1(TZ)+\frac{1}{30}c_2(W_1)\right)\cdot\widehat{I}^1_8+\left(\frac{1}{2}p_1(TZ)+\frac{1}{30}c_2(W_2)\right)\cdot\widehat{I}^2_8.$$

The purpose of this article is to show that the above anomaly factorization formulas can also be derived naturally from modularity as in the orthogonal group case dealt with in [9]. This provides a positive answer to a question of Schwarz mentioned at the beginning of the article.

To be more precise, we construct in Section 2 a modular form $\mathcal{Q}(P_i,P_j,\tau)$ of weight 14 over $SL(2,\boldsymbol{Z})$, for any $i,\ j\in\{1,\ 2\}$, such that when $i=1, j=2$, the modularity of $\mathcal{Q}(P_1,P_2,\tau)$ gives the Green-Schwarz factorization formula (1.1), while when $i=j$, the modularity of $\mathcal{Q}(P_i,P_i,\tau)$ gives the Hořava-Witten factorization formula (1.2). Actually what we construct is a more general modular form $\mathcal{Q}(P_i,P_j,\xi,\tau)$, which involves a complex line bundle (or equivalently a rank two real oriented bundle) and we are able to obtain generalizations of the Green-Schwarz formula and the Hořava-Witten formula by using the associated modularity. Our construction of the modular form $\mathcal{Q}(P_i,P_j,\xi,\tau)$ involves the basic representation of the affine Kac-Moody algebra of E_8.

Inspired by our modular method of deriving the Green-Schwarz and Hořava-Witten factorization formulas, we also construct a modular form $\mathcal{R}(P_i,\xi,\tau)$ of weight 10 over $SL(2,\boldsymbol{Z})$, the modularity of which will give us a new factorization formula of Hořava-Witten type. See Theorem 1.4 for details. It would be interesting to compare (1.8) and (1.9) with the Hořava-Witten factorization (1.2) or (1.6). Actually, another interesting question of Schwarz is to construct quantum field theories associated to the generalized anomaly factorization formulas in this paper and [9].

In the rest of this section, we present our generalized Green-Schwarz and Hořava-Witten formula, as well as the new formulas of Hořava-Witten type obtained from $\mathcal{R}(P_i,\xi,\tau)$. They are proved in Section 2

[1)] In what follows, we write characteristic forms without specifying the connections when there is no confusion (see [20]).

by using modularity after briefly reviewing some knowledge of the affine Kac-Moody algebra of E_8 in Section 1.

Let ξ be a rank two real oriented Euclidean vector bundle over X carrying a Euclidean connection ∇^{ξ}. Let $c=e(\xi,\nabla^{\xi})$ be the Euler form canonically associated to ∇^{ξ} (see [20, Subsection 3.4]). Let $T_{\boldsymbol{C}}Z$ be the complexification of TZ and $\xi_{\boldsymbol{C}}$ the complexification of ξ. For a complex vector bundle E, denote $\widetilde{E}:=E-\boldsymbol{C}^{\mathrm{rk}(E)}$.

Theorem 1.1. *For $1\leqslant i,j\leqslant 2$, the following identities hold*:

$$\begin{aligned}&\{\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(W_i+W_j)+\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(T_{\boldsymbol{C}}Z)-2\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}+\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}}\}^{(12)}\\&=\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)\cdot\left\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}-1}{p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))}\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(\mathfrak{A})\right.\\&\quad\left.+\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\right\}^{(8)},\end{aligned}\tag{1.3}$$

where $\mathfrak{A}=W_i+W_j+T_{\boldsymbol{C}}Z-2+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}}$.

