

MOD 3 CONGRUENCE AND TWISTED SIGNATURE OF 24 DIMENSIONAL STRING MANIFOLDS

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ABSTRACT. In this paper, by combining modularity of the Witten genus and the modular forms constructed by Liu and Wang, we establish mod 3 congruence properties of certain twisted signatures of 24 dimensional string manifolds.

INTRODUCTION

Let M be a $2n$ dimensional smooth closed oriented manifold. Let g^{TM} be a Riemannian metric on TM and ∇^{TM} the associated Levi-Civita connection. Let V be a complex vector bundle over M with a Hermitian metric h^V and a unitary connection ∇^V .

Let $\Lambda_{\mathbf{C}}(T^*M)$ be the complexified exterior algebra bundle of TM and let $\langle \cdot, \cdot \rangle_{\Lambda_{\mathbf{C}}(T^*M)}$ be the Hermitian metric on $\Lambda_{\mathbf{C}}(T^*M)$ induced by g^{TM} . Let dv be the Riemannian volume form associated to g^{TM} . Then $\Gamma(M, \Lambda_{\mathbf{C}}(T^*M))$ has a Hermitian metric such that for $\alpha, \alpha' \in \Gamma(M, \Lambda_{\mathbf{C}}(T^*M))$,

$$\langle \alpha, \alpha' \rangle = \int_M \langle \alpha, \alpha' \rangle_{\Lambda_{\mathbf{C}}(T^*M)} dv.$$

For $X \in TM$, let $c(X)$ be the Clifford action on $\Lambda_{\mathbf{C}}(T^*M)$ defined by $c(X) = X^* - i_X$, where $X^* \in T^*M$ corresponds to X via g^{TM} . Let $\{e_1, e_2, \dots, e_{2n}\}$ be an oriented orthogonal basis of TM . Set

$$\Omega = (\sqrt{-1})^n c(e_1) \cdots c(e_{2n}).$$

Then one can show that Ω is independent of the choice of the orthonormal basis and $\Omega_V = \Omega \otimes 1$ is a self-adjoint element acting on $\Lambda_{\mathbf{C}}(T^*M) \otimes V$ such that $\Omega_V^2 = \text{Id}|_{\Lambda_{\mathbf{C}}(T^*M) \otimes V}$.

Let d be the exterior differentiation operator and d^* be the formal adjoint of d with respect to the Hermitian metric. The operator

$$D_{Sig} := d + d^* = \sum_{i=1}^{2n} c(e_i) \nabla_{e_i}^{\Lambda_{\mathbf{C}}(T^*M)} : \Gamma(M, \Lambda_{\mathbf{C}}(T^*M)) \rightarrow \Gamma(M, \Lambda_{\mathbf{C}}(T^*M))$$

is the signature operator, and the more general twisted signature operator is defined as (cf. [6])

$$D_{Sig} \otimes V := \sum_{i=1}^{2n} c(e_i) \nabla_{e_i}^{\Lambda_{\mathbf{C}}(T^*M) \otimes V} : \Gamma(M, \Lambda_{\mathbf{C}}(T^*M) \otimes V) \rightarrow \Gamma(M, \Lambda_{\mathbf{C}}(T^*M) \otimes V).$$

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The operators $D_{Sig} \otimes V$ and Ω_V anticommute. If we decompose $\Lambda_{\mathbb{C}}(T^*M) \otimes V = \Lambda_{\mathbb{C}}^+(T^*M) \otimes V \oplus \Lambda_{\mathbb{C}}^-(T^*M) \otimes V$ into ± 1 eigenspaces of Ω_V , then $D_{Sig} \otimes V$ decomposes to define

$$(D_{Sig} \otimes V)^{\pm} : \Gamma(M, \Lambda_{\mathbb{C}}^{\pm}(T^*M) \otimes V) \rightarrow \Gamma(M, \Lambda_{\mathbb{C}}^{\mp}(T^*M) \otimes V).$$

The index of the operator $(D_{Sig} \otimes V)^+$ is called the *twisted signature* of M and denoted by $Sig(M, V)$. By the Atiyah-Singer index theorem,

$$Sig(M, V) = \int_M \widehat{L}(TM, \nabla^{TM}) \text{ch}(V, \nabla^V)$$

(see Section 1.2 for the definitions of \widehat{L} and ch as well as some explanation of the above formula).

