E_8 BUNDLES AND RIGIDITY

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Abstract. In this paper, we establish rigidity and vanishing theorems for Dirac operators twisted by E_8 bundles.

Introduction

Let X be a closed smooth connected manifold which admits a nontrivial S^1 action. Let P be an elliptic differential operator on X commuting with the S^1 action. Then the kernel and cokernel of P are finite dimensional representation of S^1. The equivariant index of P is the virtual character of S^1 defined by

$$\mathrm{Ind}(g, P) = \mathrm{tr}|_g \mathrm{ker} P - \mathrm{tr}|_g \mathrm{coker} P, \tag{0.1}$$

for $g \in S^1$. We call that P is *rigid* with respect to this circle action if $\mathrm{Ind}(g, P)$ is independent of g.

It is well known that classical operators: the signature operator for oriented manifolds, the Dolbeault operator for almost complex manifolds and the Dirac operator for spin manifolds are rigid [2]. In [30], Witten considered the indices of Dirac-like operators on the free loop space LX. The Landweber-Stong-Ochanine elliptic genus ([20], [28]) is just the index of one of these operators. Witten conjectured that these elliptic operators should be rigid. See [19] for a brief early history of the subject. Witten's conjecture were first proved by Taubes [29] and Bott-Taubes [4]. Hirzebruch [13] and Krichever [18] proved Witten's conjecture for almost complex manifold case. Various aspects of mathematics are involved in these proofs. Taubes used analysis of Fredholm operators, Krichever used cobordism, Bott-Taubes and Hirzebruch used Lefschetz fixed point formula. In [22, 23], using modularity, Liu gives simple and unified proof as well as various generalizations of the Witten conjecture. Several new vanishing theorems are also found in [22, 23]. Liu-Ma [24, 25] and Liu-Ma-Zhang [26, 27] established family versions of rigidity and vanishing theorems.

In this paper, we study rigidity and vanishing properties for Dirac operators twisted by E_8 bundles. Let X be an even dimensional closed spin manifold and D the Dirac operator on X. Let P be an (compact-)E_8 principal bundle over X. Let W be the vector bundle over X associated to the complex adjoint representation ρ of E_8. The twisted Dirac operator D^W plays a prominent role in string theory and M theory. In [31], the index of such twisted operator is discovered as part of the phase of the M-theory action. In [8], the partition function in M-theory, involving the index theory of an E_8 bundle, is compared with the partition function in type IIA string theory described by K-theory to test M-theory/Type IIA duality. In this paper, we are interested in the equivariant index of the operator D^W and establish rigidity and vanishing theorems for this operator.

More precisely, let X be a $2k$ dimensional closed spin manifold, which admits a nontrivial S^1 action. Let P be an (compact-)E_8 principal bundle over X such that the S^1 action on X can be lifted to P as a left action which commutes with the free action of E_8 on P. Let W be the complex vector bundle associated to the complex adjoint representation of E_8 mentioned above. Then the S^1 action on P naturally induces an action on W by $g\cdot[s,v]=[g\cdot s,v]$, where $[s,v]$ with $s\in P, v\in \mathbf{C}^{248}$, is the equivalent classes defining the elements in W by the equivalent relations $(s,v)\sim(s\cdot h,\rho(h^{-1})\cdot v)$ for $h\in E_8$. Let X^{S^1} be the fixed point manifold and π be the projection from X^{S^1} to a point pt. Let u be a fixed generator of $H^2(BS^1,\mathbf{Z})$. We have the following theorem:

Theorem 0.1. *Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class* $\frac{1}{30}c_2(W)_{S^1}-p_1(TX)_{S^1}$ *to* X^{S^1} *is equal to* $n\cdot\pi^*u^2$ *for some integer* n.
(i) If $n<0$, *then* $\mathrm{Ind}(g,D^W)$ *is independent of* g *and equal to* $-\mathrm{Ind}(D^{T_{\mathbf{C}}X})$, *minus the index of the Rarita-Schwinger operator. In particular, one has* $\mathrm{Ind}D^W=-\mathrm{Ind}D^{T_{\mathbf{C}}X}$ *and when* k *is odd, i.e. dim* $X\equiv 2\,(mod\,4)$, *one has* $\mathrm{Ind}(g,D^W)\equiv 0$.
(ii) If $n=0$, *then* $\mathrm{Ind}(g,D^W)$ *is independent of* g. *Moreover, when* k *is odd, one has* $\mathrm{Ind}(g,D^W)\equiv 0$.
(iii) If $n=2$ *and* k *is odd, then* $\mathrm{Ind}(g,D^W)\equiv 0$.

Actually we have established rigidity and vanishing results in more general settings concerning the twisted spin^c Dirac operators. See Theorem 2.1 and Theorem 2.2 for details. The above theorem is a corollary of Theorem 2.1. We prove our theorems by studying the modularity of Lefschetz numbers of certain elliptic operators involving the basic representation of the affine Kac-Moody algebra of E_8. In the rest of the paper, we will first briefly review the Jacobi theta functions and the basic representation for the affine E_8 by following [16] (see also [17]) as the preliminary knowledge in Section 1 and then state our theorems as well as give their proofs in Section 2.