Putting $i=j$, one has for each i,

$$\begin{aligned}&\left\{\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(W_i)+\frac{1}{2}\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(T_{\boldsymbol{C}}Z)-\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}+\frac{1}{2}\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}})\right\}^{(12)}\\&=\left(\frac{1}{2}p_1(TZ)-\frac{3}{2}c^2+\frac{1}{30}c_2(W_i)\right)\cdot\left\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{15}c_2(W_i))}-1}{p_1(TZ)-3c^2+\frac{1}{15}c_2(W_i)}\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(\mathfrak{B}_i)\right.\\&\quad\left.+\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{15}c_2(W_i))}\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\right\}^{(8)},\end{aligned}\tag{1.4}$$

where $\mathfrak{B}_i=2W_i+T_{\boldsymbol{C}}Z-2+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}}$.

If ξ is trivial, we obtain the Green-Schwarz formula (1.1) for $E_8\times E_8$ and the Hořava-Witten formula (1.2) for E_8 in the following corollary.

Corollary 1.2. *One has*

$$\begin{aligned}&\{\widehat{A}(TZ)\mathrm{ch}(W_1+W_2)+\widehat{A}(TZ)\mathrm{ch}(T_{\boldsymbol{C}}Z)-2\widehat{A}(TZ)\}^{(12)}\\&=\left(p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2))\right)\cdot\left\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2)))}-1}{p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2))}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{C})\right.\\&\quad\left.+\mathrm{e}^{\frac{1}{24}(p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2)))}\widehat{A}(TZ)\right\}^{(8)},\end{aligned}\tag{1.5}$$

where $\mathfrak{C}=W_1+W_2+T_{\boldsymbol{C}}Z-2$.

In addition, for each i,

$$\begin{aligned}&\left\{\widehat{A}(TZ)\mathrm{ch}(W_i)+\frac{1}{2}\widehat{A}(TZ)\mathrm{ch}(T_{\boldsymbol{C}}Z)-\widehat{A}(TZ)\right\}^{(12)}\\&=\left(\frac{1}{2}p_1(TZ)+\frac{1}{30}c_2(W_i)\right)\cdot\left\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)+\frac{1}{15}c_2(W_i))}-1}{p_1(TZ)+\frac{1}{15}c_2(W_i)}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{D}_i)\right.\\&\quad\left.+\mathrm{e}^{\frac{1}{24}(p_1(TZ)+\frac{1}{15}c_2(W_i))}\widehat{A}(TZ)\right\}^{(8)},\end{aligned}\tag{1.6}$$

where $\mathfrak{D}_i=2W_i+T_{\boldsymbol{C}}Z-2$.

Remark 1.3. It can be checked by direct computations that the second factors in the right-hand sides of (1.5) and (1.6) are equal to I_8 and $\widehat{I}_8^i$, respectively.

We now state a new factorization formula, which is of the Hořava-Witten type.

Theorem 1.4. *For each i, the following identity holds*:

$$\begin{aligned}&\{\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(W_i)+\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(T_{\boldsymbol{C}}Z)+246\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}+\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}})\}^{(12)}\\&=\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\cdot\bigg\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}-1}{p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)}\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\mathrm{ch}(\mathfrak{E}_i)\\&\quad+\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\mathrm{e}^{\frac{c}{2}}\bigg\}^{(8)},\end{aligned}\tag{1.7}$$

where $\mathfrak{E}_i=W_i+T_{\boldsymbol{C}}Z+246+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}}$.

If ξ is trivial, we have

$$\begin{aligned}&\{\widehat{A}(TZ)\mathrm{ch}(W_i)+\widehat{A}(TZ)\mathrm{ch}(T_{\boldsymbol{C}}Z)+246\widehat{A}(TZ)\}^{(12)}\\&=\left(p_1(TZ)+\frac{1}{30}c_2(W_i)\right)\cdot\bigg\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)+\frac{1}{30}c_2(W_i))}-1}{p_1(TZ)+\frac{1}{30}c_2(W_i)}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{F}_i)\\&\quad+\mathrm{e}^{\frac{1}{24}(p_1(TZ)+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\bigg\}^{(8)},\end{aligned}\tag{1.8}$$

where $\mathfrak{F}_i=W_i+T_{\boldsymbol{C}}Z+246$.