When V is trivial, $Sig(M, V)$ is just the signature of M , denoted by $Sig(M)$. Let $T_{\mathbb{C}}M$ be the complexification of TM . When $V = T_{\mathbb{C}}M, T_{\mathbb{C}}M \otimes T_{\mathbb{C}}M$ and $\Lambda^2 T_{\mathbb{C}}M$, simply denote $Sig(M, V)$ by $Sig(M, \mathcal{T}), Sig(M, \mathcal{T} \otimes \mathcal{T})$ and $Sig(M, \Lambda^2 \mathcal{T})$ respectively.

Further, assume that M is spin. Let O be the $SO(2n)$ bundle of oriented orthogonal frames in TM . Since TM is spin, the $SO(2n)$ bundle $O \xrightarrow{e} M$ lifts to a $Spin(2n)$ bundle $O' \xrightarrow{\sigma} O \xrightarrow{e} M$ such that σ induces the covering projection $Spin(2n) \rightarrow SO(2n)$ on each fiber. Let $\Delta(TM), \Delta(TM)^{\pm}$ denote the Hermitian bundles of spinors

$$\Delta(TM) = O' \times_{Spin(2n)} S_{2n}, \quad \Delta(TM)^{\pm} = O' \times_{Spin(2n)} S_{\pm, 2n},$$

where $S_{2n} = S_{+, 2n} \oplus S_{-, 2n}$ is the complex spinor representation. The connection ∇^{TM} on O lifts to a connection on O' . $\Delta(TM), \Delta(TM)^{\pm}$ are then naturally endowed with a unitary connection, which we simply denote by ∇ .

The elements of TM act by Clifford multiplication on $\Delta(TM) \otimes V$. Define the twisted Dirac operator $D \otimes V$ to be $\sum_{i=1}^{2n} e_i \nabla_{e_i}^{\Delta(TM) \otimes V}$. Let $(D \otimes V)^{\pm}$ denote the restriction of $D \otimes V$ to $\Delta(TM)^{\pm} \otimes V$. The twisted operator $(D \otimes T_{\mathbb{C}}M)^+$ is known as the *Rarita-Schwinger operator* [23]. By the Atiyah-Singer index theorem,

$$Ind((D \otimes V)^+) = \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(V, \nabla^V)$$

(see (1.17) for the definition of \widehat{A}).

On spin manifolds, there are divisibility properties for the signature and twisted signatures. The famous Rokhlin theorem ([20]) says that when M is a 4 dimensional smooth closed spin manifold, $Sig(M)$ is divisible by 16. Ochanine ([19]) generalizes the Rokhlin congruence to higher dimensions by proving that when M is an $8k + 4$ dimensional smooth closed spin manifold, the signature $Sig(M)$ is divisible by 16. The Hirzebruch divisibilities ([9]; cf. [5]) assert that when M is an $8k + 4$ dimensional smooth closed spin manifold, the twisted signature $Sig(M, \mathcal{T})$ is divisible by 256, while when M is $8k$ dimensional, $Sig(M, \mathcal{T})$ is divisible by 2048. In [5], the authors show that when M is an $8k + 4$ dimensional smooth closed spin manifold with $k \geq 1$, the twisted signature $Sig(M, \mathcal{T} \otimes \mathcal{T})$ is divisible by 256, while when M is $8k$ dimensional with $k \geq 2$, $Sig(M, \mathcal{T} \otimes \mathcal{T})$ is divisible by 2048.

A spin manifold M is called *string* if $\frac{p_1(M)}{2} = 0$, where $\frac{p_1(M)}{2}$ is a degree 4 integral cohomology class determined by the spin structure of M , twice of which

is equal to the first Pontryagin class $p_1(M)$. On a $4k$ dimensional smooth closed string manifold M , the *Witten genus* ([22])

$$W(M) := \int_M \widehat{A}(TM) \text{ch} \left(\bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \right)$$

is a modular form of weight $2k$ over $SL(2, \mathbf{Z})$ with integral Fourier expansion ([24]). See Section 1 for details. 24 is an interesting dimension for string manifolds. For example, the Hirzebruch prize question [10] asks for the existence of a 24 dimensional string manifold M such that $\int_M \widehat{A}(M) = 1, \int_M \widehat{A}(M) \text{ch}(T_{\mathbf{C}}M) = 0$ (answered positively by Hopkins-Mahowald [11]) and to find such a string manifold on which the Monster group acts by diffeomorphism (still open).

In this paper, we study 24 dimensional string manifolds and obtain the following mod 3 congruence of the twisted signature and the index of the Rarita-Schwinger operator by combining the modularity of the Witten genus and the modular forms constructed by Liu and Wang in [17].