1. Preliminaries

1.1. **Jacobi theta functions.** The four Jacobi theta-functions are defined as follows (cf. [5]),

$$\theta(z,\tau)=2q^{1/8}\sin(\pi z)\prod_{j=1}^{\infty}[(1-q^j)(1-e^{2\pi\sqrt{-1}z}q^j)(1-e^{-2\pi\sqrt{-1}z}q^j)], \tag{1.1}$$

(1.2) $\theta_1(z,\tau)=2q^{1/8}\cos(\pi z)\prod_{j=1}^{\infty}[(1-q^j)(1+e^{2\pi\sqrt{-1}z}q^j)(1+e^{-2\pi\sqrt{-1}z}q^j)],$

(1.3) $\theta_2(z,\tau)=\prod_{j=1}^{\infty}[(1-q^j)(1-e^{2\pi\sqrt{-1}z}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}z}q^{j-1/2})],$

(1.4) $\theta_3(z,\tau)=\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}z}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}z}q^{j-1/2})\right],$

where $q=e^{2\pi\sqrt{-1}\tau}$, $\tau\in\mathbf{H}$, the upper half plane.

They are all holomorphic functions for $(z,\tau)\in\mathbf{C}\times\mathbf{H}$, where $\mathbf{C}$ is the complex plane.

Let $\theta'(0,\tau)=\frac{\partial}{\partial z}\theta(z,\tau)|_{z=0}$. One has the following Jacobi identity (c.f. [5]),

(1.5) $$\theta'(0,\tau)=\pi\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau).$$

Let

$$SL(2,\mathbf{Z}):=\left\{\left(\begin{array}{cc}a_1 & a_2\\ a_3 & a_4\end{array}\right)\middle|\, a_1,a_2,a_3,a_4\in\mathbf{Z},\ a_1a_4-a_2a_3=1\right\}$$

be the modular group. Let $S=\left(\begin{array}{cc}0 & -1\\ 1 & 0\end{array}\right)$, $T=\left(\begin{array}{cc}1 & 1\\ 0 & 1\end{array}\right)$ be the two generators of $SL(2,\mathbf{Z})$. Their actions on $\mathbf{H}$ are given by

$$S:\tau\mapsto-\frac{1}{\tau},\quad T:\tau\mapsto\tau+1.$$

The actions on theta-functions by S and T are given by the following transformation formulas (cf. [5]),

(1.6)
$$\theta(z,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta(z,\tau),\quad \theta\left(z,-1/\tau\right)=\frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau z^2}\theta\left(\tau z,\tau\right);$$

(1.7)
$$\theta_1(z,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta_1(z,\tau),\quad \theta_1\left(z,-1/\tau\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau z^2}\theta_2(\tau z,\tau);$$

(1.8) $\theta_2(z,\tau+1)=\theta_3(z,\tau),\quad \theta_2\left(z,-1/\tau\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau z^2}\theta_1(\tau z,\tau);$

(1.9) $\theta_3(z,\tau+1)=\theta_2(z,\tau),\quad \theta_3\left(z,-1/\tau\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau z^2}\theta_3(\tau z,\tau).$

One also has the following formulas about how the theta functions vary along the lattice $\Gamma=\{a+b\tau|a,b\in\mathbf{Z}\}$ (cf. [5]),

(1.10)
$$\theta(z+a,\tau)=(-1)^a\theta(z,\tau),\ \theta(z+b\tau,\tau)=(-1)^be^{-2\pi\sqrt{-1}bz-\pi\sqrt{-1}b^2\tau}\theta(z,\tau);$$

(1.11)
$$\theta_1(z+a,\tau)=(-1)^a\theta_1(z,\tau),\ \theta_1(z+b\tau,\tau)=e^{-2\pi\sqrt{-1}bz-\pi\sqrt{-1}b^2\tau}\theta_1(z,\tau);$$
(1.12)
$$\theta_2(z+a,\tau)=\theta_2(z,\tau),\ \theta_2(z+b\tau,\tau)=(-1)^b e^{-2\pi\sqrt{-1}bz-\pi\sqrt{-1}b^2\tau}\theta_2(z,\tau);$$

$$\theta_3(z+a,\tau)=\theta_3(z,\tau),\ \theta_3(z+b\tau,\tau)=e^{-2\pi\sqrt{-1}bz-\pi\sqrt{-1}b^2\tau}\theta_3(z,\tau). \tag{1.13}$$

1.2. **The basic representation for the affine E_8.** In this subsection we briefly review the basic representation for the affine E_8 following [16] (see also [17]).