Remark 1.5. We can express (1.8) by direct computations as follows:

$$\begin{aligned}&\frac{-1}{64\pi^6}\frac{1}{1440}\left(-\frac{15}{4}\mathrm{tr}R^2\mathrm{tr}R^4-\frac{15}{16}(\mathrm{tr}R^2)^3+2\mathrm{Tr}F_i^6+\mathrm{Tr}F_i^2\left(\frac{1}{8}\mathrm{tr}R^4+\frac{5}{32}(\mathrm{tr}R^2)^2\right)-\frac{5}{4}\mathrm{Tr}F_i^4\mathrm{tr}R^2\right)\\&=\frac{-1}{4\pi^2}\frac{1}{2}\left(\mathrm{tr}R^2-\frac{1}{30}\mathrm{Tr}F_i^2\right)\cdot\frac{1}{16\pi^4}\frac{1}{180}\left(-\frac{1}{480}(\mathrm{Tr}F_i^2)^2+\frac{1}{32}\mathrm{tr}R^2\mathrm{Tr}F_i^2-\frac{15}{16}\mathrm{tr}R^4-\frac{15}{64}(\mathrm{tr}R^2)^2\right)\\&=\left(p_1(TZ)+\frac{1}{30}c_2(W_i)\right)\cdot\widehat{J}_8^i.\end{aligned}\tag{1.9}$$

Remark 1.6. As in [19], one may ask whether there is a physics model corresponding to (1.8) and (1.9).

We point out that the appearance of the factor $(\mathrm{tr}R^2-\frac{1}{30}\mathrm{Tr}F_i^2)$ in the Green-Schwarz anomaly cancellation formulas being linked to the modularity of certain q-series after being multiplied by

$$\mathrm{e}^{\frac{1}{24}E_2(\tau)\frac{-1}{8\pi^2}(\mathrm{tr}R^2-\frac{1}{30}F^2)}$$

was already realized by physicists (see [15,17]), of which we do not claim priority. However, no concrete and precise formulas were given in those work such that later when needed one still derives the Green-Schwarz type anomaly cancellation formulas by using the traditional method without applying modularity (see [11,12,18]).

Actually, the motivation of our work (as well as [9]) is simply to derive explicitly all the above mentioned formulas in a unified framework of applying modularity. Moreover, the formulas derived in this unified framework make the Green-Schwarz formulas more transparent by showing explicitly what are the factors in the anomaly factorization. For example, the second factor in the right-hand side of (1.5) explicitly gives I_8 in (1.1). This essentially comes from the application of the basic representation of affine E_8, which was not explicitly used in [15, 17]. This unified framework can also inspire many new formulas. For example, (1.8) and (1.9) give a new Hořava-Witten type formula with one E_8 bundle involved; (1.3) and (1.7) give formulas with a complex line bundle being involved, which should deal with anomalies coming from nontriviality of determinant line bundles of (twisted) spin^c Dirac operators. We hope these new formulas could be physically meaningful.

2 The basic representation of affine E_8

In this section, we briefly review the basic representation theory for the affine E_8 by following [13] (see also [14]).

Let $\mathfrak{g}$ be the Lie algebra of E_8. Let $\langle \cdot, \cdot \rangle$ be the Killing form on $\mathfrak{g}$. Let $\widetilde{\mathfrak{g}}$ be the affine Lie algebra corresponding to $\mathfrak{g}$ defined by

$$\widetilde{\mathfrak{g}} = \boldsymbol{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \boldsymbol{C}c,$$

with bracket

$$[P(t) \otimes x + \lambda c, Q(t) \otimes y + \mu c] = P(t)Q(t) \otimes [x, y] + \langle x, y \rangle \operatorname{Res}_{t=0}\left(\frac{dP(t)}{dt}Q(t)\right)c.$$

Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra obtained from $\widetilde{\mathfrak{g}}$ by adding a derivation $t\frac{d}{dt}$, which operates on $\boldsymbol{C}[t, t^{-1}] \otimes \mathfrak{g}$ in an obvious way and sends c to 0.