Theorem 0.1. *If M is a 24 dimensional smooth closed string manifold, then*

$$(0.1) \quad \text{Sig}(M, \Lambda^2 \mathcal{T}) \equiv \text{Ind}((D \otimes T_{\mathbf{C}}M)^+) \pmod{3\mathbf{Z}}.$$

Let Ω_{4k}^{String} be the string cobordism group in dimension $4k$ and tmf be the theory of topological modular form developed by Hopkins and Miller ([11]). Let $MF_{2k}^{\mathbf{Z}}(SL(2, \mathbf{Z}))$ be the space of modular forms of weight $2k$ over $SL(2, \mathbf{Z})$ with integral Fourier expansion. The Witten genus W is equal to the composition of the maps ([11])

$$\Omega_{4k}^{String} \xrightarrow{\sigma} tmf^{-4k}(pt) \xrightarrow{e} MF_{2k}^{\mathbf{Z}}(SL(2, \mathbf{Z})),$$

where σ is the *refined Witten genus* and e is the edge homomorphism in a spectral sequence. Hopkins and Mahowald ([11]) show that σ is surjective. For $i, l \geq 0, j = 0, 1$, define

$$a_{i,j,l} = \begin{cases} 1 & i > 0, j = 0, \\ 2 & j = 1, \\ 24/\text{gcd}(24, l) & i, j = 0. \end{cases}$$

Hopkins and Mahowald also show that the image of e (and therefore the image of the Witten genus) has a basis given by monomials

$$(0.2) \quad \alpha_{i,j,l} E_4(\tau)^i E_6(\tau)^j \Delta(\tau)^l, \quad i, l \geq 0, j = 0, 1,$$

where

$$E_4(\tau) = 1 + 240(q + 9q^2 + 28q^3 + \dots),$$

$$E_6(\tau) = 1 - 504(q + 33q^2 + 244q^3 + \dots)$$

are the Eisenstein series and $\Delta(\tau) = q \prod_{n \geq 0} (1 - q^n)^{24}$ is the modular discriminant (see Section 1.1). Their weights are 4, 6, 12 respectively. In dimension 24, the image of the Witten genus is spanned by the monomials $E_4(\tau)^3, 24\Delta(\tau)$, and since $\int_M \widehat{A}(M) \text{ch}(T_{\mathbf{C}}M) - 24 \int_M \widehat{A}(M)$ is the coefficient of q in the expansion of the Witten genus, one has that $\text{Ind}((D \otimes T_{\mathbf{C}}M)^+) = \int_M \widehat{A}(M) \text{ch}(T_{\mathbf{C}}M)$ is divisible by 24 (this observation is due to Teichner [21]). Therefore, by Theorem 0.1, we have

Corollary 0.1. *If M is a 24 dimensional smooth closed string manifold, then*

$$(0.3) \quad 3 | \text{Sig}(M, \Lambda^2 \mathcal{T}).$$

One naturally asks if the string condition is indispensable for the mod 3 divisibility in Corollary 0.1. We answer this question as follows. Let B^8 be such a Bott manifold, which is 8 dimensional and spin with the A -hat genus $\widehat{A}(B^8) = 1$, $Sig(B^8) = 0$ ([15]). Let \mathbf{HP}^2 be a quaternionic projective plane. $B^8 \times \mathbf{HP}^2 \times \mathbf{HP}^2$ is a 24 dimensional spin manifold but not string. In Section 3, we will show that

$$(0.4) \quad 3 \nmid Sig(B^8 \times \mathbf{HP}^2 \times \mathbf{HP}^2, \Lambda^2 \mathcal{T}),$$

and therefore the string condition is indispensable.

One can also show that the power of 3 cannot be increased for the divisibility in Corollary 0.1. Let M_0^8 be the 8 dimensional Milnor-Kervaire almost-parallelizable manifold. It is a string manifold. Consider the 24 dimensional string manifold $M_0^8 \times M_0^8 \times M_0^8$. In Section 3, we will show that

$$(0.5) \quad 3 \mid Sig(M_0^8 \times M_0^8 \times M_0^8, \Lambda^2 \mathcal{T})$$

but

$$(0.6) \quad 3^2 \nmid Sig(M_0^8 \times M_0^8 \times M_0^8, \Lambda^2 \mathcal{T}).$$

We would like to point out that $M_0^8 \times M_0^8 \times M_0^8$ is an interesting 24 dimensional string manifold with $W(M_0^8 \times M_0^8 \times M_0^8) = -E_4(\tau)^3$. See Section 3 for details.

The paper is organized as follows. In Section 1, we review some basic knowledge of Jacobi theta functions, modular forms and then review the Witten genus as well as the modular forms constructed by Liu and Wang. In Section 2, we prove Theorem 0.1 by combining modularity of the Witten genus and the Liu-Wang modular forms. The examples and computation are included in Section 3.