Let $\mathfrak{g}$ be the (complex) Lie algebra of E_8. Let $\langle,\rangle$ be the Killing form on $\mathfrak{g}$. Let $\widetilde{\mathfrak{g}}$ be the affine Lie algebra corresponding to $\mathfrak{g}$ defined by

$$\widetilde{\mathfrak{g}}=\mathbf{C}[t,t^{-1}]\otimes\mathfrak{g}\oplus\mathbf{C}c,$$

with bracket

$$[P(t)\otimes x+\lambda c, Q(t)\otimes y+\mu c]=P(t)Q(t)\otimes[x,y]+\langle x,y\rangle\,\mathrm{Res}_{t=0}\left(\frac{dP(t)}{dt}Q(t)\right)c.$$

Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra obtained from $\widetilde{\mathfrak{g}}$ by adding a derivation $t\frac{d}{dt}$ which operates on $\mathbf{C}[t,t^{-1}]\otimes\mathfrak{g}$ in an obvious way and sends c to 0.

The basic representation $V(\Lambda_0)$ is the $\widehat{\mathfrak{g}}$-module defined by the property that there is a nonzero vector v_0 (highest weight vector) in $V(\Lambda_0)$ such that $cv_0=v_0, (\mathbf{C}[t]\otimes\mathfrak{g}\oplus\mathbf{C}t\frac{d}{dt})v_0=0$. Setting $V_i:=\{v\in V(\Lambda_0)|t\frac{d}{dt}v=-iv\}$ gives a $\mathbf{Z}_+$-gradation by finite dimensional subspaces. Since $[\mathfrak{g},t\frac{d}{dt}]=0$, each V_i is a representation of $\mathfrak{g}$. Moreover, V_1 is the adjoint representation of E_8.

Fix a basis $\{Z_i\}_{i=1}^8$ for the Cartan subalgebra. The character of the basic representation is given by
(1.14)
$$\mathrm{ch}(z_1,z_2,\cdots,z_8,\tau):=\sum_{i=0}^{\infty}(\mathrm{ch}V_i)(z_1,z_2,\cdots,z_8)q^i=\varphi(\tau)^{-8}\Theta_{\mathfrak{g}}(z_1,z_2,\cdots,z_8,\tau),$$

where $\varphi(\tau)=\prod_{n=1}^{\infty}(1-q^n)$ so that $\eta(\tau)=q^{1/24}\varphi(\tau)$ is the Dedekind η function; $\Theta_{\mathfrak{g}}(z_1,z_2,\cdots,z_8,\tau)$ is the theta function defined on the root lattice Q by

$$\Theta_{\mathfrak{g}}(z_1,z_2,\cdots,z_8,\tau)=\sum_{\gamma\in Q}q^{|\gamma|^2/2}e^{2\pi\sqrt{-1}\gamma(\sum_{l=1}^8 z_lZ_l)}. \tag{1.15}$$

It is proved in [10] (cf. [11]) that there is a basis for the E_8 root lattice such that
(1.16)
$$\Theta_{\mathfrak{g}}(z_1,\cdots .z_8,\tau)=\frac{1}{2}\left(\prod_{l=1}^8\theta(z_l,\tau)+\prod_{l=1}^8\theta_1(z_l,\tau)+\prod_{l=1}^8\theta_2(z_l,\tau)+\prod_{l=1}^8\theta_3(z_l,\tau)\right).$$

2. E_8 Bundles and Rigidity

In this section we prove two rigidity and vanishing theorems for spin^c manifolds with E_8 principal bundles. Theorem 0.1 is deduced from the first one (Theorem 2.1).

Let X be a $2k$ dimensional closed spin^c manifold, which admits a nontrivial S^1 action that preserves the spin^c structure. Let L be the complex line bundle associated with the spin^c structure of X. It's the associated line bundle of the $U(1)$-bundle $Q/spin(2k) \to Q/spin^c(2k) \cong X$, where Q is the $spin^c(2k)$ principal bundle over X determined by the $spin^c$ structure. We denote the first equivariant Chern class of L by $c_1(X)_{S^1}$. Let P be an E_8 principal bundle over X such that the S^1 action on X can be lifted to P as a left action which commutes with the free action of E_8 on P. Let W be the vector bundle associated to the complex adjoint representation of E_8 mentioned above. Then the S^1 action on P naturally induces an action on W as described in the introduction.

Let g^{TX} be a Riemannian metric on X. Let ∇^{TX} be the Levi-Civita connection associated to g^{TX}. Denote the complexification of TX by $T_{\mathbf{C}}X$. Let $g^{T_{\mathbf{C}}X}$ and $\nabla^{T_{\mathbf{C}}X}$ be the induced Hermitian metric and Hermitian connection on $T_{\mathbf{C}}X$. Let h^L be a Hermitian metric on L and ∇^L be a Hermitian connection. Let $\overline{L}$ be the complex conjugate of L with the induced Hermitian metric and connection. Assume that the S^1 action on X preserves the metrics and connections involved. Let $S_c(TX) = S_{c,+}(TX) \oplus S_{c,-}(TX)$ denote the bundle of spinors associated to the spin^c structure, (TX, g^{TX}) and (L, h^L). Then $S_c(TX)$ carries induced Hermitian metric and connection preserving the above $\mathbf{Z}_2$-grading. Let $D_{c,\pm} : \Gamma(S_{c,\pm}(TX)) \to \Gamma(S_{c,\mp}(TX))$ denote the induced spin^c Dirac operators (cf. [21]). If V is an equivariant complex vector bundle over X with equivariant Hermitian metric h^V and Hermitian connection ∇^V, let $D^V_{c,\pm} : \Gamma(S_{c,\pm}(TX) \otimes V) \to \Gamma(S_{c,\mp}(TX) \otimes V)$ denote the induced twisted spin^c Dirac operators.