The basic representation $V(\Lambda_0)$ is the $\widehat{\mathfrak{g}}$-module defined by the property that there is a nonzero vector v_0 (highest weight vector) in $V(\Lambda_0)$ such that $cv_0 = v_0, (\boldsymbol{C}[t] \oplus \boldsymbol{C}t\frac{d}{dt})v_0 = 0$. Setting $V_k := \{v \in V(\Lambda_0) \mid t\frac{d}{dt} = -kv\}$ gives a $\boldsymbol{Z}_+$-gradation by finite spaces. Since $[g, d] = 0$, each V_k is a representation of $\mathfrak{g}$. Moreover, V_1 is the adjoint representation of E_8.

Let $q = \mathrm{e}^{2\pi\sqrt{-1}\tau}$. Fix a basis for the Cartan subalgebra and let $\{z_i\}_{i=1}^8$ be the corresponding coordinates. The character of the basic representation is given by

$$\mathrm{ch}(z_1, z_2, \dots, z_8, \tau) := \sum_{k=0}^{\infty}(\mathrm{ch}V_k)(z_1, z_2, \dots, z_8)q^k = \varphi(\tau)^{-8}\Theta_{\mathfrak{g}}(z_1, z_2, \dots, z_8, \tau),$$

where $\varphi(\tau) = \prod_{n=1}^{\infty}(1 - q^n)$ so that $\eta(\tau) = q^{1/24}\varphi(\tau)$ is the Dedekind η function; $\Theta_{\mathfrak{g}}(z_1, z_2, \dots, z_8, \tau)$ is the theta function defined on the root lattice Q by

$$\Theta_{\mathfrak{g}}(z_1, z_2, \dots, z_8, \tau) = \sum_{\gamma \in Q} q^{|\gamma|^2/2}\mathrm{e}^{2\pi\sqrt{-1}\gamma(\overrightarrow{z})}.$$

It is proved in [6] (see [10]) that there is a basis for the E_8 root lattice such that

$$\Theta_{\mathfrak{g}}(z_1, \dots, z_8, \tau) = \frac{1}{2}\left(\prod_{l=1}^{8}\theta(z_l, \tau) + \prod_{l=1}^{8}\theta_1(z_l, \tau) + \prod_{l=1}^{8}\theta_2(z_l, \tau) + \prod_{l=1}^{8}\theta_3(z_l, \tau)\right), \tag{2.1}$$

where θ and θ_i $(i = 1, 2, 3)$ are the Jacobi theta functions (see [4, 8]).

3 Derivation of Green-Schwarz and Horava-Witten type anomaly factorizations via modularity

In this section, we derive the Green-Schwarz and Hořava-Witten type factorization formulas in Theorems 1.1 and 1.4 via modularity.

For the principal E_8 bundles P_i, $i = 1, 2$, consider the associated bundles

$$\mathcal{V}_i = \sum_{k=0}^{\infty}(P_i \times_{\rho_k} V_k)q^k \in K(X)[[q]].$$

Since ρ_1 is the adjoint representation of E_8, we have $W_i = P_i \times_{\rho_1} V_1$.

Following [5], set

$$\Theta(T_{\boldsymbol{C}}Z, \xi_{\boldsymbol{C}}) := \left(\bigotimes_{m=1}^{\infty}S_{q^m}(\widetilde{T_{\boldsymbol{C}}Z})\right) \otimes \left(\bigotimes_{n=1}^{\infty}\Lambda_{q^n}(\widetilde{\xi_{\boldsymbol{C}}})\right) \otimes \left(\bigotimes_{u=1}^{\infty}\Lambda_{-q^{u-1/2}}(\widetilde{\xi_{\boldsymbol{C}}})\right) \otimes \left(\bigotimes_{v=1}^{\infty}\Lambda_{q^{v-1/2}}(\widetilde{\xi_{\boldsymbol{C}}})\right) \in K(X)[[q]],$$

where $\xi_{\boldsymbol{C}}$ is the complexification of ξ.