1. MODULAR FORMS AND CHARACTERISTIC FORMS

1.1. Preliminary on the Jacobi theta functions and modular forms. Let

$$SL_2(\mathbf{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z}, ad - bc = 1 \right\}$$

as usual be the modular group. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the two generators of $SL_2(\mathbf{Z})$. Their actions on \mathbf{H} , the upper half plane, are given by

$$S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1.$$

The four Jacobi theta functions are defined as follows (cf. [4]):

$$\theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} \left[(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^j)(1 - e^{-2\pi\sqrt{-1}v} q^j) \right],$$

$$\theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} \left[(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^j)(1 + e^{-2\pi\sqrt{-1}v} q^j) \right],$$

$$\theta_2(v, \tau) = \prod_{j=1}^{\infty} \left[(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 - e^{-2\pi\sqrt{-1}v} q^{j-1/2}) \right],$$

$$\theta_3(v, \tau) = \prod_{j=1}^{\infty} \left[(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 + e^{-2\pi\sqrt{-1}v} q^{j-1/2}) \right],$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$. They are holomorphic functions for $(v, \tau) \in \mathbf{C} \times \mathbf{H}$.

If we act on the theta-functions by S and T , they obey the following transformation laws (cf. [4]):

(1.1)

$$\theta(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta(v, \tau), \quad \theta(v, -1/\tau) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau) ;$$

(1.2) $\theta_1(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta_1(v, \tau), \quad \theta_1(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_2(\tau v, \tau) ;$

(1.3) $\theta_2(v, \tau + 1) = \theta_3(v, \tau), \quad \theta_2(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_1(\tau v, \tau) ;$

(1.4) $\theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_3(\tau v, \tau) .$

Definition 1.1. Let Γ be a subgroup of $SL_2(\mathbf{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbf{H} \cup \{\infty\}$ such that for any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma ,$$

the following property holds:

$$f(g\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^l f(\tau),$$

where $\chi : \Gamma \rightarrow \mathbf{C}^*$ is a character of Γ and l is called the weight of f .

Let

(1.5)
$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2k-1} \right) q^n$$

be the Eisenstein series, where B_{2k} is the $2k$ -th Bernoulli number. When $k > 1$, $E_{2k}(\tau)$ is a modular form of weight $2k$ over $SL_2(\mathbf{Z})$. However, unlike other Eisenstein theories, $E_2(\tau)$ is not a modular form over $SL(2, \mathbf{Z})$. Instead $E_2(\tau)$ is a quasimodular form over $SL(2, \mathbf{Z})$, satisfying

(1.6)
$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6\sqrt{-1}c(c\tau + d)}{\pi}.$$

In particular, we have

(1.7)
$$E_2(\tau + 1) = E_2(\tau),$$

(1.8)
$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) - \frac{6\sqrt{-1}\tau}{\pi}.$$

For the precise definition of quasimodular forms, see [12].

Explicitly, we have

$$(1.9) \quad E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \left(\sum_{d|n} d \right) q^n = 1 - 24q - 72q^2 - 96q^3 - \dots$$

and

$$(1.10) \quad \begin{aligned} E_4(\tau) &= 1 + 240(q + 9q^2 + 28q^3 + \dots), \\ E_6(\tau) &= 1 - 504(q + 33q^2 + 244q^3 + \dots). \end{aligned}$$

Let

$$(1.11) \quad \Delta(\tau) = \frac{1}{1728} (E_4(\tau)^3 - E_6(\tau)^2) = q \prod_{n \geq 0} (1 - q^n)^{24} = q - 24q^2 + 252q^3 + \dots$$

be the modular discriminant.

Theorem 1.1 (Tate). *The ring of integral modular forms is*

$$MF_*^{\mathbf{Z}} \cong \mathbf{Z}[E_4(\tau), E_6(\tau), \Delta(\tau)] / (E_4(\tau)^3 - E_6(\tau)^2 = 1728\Delta(\tau)).$$

In the following, let's briefly review some level 2 modular forms. Let

$$\begin{aligned} \Gamma_0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \pmod{2} \right\}, \\ \Gamma^0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid b \equiv 0 \pmod{2} \right\} \end{aligned}$$

be the two modular subgroups of $SL_2(\mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST and the generators of $\Gamma^0(2)$ are STS, T^2STS (cf. [4]).