Theorem 2.1. *Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class*

$$\frac{1}{30}c_2(W)_{S^1} + 3c_1(X)^2_{S^1} - p_1(TX)_{S^1}$$

to X^{S^1} is equal to $n \cdot \pi^* u^2$ for some integer n.
(i) If $n < 0$, then

$$\mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X - (L^2+\overline{L}^2)+(L+\overline{L}))}) \equiv 0.$$

In particular,

$$\mathrm{Ind}D_{c,+}^{(1+\overline{L})\otimes W} + \mathrm{Ind}D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X - (L^2+\overline{L}^2)+(L+\overline{L}))} = 0.$$

(ii) If $n = 0$, then

$$\mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X - (L^2+\overline{L}^2)+(L+\overline{L}))})$$

is independent of g. Moreover, when k is odd, one has

$$\mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L}))}) \equiv 0.$$

(iii) If $n = 2$ and k is odd, then

$$\mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L}))}) \equiv 0.$$

Proof. Let $g = e^{2\pi\sqrt{-1}t} \in S^1$ be the generator of the action group. Let $X^{S^1} = \{p\}$ be the set of fixed points. Let $TX|p = E_1 \oplus \cdots \oplus E_k$ be the decomposition of the tangent bundle into the S^1-invariant 2-planes. Assume that g acts on E_j by $e^{2\pi\sqrt{-1}\alpha_j t}$, $\alpha_j \in \mathbf{Z}$. Assume g acts on $L|_p$ by $e^{2\pi\sqrt{-1}ct}$, $c \in \mathbf{Z}$. Clearly,

$$p_1(TM|_p)_{S^1} = (2\pi\sqrt{-1})^2 \sum_{j=1}^{k} {\alpha_j}^2 t^2,\ c_1(L|_p)_{S^1} = 2\pi\sqrt{-1}ct. \tag{2.1}$$

Denote $L \oplus \overline{L}$ by $L_{\mathbf{C}}$. If E is a complex vector bundle over X, set $\widetilde{E} = E - \mathbf{C}^{\mathrm{rk}(E)} \in K(X)$.

Let $\Theta(X, L, \tau)$ be the virtual complex vector bundle over X defined by

$$\begin{aligned}\Theta(X,L,\tau) := &\left(\bigotimes_{m=1}^{\infty} S_{q^m}(\widetilde{T_{\mathbf{C}}X})\right) \otimes \left(\bigotimes_{u=1}^{\infty} \Lambda_{q^u}(\widetilde{L_{\mathbf{C}}})\right) \\ &\otimes \left(\bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-1/2}}(\widetilde{L_{\mathbf{C}}})\right) \otimes \left(\bigotimes_{w=1}^{\infty} \Lambda_{q^{w-1/2}}(\widetilde{L_{\mathbf{C}}})\right),\end{aligned}$$

Let W_i $(i = 0, 1. \cdots)$ be the associated bundles $P \times_{\rho_i} V_i$, where V_i's are the representations of E_8 as in §1.2. Then $W = W_1$.

Consider the twisted operator

$$D_{c,+}^{(1+\overline{L})\otimes\Theta(X,L,\tau)\otimes(\varphi^8(\tau)\sum_{i=0}^{\infty} W_i q^i)}. \tag{2.2}$$

Expanding q-series, we have

(2.3)

$$\begin{aligned}&\Theta(X,L,\tau) \otimes (\varphi^8(\tau)\sum_{i=0}^{\infty} W_i q^i) \\ =&(1 + (T_{\mathbf{C}}X - 2k)q + O(q^2)) \otimes (1 + \widetilde{L_{\mathbf{C}}}q + O(q^2)) \\ &\otimes (1 - \widetilde{L_{\mathbf{C}}}q^{1/2} - 2\widetilde{L_{\mathbf{C}}}q + O(q^{3/2})) \otimes (1 + \widetilde{L_{\mathbf{C}}}q^{1/2} - 2\widetilde{L_{\mathbf{C}}}q + O(q^{3/2})) \\ &\otimes (1 - 8q + O(q^2)) \otimes (1 + Wq + O(q^2)) \\ =&1 + (W - 8 + T_{\mathbf{C}}X - 2k - 3\widetilde{L_{\mathbf{C}}} - \widetilde{L_{\mathbf{C}}} \otimes \widetilde{L_{\mathbf{C}}})q + O(q^2).\end{aligned}$$

It's not hard to see that $\widetilde{L_{\mathbf{C}}} \otimes \widetilde{L_{\mathbf{C}}} = L^2 + \overline{L}^2 - 4(L + \overline{L}) + 6$. So

$$\begin{aligned}&D_{c,+}^{(1+\overline{L})\otimes\Theta(M,L,\tau)\otimes(\varphi^8(\tau)\sum_{i=0}^{\infty} W_i q^i)} \\ =&D_{c,+}^{(1+\overline{L})} + D_{c,+}^{(1+\overline{L})\otimes(W+T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L})-8-2k)}q + O(q^2).\end{aligned} \tag{2.4}$$