Clearly, $\Theta(T_{\boldsymbol{C}}Z, \xi_{\boldsymbol{C}})$ admits a formal Fourier expansion in q as

$$\Theta(T_{\boldsymbol{C}}Z, \xi_{\boldsymbol{C}}) = \boldsymbol{C} + B_1q + B_2q^2 + \cdots, \tag{3.1}$$

where the B_j's are elements in the semi-group formally generated by complex vector bundles over X. Moreover, they carry canonically induced connections denoted by ∇^{B_j}. Let ∇^{Θ} be the induced connection with q-coefficients on Θ.

For $1 \leqslant i, j \leqslant 2$, set

$$\mathcal{Q}(P_i, P_j, \xi, \tau) := \Big\{ \mathrm{e}^{\frac{1}{24}E_2(\tau)(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ) \times \cosh\left(\frac{c}{2}\right)\mathrm{ch}(\Theta(T_{\boldsymbol{C}}Z, \xi_{\boldsymbol{C}}))\varphi(\tau)^{16}\mathrm{ch}(\mathcal{V}_i)\mathrm{ch}(\mathcal{V}_j)\Big\}^{(12)}. \tag{3.2}$$

Theorem 3.1. *$\mathcal{Q}(P_i, P_j, \xi, \tau)$ is a modular form of weight 14 over $SL(2, \boldsymbol{Z})$.*

Proof. By the knowledge reviewed in Section 2, we see that there are formal two forms $y_l^i, 1 \leqslant l \leqslant 8$, $i = 1, 2$ such that

$$\varphi(\tau)^8\mathrm{ch}(\mathcal{V}_i) = \frac{1}{2}\left(\prod_{l=1}^{8}\theta(y_l^i, \tau) + \prod_{l=1}^{8}\theta_1(y_l^i, \tau) + \prod_{l=1}^{8}\theta_2(y_l^i, \tau) + \prod_{l=1}^{8}\theta_3(y_l^i, \tau)\right). \tag{3.3}$$

Since $\theta(z, \tau)$ is an odd function about z, one can see that up to degree 12, the term $\prod_{l=1}^{8}\theta(y_l^i, \tau)$ can be dropped and we have

$$\varphi(\tau)^8\mathrm{ch}(\mathcal{V}_i) = \frac{1}{2}\left(\prod_{l=1}^{8}\theta_1(y_l^i, \tau) + \prod_{l=1}^{8}\theta_2(y_l^i, \tau) + \prod_{l=1}^{8}\theta_3(y_l^i, \tau)\right). \tag{3.4}$$

Since $\theta_1(z, \tau), \theta_2(z, \tau)$ and $\theta_3(z, \tau)$ are all even functions about z, the right-hand side of the above equality only contains even powers of y_j^i's. Therefore, $\mathrm{ch}(W_i)$ only consists of forms of degrees divisible by 4 (this is actually a basic fact about E_8). So

$$\mathrm{ch}(\mathcal{V}_i) = 1 + \mathrm{ch}(W_i)q + \cdots = 1 + (248 - c_2(W_i) + \cdots)q + \cdots. \tag{3.5}$$

On the other hand,

$$\frac{1}{2}\left(\prod_{l=1}^{8}\theta_1(y_l^i, \tau) + \prod_{l=1}^{8}\theta_2(y_l^i, \tau) + \prod_{l=1}^{8}\theta_3(y_l^i, \tau)\right) = 1 + \left(240 + 30\sum_{l=1}^{8}(y_l^i)^2 + \cdots\right)q + O(q^2). \tag{3.6}$$

From (3.4)–(3.6), we have

$$\sum_{l=1}^{8}\left(y_l^i\right)^2 = -\frac{1}{30}c_2(W_i). \tag{3.7}$$

Note that this is also a basic fact about representations of E_8, although it could be deduced in this interesting way by playing with modular forms.