Simply write $\theta_j = \theta_j(0, \tau)$, $1 \leq j \leq 3$. Define (cf. [16])

$$\begin{aligned} \delta_1(\tau) &= \frac{1}{8}(\theta_2^4 + \theta_3^4), & \varepsilon_1(\tau) &= \frac{1}{16}\theta_2^4\theta_3^4, \\ \delta_2(\tau) &= -\frac{1}{8}(\theta_1^4 + \theta_3^4), & \varepsilon_2(\tau) &= \frac{1}{16}\theta_1^4\theta_3^4. \end{aligned}$$

More explicitly, we have

$$(1.12) \quad \delta_1(\tau) = \frac{1}{4} + 6 \sum_{n=1}^{\infty} \sum_{d|n, d \text{ odd}} d \cdot q^n = \frac{1}{4} + 6q + 6q^2 + \dots,$$

$$(1.13) \quad \varepsilon_1(\tau) = \frac{1}{16} + \sum_{n=1}^{\infty} \sum_{d|n} (-1)^d d^3 \cdot q^n = \frac{1}{16} - q + 7q^2 + \dots,$$

$$(1.14) \quad \delta_2(\tau) = -\frac{1}{8} - 3 \sum_{n=1}^{\infty} \sum_{d|n, d \text{ odd}} d \cdot q^{\frac{n}{2}} = -\frac{1}{8} - 3q^{1/2} - 3q - 12q^{3/2} + \dots$$

and

$$(1.15) \quad \varepsilon_2(\tau) = \sum_{n=1}^{\infty} \sum_{d|n, n/d \text{ odd}} d^3 \cdot q^{\frac{n}{2}} = q^{1/2} + 8q + 28q^{3/2} \dots,$$

where the “...” terms are the higher degree terms, all of which have integral coefficients.

If Γ is a modular subgroup, let $M_*^{\mathbf{Z}}(\Gamma)$ denote the ring of modular forms over Γ with integral Fourier coefficients.

Theorem 1.2 (cf. [16]). *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$; $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, and moreover $M_*^{\mathbf{Z}}(\Gamma^0(2)) = \mathbf{Z}[8\delta_2(\tau), \varepsilon_2(\tau)]$. Furthermore, we have the transformation laws*

$$(1.16) \quad \delta_2\left(-\frac{1}{\tau}\right) = \tau^2\delta_1(\tau), \quad \varepsilon_2\left(-\frac{1}{\tau}\right) = \tau^4\varepsilon_1(\tau).$$

1.2. Modular characteristic forms. Let M be a $4k$ dimensional smooth Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM} . ∇^{TM} extends canonically to a Hermitian connection $\nabla^{T_{\mathbf{C}}M}$ on $T_{\mathbf{C}}M = TM \otimes \mathbf{C}$.

Let $\widehat{A}(TM, \nabla^{TM})$ and $\widehat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined respectively by (cf. [25])

$$(1.17) \quad \begin{aligned} \widehat{A}(TM, \nabla^{TM}) &= \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi} R^{TM}\right)} \right), \\ \widehat{L}(TM, \nabla^{TM}) &= \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{4\pi} R^{TM}\right)} \right). \end{aligned}$$

Note that $\widehat{L}(TM, \nabla^{TM})$ defined here is different from the classical Hirzebruch L -form defined by

$$L(TM, \nabla^{TM}) = \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{2\pi} R^{TM}\right)} \right).$$

However, they give the same top (degree $4k$) forms, and therefore when M is oriented

$$\int_M \widehat{L}(TM, \nabla^{TM}) = \int_M L(TM, \nabla^{TM}).$$

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^E, ∇^F respectively. Let $R^E = (\nabla^E)^2$ (resp. $R^F = (\nabla^F)^2$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as (cf. [25])

$$(1.18) \quad \text{ch}(G, \nabla^G) = \text{tr} \left[\exp\left(\frac{\sqrt{-1}}{2\pi} R^E\right) \right] - \text{tr} \left[\exp\left(\frac{\sqrt{-1}}{2\pi} R^F\right) \right].$$

Let $\text{ch}(G, \nabla^G) = \sum_{i=0}^{2k} \text{ch}^i(G, \nabla^G)$ such that $\text{ch}^i(G, \nabla^G)$ is the degree $2i$ component. Define

$$\text{ch}_2(G, \nabla^G) = \sum_{i=0}^{2k} 2^i \text{ch}^i(G, \nabla^G).$$

It's not hard to see that

$$\int_M \widehat{L}(TM, \nabla^{TM}) \text{ch}(E, \nabla^E) = \int_M L(TM, \nabla^{TM}) \text{ch}_2(E, \nabla^E).$$

Note that in the book [14] (Theorem 13.9), the following formula is given:

$$\text{Sig}(M, E) = \int_M L(TM, \nabla^{TM}) \text{ch}_2(E, \nabla^E).$$

Here we use $\int_M \widehat{L}(TM, \nabla^{TM}) \text{ch}(E, \nabla^E)$ to avoid ch_2 . (However, we would like to point out that our \widehat{L} is different from the \widehat{L} in [14].)