By the Atiyah-Bott-Segal-Singer Letschetz fixed point formula, for the twisted operator $D_{c,+}^{(1+\overline{L})\otimes\Theta(X,L,\tau)\otimes(\varphi^8(\tau)\sum_{i=0}^{\infty}W_iq^i)}$, the equivariant index

$$\begin{aligned}I(t,\tau)=&2\sum_p\left\{\frac{1}{(2\pi\sqrt{-1})^k}\prod_{j=1}^k\frac{\theta'(0,\tau)}{\theta(\alpha_jt,\tau)}\frac{\theta_1(ct,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(ct,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(ct,\tau)}{\theta_3(0,\tau)}\right.\\&\left.\cdot\varphi^8(\tau)\cdot\left(\sum_{i=0}^{\infty}\mathrm{ch}(W_i|_p)_{S^1}q^i\right)\right\}.\end{aligned}\tag{2.5}$$

On the fixed point p, fixing an element $s\in P|p$, one can define a map $f_s:S^1\to E_8$ by $g\cdot s=s\cdot f_s(g)$. It's not hard to check that f_s is a group homomorphism. Moreover, for $h\in E_8$, we have

$$g\cdot(s\cdot h)=(g\cdot s)\cdot h=s\cdot f_s(g)\cdot h=(s\cdot h)\cdot(h^{-1}f_s(g)h).$$

As all the maximal tori in E_8 are conjugate, then one may choose $s\in P|p$ such that $f_s:S^1\to E_8$ maps S^1 into the maximal torus $\mathfrak{t}$ that corresponds to the Cartan subalgebra such that the theta function $\Theta_{\mathfrak{g}}(z_1,\cdots,z_8,\tau)$ appears as in (1.16). For any unitary representation $\rho:E_8\to U(N)$, let $\mathfrak{T}$ be a maximal torus of $U(N)$ that contains $\rho(\mathfrak{t})$. Let

$$\widehat{\mathfrak{T}}\xrightarrow{\widehat{\rho}}\widehat{\mathfrak{t}}\xrightarrow{\widehat{f_s}}\widehat{S^1}$$

be the induced maps on the character groups. Assume $\widehat{f_s}(z_i)=\beta_it$. Let $\{x_i\}$ are basis for $\widehat{\mathfrak{T}}$. By definition,

$$(\mathrm{ch}\rho)(z_1,z_2,\cdots,z_8)=\sum_{i=1}^Ne^{\widehat{\rho}(x_i)},$$

and therefore

$$\begin{aligned}&(\mathrm{ch}\rho)(\beta_1t,\beta_2t,\cdots,\beta_8t)\\=&\widehat{f_s}((\mathrm{ch}\rho)(z_1,z_2,\cdots,z_8))\\=&\sum_{i=1}^Ne^{(\widehat{f_s}\circ\widehat{\rho})(x_i)}\\=&\mathrm{ch}((P\times_\rho\mathbf{C}^N)|_p)_{S^1}.\end{aligned}$$

So for each i, we have $\mathrm{ch}(W_i|p)_{S^1}=(\mathrm{ch}V_i)(\beta_1t,\beta_2t,\cdots,\beta_8t)$. Then by (1.14) and (1.16), we have

$$\begin{aligned}&\varphi^8(\tau)\cdot\left(\sum_{i=0}^{\infty}\mathrm{ch}(W_i|_p)_{S^1}q^i\right)\\=&\frac{1}{2}\left(\prod_{l=1}^8\theta(\beta_lt,\tau)+\prod_{l=1}^8\theta_1(\beta_lt,\tau)+\prod_{l=1}^8\theta_2(\beta_lt,\tau)+\prod_{l=1}^8\theta_3(\beta_lt,\tau)\right).\end{aligned}\tag{2.6}$$

Comparing both sides of (2.6), we can see by direct computation that

$$30\cdot(2\pi\sqrt{-1})^2\sum_{l=1}^{8}\beta_l^2t^2=c_2(W|_p)_{S^1}. \tag{2.7}$$

By (2.5) and (2.6), we have

(2.8)
$$I(t,\tau)=\sum_p\left\{\frac{1}{(2\pi\sqrt{-1})^k}\prod_{j=1}^{k}\frac{\theta'(0,\tau)}{\theta(\alpha_jt,\tau)}\frac{\theta_1(ct,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(ct,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(ct,\tau)}{\theta_3(0,\tau)}\right.$$
$$\left.\cdot\left(\prod_{l=1}^{8}\theta(\beta_lt,\tau)+\prod_{l=1}^{8}\theta_1(\beta_lt,\tau)+\prod_{l=1}^{8}\theta_2(\beta_lt,\tau)+\prod_{l=1}^{8}\theta_3(\beta_lt,\tau)\right)\right\}.$$