Let $\{\pm 2\pi\sqrt{-1}x_k\}$ $(1 \leqslant k \leqslant 5)$ be the formal Chern roots for $(TZ_{\boldsymbol{C}}, \nabla^{TZ_{\boldsymbol{C}}})$. Let $c = e(\xi, \nabla^{\xi}) = 2\pi\sqrt{-1}u$ be the Euler form canonically associated to ∇^{ξ}. One has

$$\begin{aligned}
&\mathcal{Q}(P_i, P_j, \xi, \tau)\\
&= \Big\{\mathrm{e}^{\frac{1}{24}E_2(\tau)(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\Theta(T_{\boldsymbol{C}}Z, \xi_{\boldsymbol{C}}))\varphi(\tau)^{16}\mathrm{ch}(\mathcal{V}_i)\mathrm{ch}(\mathcal{V}_j)\Big\}^{(12)}\\
&= \Big\{\mathrm{e}^{\frac{1}{24}E_2(\tau)(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\left(\prod_{k=1}^{5}\left(x_k\frac{\theta'(0,\tau)}{\theta(x_k,\tau)}\right)\right)\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\\
&\quad\times\frac{1}{4}\left(\prod_{l=1}^{8}\theta_1(y_l^i, \tau) + \prod_{l=1}^{8}\theta_2(y_l^i, \tau) + \prod_{l=1}^{8}\theta_3(y_l^i, \tau)\right)\\
&\quad\times\left(\prod_{l=1}^{8}\theta_1(y_l^j, \tau) + \prod_{l=1}^{8}\theta_2(y_l^j, \tau) + \prod_{l=1}^{8}\theta_3(y_l^j, \tau)\right)\Big\}^{(12)}.
\end{aligned} \tag{3.8}$$

Then we can perform the transformation formulas for the theta functions and $E_2(\tau)$ (see [4,8]) to show that $\mathcal{Q}(P_i,P_j,\xi,\tau)$ is a modular form of weight 14 over $SL(2,\boldsymbol{Z})$. $\square$

Proof of Theorem 1.1. Expanding the q-series, we have

$$\begin{aligned}
&\mathrm{e}^{\frac{1}{24}E_2(\tau)(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\Theta(T_{\boldsymbol{C}}Z,\xi_{\boldsymbol{C}}))\varphi(\tau)^{16}\mathrm{ch}(\mathcal{V}_i)\mathrm{ch}(\mathcal{V}_j)\\
&\quad=\bigg(\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}-\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\\
&\qquad\times\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)q+O(q^2)\bigg)\cdot\widehat{A}(TZ)\\
&\qquad\times\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\boldsymbol{C}+B_1q+O(q^2))(1-16q+O(q^2))(1+\mathrm{ch}(W_i)q+O(q^2))(1+\mathrm{ch}(W_j)q+O(q^2))\\
&\quad=\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\\
&\qquad+q\bigg(\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1-16+W_i+W_j)\\
&\qquad-\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\bigg)\\
&\qquad+O(q^2).
\end{aligned}\tag{3.9}$$

It is well known that modular forms over $SL(2,\boldsymbol{Z})$ can be expressed as polynomials of the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, where

$$E_4(\tau)=1+240q+2160q^2+6720q^3+\cdots,\tag{3.10}$$

$$E_6(\tau)=1-504q-16632q^2-122976q^3+\cdots.\tag{3.11}$$

Their weights are 4 and 6, respectively.

Since the weight of the modular form $\mathcal{Q}(P_i,P_j,\xi,\tau)$ is 14, it must be a multiple of

$$E_4(\tau)^2E_6(\tau)=1-24q+\cdots.\tag{3.12}$$

So from (3.9) and (3.12), we have

$$\begin{aligned}
&\left\{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1-16+W_i+W_j)\right\}^{(12)}\\
&\quad-\left\{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(12)}\\
&\qquad=-24\left\{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(12)}.
\end{aligned}\tag{3.13}$$