By the Chern-Weil theory, the cohomology classes represented by the characteristic forms defined above are independent of the choice of connections. In the rest of this chapter, we simply write characteristic forms without writing connections.

For any complex number t , let

$$S_t(E) = \mathbf{C}|_M + tE + t^2 S^2(E) + \dots, \Lambda_t(E) = \mathbf{C}|_M + tE + t^2 \Lambda^2(E) + \dots$$

denote respectively the total symmetric and exterior powers of E , which lie in $K(M)[[t]]$. The following relations between these two operations [1] hold:

$$(1.19) \quad S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \quad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}.$$

Let $\{\omega_i\}, \{\omega_j'\}$ be formal Chern roots for Hermitian vector bundles E, F respectively. Then [8]

$$(1.20) \quad \text{ch}(\Lambda_t(E)) = \prod_i (1 + e^{\omega_i t}).$$

Therefore, we have the following formulas for Chern character forms:

$$(1.21) \quad \text{ch}(S_t(E)) = \frac{1}{\text{ch}(\Lambda_{-t}(E))} = \frac{1}{\prod_i (1 - e^{\omega_i t})},$$

$$(1.22) \quad \text{ch}(\Lambda_t(E - F)) = \frac{\text{ch}(\Lambda_t(E))}{\text{ch}(\Lambda_t(F))} = \frac{\prod_i (1 + e^{\omega_i t})}{\prod_j (1 + e^{\omega_j' t})}.$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W , then its complexification $W_{\mathbf{C}} = W \otimes \mathbf{C}$ is a complex vector bundle over M carrying a canonically induced Hermitian metric from that of W , as well as a Hermitian connection $\nabla^{W_{\mathbf{C}}}$ induced from ∇^W . If E is a complex vector bundle over M , set $\widetilde{E} = E - \mathbf{C}^{\text{rk}(E)}$ in $K(M)$.

Set

$$(1.23) \quad \Theta(T_{\mathbf{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}).$$

$\Theta(T_{\mathbf{C}}M)$ carries the induced connection from $\nabla^{T_{\mathbf{C}}M}$.

When M is a closed string manifold, the *Witten genus* ([22])

$$W(M) := \int_M \widehat{A}(TM) \text{ch}(\Theta(T_{\mathbf{C}}M))$$

is a modular form of weight $2k$ over $SL(2, \mathbf{Z})$ with integral Fourier expansion ([24]).

Let V be a $2l$ dimensional real Euclidean vector bundle over M carrying a Euclidean connection. Let a, b be two integers. Liu and Wang introduce the following

elements ([17]) in $K(M)[[q^{\frac{1}{2}}]]$ which consist of formal power series in $q^{\frac{1}{2}}$ with coefficients in the K -group of M :

$$(1.24) \quad \Theta_1(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \left(\bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{V_{\mathbf{C}}}) \right)^a \otimes \left(\bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{V_{\mathbf{C}}}) \right)^b \otimes \left(\bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{V_{\mathbf{C}}}) \right)^b,$$

$$(1.25) \quad \Theta_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \left(\bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{V_{\mathbf{C}}}) \right)^b \otimes \left(\bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{V_{\mathbf{C}}}) \right)^b \otimes \left(\bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{V_{\mathbf{C}}}) \right)^a.$$

$\Theta_i(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b), i = 1, 2$, carry the induced connections from $\nabla^{T_{\mathbf{C}}M}$ and $\nabla^{V_{\mathbf{C}}}$.

Now assume V is spin and denote the spinor bundle of V by $\Delta(V)$, which carries the induced connection from $\nabla^{V_{\mathbf{C}}}$.

Let $p_1(TM)$ and $p_1(V)$ be the first Pontrjagin forms of TM and V respectively.

If ω is a differential form on M , we denote by $\omega^{(i)}$ its degree i component.