From the transformation laws of theta functions (1.10)-(1.13), for $a,b\in 2\mathbf{Z}$, it's not hard to see that

$$I(t+a\tau+b,\tau)=e^{-\pi\sqrt{-1}(\sum_{l=1}^{8}\beta_l^2+3c^2-\sum_{j=1}^{k}m_j^2)(b^2\tau+2b\tau)}I(t,\tau).$$

Since when restricted to fixed points, $\frac{1}{30}c_2(W)_{S^1}+3c_1(L)_{S^1}^2-p_1(TX)_{S^1}$ is equal to $n\cdot\pi^*u^2$, then for each fixed point, from (2.1) and (2.7) we have

$$\sum_{l=1}^{8}\beta_l^2+3c^2-\sum_{j=1}^{k}\alpha_j^2=n$$

and therefore

$$I(t+a\tau+b,\tau)=e^{-\pi\sqrt{-1}n(b^2\tau+2b\tau)}I(t,\tau). \tag{2.9}$$

It's easy to deduce from (1.6) that

$$\theta'(0,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta'(0,\tau),\quad \theta'\left(0,-1/\tau\right)=\frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}\tau\theta'\left(0,\tau\right).$$

Using the above two formulas and the transformation laws of theta functions (1.6)-(1.9), we have

$$I(t,\tau+1)=I(t,\tau) \tag{2.10}$$

and

(2.11)
$$I\left(\frac{t}{\tau},-\frac{1}{\tau}\right)=\tau^{k+4}e^{\frac{\pi\sqrt{-1}\left(\sum_{l=1}^{8}\beta_l^2+3c^2-\sum_{j=1}^{k}\alpha_j^2\right)t^2}{\tau}}I(t,\tau)=\tau^{k+4}e^{\frac{\pi\sqrt{-1}nt^2}{\tau}}I(t,\tau).$$

(2.9)-(2.11) tell us that $I(t,\tau)$ obeys the transformation laws that a Jacobi form (see [9]) should satisfy.

Next we shall prove that $I(t,\tau)$ is holomorphic for $(t,\tau)\in\mathbf{C}\times\mathbf{H}$. First, we have the following lemma:

Lemma 2.1. *$I(t,\tau)$ is holomorphic for $(t,\tau)\in\mathbf{R}\times\mathbf{H}$.*

The proof of this lemma is almost verbatimly same as the proof of Lemma 1.3 in [22]. We shall prove that $I(t,\tau)$ is actually holomorphic on $\mathbf{C}\times\mathbf{H}$. The possible polar divisor of $I(t,\tau)$ can be written in the form $t=\frac{m(c\tau+d)}{l}$ for integers m,l,c,d with $(c,d)=1$. Assume $\frac{m(c\tau+d)}{l}$ is a pole for $I(t,\tau)$. Find integers a,b such that $ad-bc=1$. Consider the function $I\left(\frac{t}{-c\tau+a},\frac{d\tau-b}{-c\tau+a}\right)$. By (2.10) and (2.11), it's easy to see that

$$I\left(\frac{t}{-c\tau+a},\frac{d\tau-b}{-c\tau+a}\right)=f(t,\tau)\cdot I(t,\tau), \tag{2.12}$$

where $f(t,\tau)$ is an entire function of t for every $\tau\in\mathbf{H}$. If $\tau'=\frac{a\tau+b}{c\tau+d}$, then $\tau=\frac{d\tau'-b}{-c\tau'+a}$ and $\frac{m\left(c\frac{d\tau'-b}{-c\tau'+a}+d\right)}{l}$ is a pole for the function $I\left(t,\frac{d\tau'-b}{-c\tau'+a}\right)$. However by (2.12), we have

$$\begin{aligned}
&I\left(\frac{m\left(c\frac{d\tau'-b}{-c\tau'+a}+d\right)}{l},\frac{d\tau'-b}{-c\tau'+a}\right)\\
=&I\left(\frac{\frac{m}{l}}{-c\tau'+a},\frac{d\tau'-b}{-c\tau'+a}\right)\\
=&f\left(\frac{m}{l},\tau'\right)\cdot I\left(\frac{m}{l},\tau'\right).
\end{aligned}$$

As $\frac{m}{l}$ is real, by Lemma 2.1, we get a contradiction. Therefore $I(t,\tau)$ is holomorphic for $(t,\tau)\in\mathbf{C}\times\mathbf{H}$.

Combining the transformation formulas (2.9)-(2.11) and the holomorphicity of $I(t,\tau)$ on $\mathbf{C}\times\mathbf{H}$, we see that $I(t,\tau)$ is a weak Jacobi form of index $\frac{n}{2}$ and weight $k+4$ over $(2\mathbf{Z})^2\rtimes SL(2,\mathbf{Z})$. Here by weak Jacobi form, we don't require the regularity condition at the cusp but only require that at the cusp q appears with nonnegative powers only. We refer to [9] for the precise definition of the Jacobi forms.