Therefore,

$$\begin{aligned}
&\left\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+W_j+B_1+8)\right\}^{(12)}\\
&\quad=\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)\\
&\qquad\times\bigg\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}-1}{p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+W_j+B_1+8)\\
&\qquad+\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\bigg\}^{(8)}.
\end{aligned}\tag{3.14}$$

To find B_1, we have

$$\begin{aligned}
&\Theta(T_{\boldsymbol{C}}Z,\xi_{\boldsymbol{C}})\\
&=\bigg(\bigotimes_{m=1}^{\infty}S_{q^m}(\widetilde{T_{\boldsymbol{C}}Z})\bigg)\otimes\bigg(\bigotimes_{n=1}^{\infty}\Lambda_{q^n}(\widetilde{\xi_{\boldsymbol{C}}})\bigg)\otimes\bigg(\bigotimes_{u=1}^{\infty}\Lambda_{-q^{u-1/2}}(\widetilde{\xi_{\boldsymbol{C}}})\bigg)\otimes\bigg(\bigotimes_{v=1}^{\infty}\Lambda_{q^{v-1/2}}(\widetilde{\xi_{\boldsymbol{C}}})\bigg)\\
&=(1+(T_{\boldsymbol{C}}Z-10)q+O(q^2))\otimes(1+\widetilde{\xi_{\boldsymbol{C}}}q+O(q^2))\\
&\quad\otimes(1-\widetilde{\xi_{\boldsymbol{C}}}q^{1/2}-2\widetilde{\xi_{\boldsymbol{C}}}q+O(q^{3/2}))\otimes(1+\widetilde{\xi_{\boldsymbol{C}}}q^{1/2}-2\widetilde{\xi_{\boldsymbol{C}}}q+O(q^{3/2}))\\
&=1+(T_{\boldsymbol{C}}Z-10+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}})q+O(q^2).
\end{aligned}\tag{3.15}$$

So

$$B_1=T_{\boldsymbol{C}}Z-10+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}}.$$

Plugging B_1 into (3.14), we have

$$\begin{aligned}
&\bigg\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+W_j+T_{\boldsymbol{C}}Z-2+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}})\bigg\}^{(12)}\\
&\quad=\bigg(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\bigg)\\
&\quad\times\bigg\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}-1}{p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))}\widehat{A}(TZ)\\
&\quad\times\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+W_j+T_{\boldsymbol{C}}Z-2+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}})\\
&\quad+\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\bigg\}^{(8)}.
\end{aligned}\tag{3.16}$$

Since $\mathrm{ch}(W_i)$ and $\mathrm{ch}(W_j)$ only contribute degree $4l$ forms, we can replace $\cosh(\frac{c}{2})$ by $\mathrm{e}^{\frac{c}{2}}$. Then in (2.16), putting $i=1,j=2$ gives (1.4) and putting $i=j$ gives (1.5). □

To prove Theorem 1.4, for each i, set

$$\mathcal{R}(P_i,\xi,\tau):=\bigg\{\mathrm{e}^{\frac{1}{24}E_2(\tau)(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\Theta(T_{\boldsymbol{C}}Z,\xi_{\boldsymbol{C}}))\varphi(\tau)^8\mathrm{ch}(\mathcal{V}_i)\bigg\}^{(12)}.\tag{3.17}$$

Theorem 3.2. *$\mathcal{R}(P_i,\xi,\tau)$ is a modular form of weight* 10 *over $SL(2,\boldsymbol{Z})$.*