Set ([7, 17])

$$(1.26) \quad Q_1(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau) = \left\{ e^{\frac{1}{24}E_2(\tau)[p_1(TM) - (a+2b)p_1(V)]} \widehat{A}(TM) \text{ch}((\Delta(V))^a) \text{ch}(\Theta_1(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b)) \right\}^{(4k)},$$

$$(1.27) \quad Q_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau) = \left\{ \widehat{A}(TM) \text{ch}((\Delta(V))^b) \text{ch}(\Theta_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b)) \right\}^{(4k)},$$

$$(1.28) \quad \overline{Q}_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau) = \left\{ \frac{e^{\frac{1}{24}E_2(\tau)[p_1(TM) - (a+2b)p_1(V)]} - 1}{p_1(TM) - (a+2b)p_1(V)} \cdot \widehat{A}(TM) \text{ch}((\Delta(V))^b) \text{ch}(\Theta_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b)) \right\}^{(4k-4)}.$$

Liu and Wang prove following the theorem in [17]:

Theorem 1.3 ([17]). $\int_M Q_1(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau)$ is a modular form weight $2k$ over $\Gamma_0(2)$, while

$$\int_M \{Q_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau) + [p_1(TM) - (a+2b)p_1(V)]\overline{Q}_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau)\}$$

is a modular form weight $2k$ over $\Gamma^0(2)$. Moreover, the following identity holds:

$$(1.29) \quad \int_M Q_1\left(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, -\frac{1}{\tau}\right) = 2^{(a-b)l} \tau^{2k} \int_M \{Q_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau) + [p_1(TM) - (a+2b)p_1(V)]\overline{Q}_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau)\}.$$

2. PROOF OF THEOREM 0.1

In this section, we give the proof of Theorem 0.1 by combining the modularity of the Witten genus and the Liu-Wang modular forms.

Let M be a 24 dimensional smooth closed string manifold.

Lemma 2.1.

$$(2.1) \quad \int_M \widehat{A}(TM)\text{ch}(S^2T_{\mathbf{C}}M) = \int_M \widehat{A}(TM)\text{ch}(-T_{\mathbf{C}}M + 196884),$$

and therefore

$$(2.2) \quad \int_M \widehat{A}(TM)\text{ch}(S^2T_{\mathbf{C}}M) \equiv - \int_M \widehat{A}(TM)\text{ch}(T_{\mathbf{C}}M) \pmod{3\mathbf{Z}}.$$

Proof. Since the Witten genus $W(M)$ is a weight 12 modular form over $SL(2, \mathbf{Z})$, by Tate’s Theorem, we have

$$(2.3) \quad \int_M \widehat{A}(TM)\text{ch}(\Theta(T_{\mathbf{C}}M)) = mE_4(\tau)^3 + n\Delta(\tau).$$

Expanding $\Theta(T_{\mathbf{C}}M)$, we have

$$(2.4) \quad \begin{aligned} \Theta(T_{\mathbf{C}}M) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \\ &= \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbf{C}}M)\Lambda_{-q^n}(\mathbf{C}^{24}) \\ &= (1 + T_{\mathbf{C}}Mq + S^2T_{\mathbf{C}}Mq^2) \otimes (1 + T_{\mathbf{C}}Mq^2) \otimes (1 - 24q + 276q^2) \\ &\quad \otimes (1 - 24q^2) + O(q^3) \\ &= [1 + T_{\mathbf{C}}Mq + (S^2T_{\mathbf{C}}M + T_{\mathbf{C}}M)q^2] \otimes (1 - 24q + 252q^2) + O(q^3) \\ &= 1 + (T_{\mathbf{C}}M - 24)q + (S^2T_{\mathbf{C}}M - 23T_{\mathbf{C}}M + 252)q^2 + O(q^3). \end{aligned}$$

Since

$$(2.5) \quad \begin{aligned} E_4(\tau)^3 &= 1 + 720q + 179280q^2 + O(q^3), \\ \Delta(\tau) &= q - 24q^2 + O(q^3), \end{aligned}$$

we have

$$(2.6) \quad \int_M \widehat{A}(TM) = m,$$

$$(2.7) \quad \int_M \widehat{A}(TM)\text{ch}(T_{\mathbf{C}}M - 24) = 720m + n,$$

$$(2.8) \quad \int_M \widehat{A}(TM)\text{ch}(S^2T_{\mathbf{C}}M - 23T_{\mathbf{C}}M + 252) = 179280m - 24n.$$

By solving these relations, it’s not hard to get (2.1). □

Remark 2.1. We would like to point out that (2.1) is implicitly derived in [10] by using a different basis for weight 12 modular forms. Our contribution here is to observe (2.2) and use it to prove mod 3 congruence of the twisted signature.

Using the string condition and putting $a = 0, b = 1$ and $V = TM$ in Liu-Wang’s construction, we get a pair of modular forms by using the modularity from which we can prove the following lemma.