If $n=0$, by (2.9), we see that $I(t,\tau)$ is holomorphic on the torus

$$\mathbf{C}/2\mathbf{Z}+2\mathbf{Z}\tau$$

and therefore must be independent of t. So, by (2.4), we see that

$$\mathrm{Ind}(g,D_{c,+}^{(1+\overline{L})}),$$

$$\mathrm{Ind}(g,D_{c,+}^{(1+\overline{L})\otimes(W+T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L})-8-2k)})$$

are both independent of g. So

$$\mathrm{Ind}(g,D_{c,+}^{(1+\overline{L})\otimes W})+\mathrm{Ind}(g,D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L}))})$$

must be independent of g. The index density of the operator

$$D_{c,+}^{(1+\overline{L})\otimes W}+D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L}))}$$

involves the characteristic forms

$$\widehat{A}(TM), e^{c_1(L)/2}(1+e^{-c_1(L)}), \mathrm{ch}(W), \mathrm{ch}(T_{\mathbf{C}}M), \mathrm{ch}(L+\overline{L}), \mathrm{ch}(L^2+\overline{L}^2),$$

which are all of degree $4l$ (noting that W is the complexification of the real adjoint representation of compact E_8). Therefore by the Atiyah-Singer index theorem, $\mathrm{Ind}D_{c,+}^{(1+\overline{L})\otimes W} + \mathrm{Ind}D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L}))}$ (i.e. when $g=id$) must be 0 when the dimension of the manifold is not divisible by 4. So when k is odd,

$$\mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes(T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L}))}) \equiv 0.$$

This finishes the proof of part (ii).

If $n \neq 0$, i.e in the case of nonzero anomaly, we need the following two lemmas.

Lemma 2.2 (Theorem 1.2 in [9]). *Let I be a weak Jacobi form of index m and weight h. Then for fixed τ, if not identically 0, I has exactly $2m$ zeros in any fundamental domain for the action of the lattice on $\mathbf{C}$.*

Lemma 2.3 (Theorem 2.2 in [9]). *Let I be a weak Jacobi form of index m and weight h. If $m=1$ and h is odd, then I is identically 0.*

We would like to point that Lemma 2.2 and Lemma 2.3 are stated in [9] for Jacobi forms. However, as in the proofs of them no regularity condition at the cusp are used, we state them here for weak Jacobi forms. See [9] for details.

If $n<0$, then by Lemma 2.2, $I(t,\tau) \equiv 0$, therefore

$$\mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})}) \equiv 0,$$

$$\mathrm{Ind}(g, D_{c,+}^{(1+\overline{L})\otimes(W+T_{\mathbf{C}}X-(L^2+\overline{L}^2)+(L+\overline{L})-8-2k)}) \equiv 0.$$

So part (i) follows.

If $n=2$, as the the the weight of $I(t,\tau)$ is $k+4$, so part (iii) similarly follows clearly from Lemma 2.3.

□

Theorem 0.1 can be easily deduced from Theorem 2.1 as follows.

Proof of Theorem 0.1: When X is a spin manifold, L is trivial and $D_{c,+}=D$. By the Atiyah-Hirzebruch vanishing theorem ([2]), we have $\mathrm{Ind}(g,D) \equiv 0$. Moreover by the Witten rigidity theorem ([29, 4, 22], the operator $D^{T_{\mathbf{C}}X}$ is rigid. i.e. $\mathrm{Ind}(g, D^{T_{\mathbf{C}}X}) \equiv \mathrm{Ind}D^{T_{\mathbf{C}}X}$. Also note that $\mathrm{Ind}D^{T_{\mathbf{C}}X}$ equals to 0 when k is odd. Then the three parts in Theorem 0.1 easily follow from the corresponding three parts in Theorem 2.1. □

For Spin^c manifolds, we have rigidity and vanishing theorem for another type of twisted operators.

Theorem 2.2. *Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class*

$$\frac{1}{30}c_2(W)_{S^1} + c_1(X)^2_{S^1} - p_1(TX)_{S^1}$$

to X^{S^1} is equal to $n \cdot \pi^* u^2$ for some integer n.
(i) If $n < 0$, then

$$\mathrm{Ind}(g, D_{c,+}^{(1-\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1-\overline{L})\otimes(T_{\mathbf{C}}X-(L+\overline{L}))}) \equiv 0.$$

In particular,

$$\mathrm{Ind} D_{c,+}^{(1-\overline{L})\otimes W} + \mathrm{Ind} D_{c,+}^{(1-\overline{L})\otimes(T_{\mathbf{C}}X-(L+\overline{L}))} = 0.$$

(ii) If $n = 0$, then

$$\mathrm{Ind}(g, D_{c,+}^{(1-\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1-\overline{L})\otimes(T_{\mathbf{C}}X-(L+\overline{L}))})$$

is independent of g. Moreover, when k is even, one has

$$\mathrm{Ind}(g, D_{c,+}^{(1-\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1-\overline{L})\otimes(T_{\mathbf{C}}X-(L+\overline{L}))}) \equiv 0.$$

(iii) If $n = 2$ and k is even, then

$$\mathrm{Ind}(g, D_{c,+}^{(1-\overline{L})\otimes W}) + \mathrm{Ind}(g, D_{c,+}^{(1-\overline{L})\otimes(T_{\mathbf{C}}X-(L+\overline{L}))}) \equiv 0.$$

Proof. We will use same notations as in the proof of Theorem 2.1.