Proof. This can be similarly proved to Theorem 3.1 by seeing that

$$\begin{aligned}
&\mathcal{R}(P_i,\xi,\tau)\\
&=\bigg\{\mathrm{e}^{\frac{1}{24}E_2(\tau)(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\Theta(T_{\boldsymbol{C}}Z,\xi_{\boldsymbol{C}}))\varphi(\tau)^8\mathrm{ch}(\mathcal{V}_i)\bigg\}^{(12)}\\
&=\bigg\{\mathrm{e}^{\frac{1}{24}E_2(\tau)(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\bigg(\prod_{l=1}^{5}\bigg(x_l\frac{\theta'(0,\tau)}{\theta(x_l,\tau)}\bigg)\bigg)\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\\
&\quad\times\frac{1}{2}\bigg(\prod_{l=1}^{8}\theta_1(y_l^i,\tau)+\prod_{l=1}^{8}\theta_2(y_l^i,\tau)+\prod_{l=1}^{8}\theta_3(y_l^i,\tau)\bigg)\bigg\}^{(12)},
\end{aligned}\tag{3.18}$$

and then applying the transformation laws of theta functions. □

Proof of Theorem 1.4. Similar to that in the proof of Theorem 1.1, expanding the q-series, we have

$$\begin{aligned}
&\mathrm{e}^{\frac{1}{24}E_2(\tau)(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\Theta(T_{\boldsymbol{C}}Z,\xi_{\boldsymbol{C}}))\varphi(\tau)^8\mathrm{ch}(\mathcal{V}_i)\\
&=\bigg(\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}-\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\bigg(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\bigg)q+O(q^2)\bigg)
\end{aligned}$$

$$
\begin{aligned}
&\times \widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\boldsymbol{C}+B_1q+O(q^2))(1-8q+O(q^2))(1+\mathrm{ch}(W_i)q+O(q^2))\\
=\ &\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\\
&+q\bigg(\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1-8+W_i)\\
&-\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\bigg)\\
&+O(q^2).
\end{aligned}
\tag{3.19}
$$

However, modular form of weight 10 must be a multiple of $E_4(\tau)E_6(\tau)=1-264q+\cdots$, so we have

$$
\begin{aligned}
&\left\{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1-8+W_i)\right\}^{(12)}\\
&\quad-\left\{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(12)}\\
&\quad=-264\left\{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(12)}.
\end{aligned}
\tag{3.20}
$$

Therefore,

$$
\begin{aligned}
&\left\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+B_1+256)\right\}^{(12)}\\
&=\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\cdot\Bigg\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}-1}{p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)}\widehat{A}(TZ)\\
&\quad\times\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+B_1+256)+\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\Bigg\}^{(8)}.
\end{aligned}
\tag{3.21}
$$

Plugging in B_1, we have

$$
\begin{aligned}
&\left\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+T_{\boldsymbol{C}}Z+246+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}})\right\}^{(12)}\\
&=\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\\
&\quad\times\Bigg\{-\frac{\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}-1}{p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+T_{\boldsymbol{C}}Z+246+\widetilde{\xi_{\boldsymbol{C}}}+3\widetilde{\xi_{\boldsymbol{C}}}\otimes\widetilde{\xi_{\boldsymbol{C}}})\\
&\quad+\mathrm{e}^{\frac{1}{24}(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\Bigg\}^{(8)}.
\end{aligned}
\tag{3.22}
$$

Since $\mathrm{ch}(W_i)$ only contribute degree $4l$ forms, we can replace $\cosh(\frac{c}{2})$ by $\mathrm{e}^{\frac{c}{2}}$, and (3.22) gives (1.7). □

4 Discussion

Combining with the results in this paper and our previous work in [9], it is interesting to see that the fundamental anomaly cancellation formulas in various string theories can be unified in the framework of modular forms and modular transformations. This phenomena has its roots in the hidden symmetry of the much larger configuration space of string theory, namely, loop space, double loop space, path space, etc. We expect that many other anomaly cancellation formulas in string theory and M-theory can also be derived from the modular form method.

On the other hand, the modular form method can help one detect new cancellation formulas, for example, in [9] one finds similar cancellation formulas for general gauge group $SO(N)$, not restricted in

$SO(32)$; and in this paper, for a single E_8 bundle, we find (1.9), which is different from the Hořava-Witten's formula (1.2). Moreover, we find formulas with a complex line bundle involved, which give cancellations of anomalies coming from families of (twisted) spin^c Dirac operators instead of spin Dirac operators. We hope these new formulas can find applications in physics.

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