Let $\Theta^*(X, L, \tau)$ be the virtual complex vector bundles over X defined by

$$\Theta^*(X, L, \tau) := \left(\bigotimes_{m=1}^{\infty} S_{q^m}(\widetilde{T_{\mathbf{C}}X}) \right) \otimes \left(\bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\widetilde{L_{\mathbf{C}}}) \right).$$

Consider the twisted operator

$$D_{c,+}^{(1-\overline{L})\otimes\Theta^*(X,L,\tau)\otimes(\varphi^8(\tau)\sum_{i=0}^{\infty} W_i q^i)}. \tag{2.13}$$

Expanding q-series, we have

$$\begin{aligned} &\Theta^*(X, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i) \\ =&(1 + (T_{\mathbf{C}}X - 2k)q + O(q^2)) \otimes (1 - \widetilde{L_{\mathbf{C}}}q + O(q^2)) \\ &\otimes (1 - 8q + O(q^2)) \otimes (1 + Wq + O(q^2)) \\ =&1 + (W + T_{\mathbf{C}}X - (L + \overline{L}) - 2k - 6)q + O(q^2). \end{aligned} \tag{2.14}$$

So

$$\begin{aligned} &D_{c,+}^{(1-\overline{L})\otimes\Theta^*(X,L,\tau)\otimes(\varphi^8(\tau)\sum_{i=0}^{\infty} W_i q^i)} \\ =&D_{c,+}^{(1-\overline{L})} + D_{c,+}^{(1-\overline{L})\otimes(W+T_{\mathbf{C}}X-(L+\overline{L})-2k-6)} q + O(q^2). \end{aligned} \tag{2.15}$$

By the Atiyah-Bott-Segal-Singer Letschetz fixed point formula, for this twisted operator $D_{c,+}^{(1-\overline{L})\otimes\Theta^*(X,L,\tau)\otimes(\varphi^8(\tau)\sum_{i=0}^\infty W_i q^i)}$, the equivariant index

(2.16)

$$\begin{aligned} J(t,\tau) =& 2\sum_p \left\{ \frac{1}{(2\pi\sqrt{-1})^k}\prod_{j=1}^k \frac{\theta'(0,\tau)}{\theta(\alpha_j t,\tau)}\frac{\theta(ct,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)} \right. \\ & \left. \cdot\varphi^8(\tau)\cdot\left(\sum_{i=0}^\infty \mathrm{ch}(W_i|_p)_{S^1} q^i\right)\right\} \\ =& \sum_p \left\{ \frac{1}{(2\pi\sqrt{-1})^k}\prod_{j=1}^k \frac{\theta'(0,\tau)}{\theta(\alpha_j t,\tau)}\frac{\theta(ct,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)} \right. \\ & \left. \cdot\left(\prod_{l=1}^8\theta(\beta_l t,\tau)+\prod_{l=1}^8\theta_1(\beta_l t,\tau)+\prod_{l=1}^8\theta_2(\beta_l t,\tau)+\prod_{l=1}^8\theta_3(\beta_l t,\tau)\right)\right\}. \end{aligned}$$

As when restricted to fixed points, $\frac{1}{30}c_2(W)_{S^1}+c_1(L)_{S^1}^2-p_1(TX)_{S^1}$ is equal to $n\cdot\pi^*u^2$, then for each fixed point, we have

$$\sum_{l=1}^8\beta_l^2+c^2-\sum_{j=1}^k\alpha_j^2=n.$$

Therefore, similar to (2.9), one can show that for $a,b\in 2\mathbf{Z}$

$$J(t+a\tau+b,\tau)=e^{-\pi\sqrt{-1}n(b^2\tau+2b\tau)}J(t,\tau). \tag{2.17}$$

One can also show that

$$J(t,\tau+1)=J(t,\tau) \tag{2.18}$$

and

$$J\left(\frac{t}{\tau},-\frac{1}{\tau}\right)=\tau^{k+3}e^{\frac{\pi\sqrt{-1}nt^2}{\tau}}J(t,\tau). \tag{2.19}$$

So similar to $I(t,\tau)$ in the proof of Theorem 2.1, combing Lemma 2.1 and the above transformation laws, we can prove that $J(t,\tau)$ is a weak Jacobi form of index $\frac{n}{2}$ and weight $k+3$ over $(2\mathbf{Z})^2\rtimes SL(2,\mathbf{Z})$.

Then one can prove the three parts of Theorem 2.2 almost the same as those in Theorem 2.1. The only difference one needs to notice is that by the Atiyah-Singer index theorem, $\mathrm{Ind}D_{c,+}^{(1-\overline{L})\otimes W}+\mathrm{Ind}D_{c,+}^{(1-\overline{L})\otimes(T_{\mathbf{C}}X-(L+\overline{L}))}$ must be 0 when the dimension of the manifold is divisible by 4 as the index density of the operator

$$D_{c,+}^{(1-\overline{L})\otimes W}+D_{c,+}^{(1-\overline{L})\otimes(T_{\mathbf{C}}X-(L+\overline{L}))}$$

is a differential form of degree $4l+2$. □

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