Lemma 2.2.

$$(2.9) \quad \int_M \widehat{L}(TM) \text{ch}(\Lambda^2 T_{\mathbb{C}}M - T_{\mathbb{C}}M) \equiv \int_M \widehat{A}(TM) \text{ch}(\Lambda^2 T_{\mathbb{C}}M - S^2 T_{\mathbb{C}}M + T_{\mathbb{C}}M) \pmod{3\mathbb{Z}}.$$

Proof. Putting $a = 0, b = 1$ and $V = TM$ in Liu-Wang’s construction, we have

$$(2.11) \quad \Theta_1(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{T_{\mathbb{C}}M}),$$

$$(2.12) \quad \Theta_2(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{T_{\mathbb{C}}M}),$$

$$(2.13) \quad \begin{aligned} Q_1(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1, \tau) &= \left\{ e^{-\frac{1}{24}E_2(\tau)p_1(TM)} \widehat{A}(TM) \text{ch}(\Theta_1(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1)) \right\}^{(24)}, \\ Q_2(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1, \tau) - p_1(TM) \overline{Q_2}(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1, \tau) &= \left\{ e^{-\frac{1}{24}E_2(\tau)p_1(TM)} \widehat{L}(TM) \text{ch}(\Theta_2(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1)) \right\}^{(24)}. \end{aligned}$$

Note that we have used $\widehat{L}(TM) = \widehat{A}(TM) \text{ch}(\Delta(TM))$.

Let

$$(2.14) \quad \Theta_1(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1) = A_0 + A_1 q^{1/2} + A_2 q + \dots,$$

$$(2.15) \quad \Theta_2(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1) = B_0 + B_1 q^{1/2} + B_2 q + \dots.$$

Since M is string, we have

$$(2.16) \quad \begin{aligned} R_1(\tau) &:= \int_M Q_1(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1, \tau) \\ &= \int_M e^{-\frac{1}{24}E_2(\tau)p_1(TM)} \widehat{A}(TM) \text{ch}(\Theta_1(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1)) \\ &= \int_M \widehat{A}(TM) \text{ch}(\Theta_1(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1)), \end{aligned}$$

$$(2.17) \quad \begin{aligned} R_2(\tau) &:= \int_M \{ Q_2(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1, \tau) - p_1(TM) \overline{Q_2}(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1, \tau) \} \\ &= \int_M e^{-\frac{1}{24}E_2(\tau)p_1(TM)} \widehat{L}(TM) \text{ch}(\Theta_2(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1)) \\ &= \int_M \widehat{L}(TM) \text{ch}(\Theta_2(T_{\mathbb{C}}M, T_{\mathbb{C}}M, 0, 1)). \end{aligned}$$

By Theorem 1.3, we see that $R_1(\tau)$ is an integral modular form of weight 12 over $\Gamma_0(2)$, while $R_2(\tau)$ is an integral modular form of weight 12 over $\Gamma^0(2)$.

So by Theorem 1.2, we have the following expansion:

$$(2.18) \quad R_2(\tau) = h_0(8\delta_2)^6 + h_1(8\delta_2)^4 \varepsilon_2 + h_2(8\delta_2)^2 \varepsilon_2^2 + h_3 \varepsilon_2^3,$$

where each $h_r = \int_M \widehat{A}(TM)\text{ch}(b_r(T_{\mathbf{C}}M))$, $0 \leq r \leq 3$, and each $b_r(T_{\mathbf{C}}M)$ is a canonical integral linear combination of $B_j(T_{\mathbf{C}}M)$, $0 \leq j \leq r$.

From (2.18) and (1.14), (1.15), one has

$$(2.19) \quad \int_M \widehat{L}(TM)\text{ch}(B_0) = h_0,$$

$$(2.20) \quad \int_M \widehat{L}(TM)\text{ch}(B_1) = 144h_0 + h_1,$$

$$(2.21) \quad \int_M \widehat{L}(TM)\text{ch}(B_2) = 8784h_0 + 104h_1 + h_2.$$

From (1.19) and (2.11), one can compute the B_i 's explicitly as follows:

$$(2.22) \quad \begin{aligned} & B_0 + B_1q^{1/2} + B_2q + O(q^{3/2}) \\ &= \left(\bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M})\right) \otimes \left(\bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbf{C}}M})\right) \otimes \left(\bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}M})\right) \\ &= \left(\bigotimes_{n=1}^{\infty} \frac{\Lambda_{-q^n}(\mathbf{C}^{24})}{\Lambda_{-q^n}(T_{\mathbf{C}}M)}\right) \otimes \left(\bigotimes_{m=1}^{\infty} \frac{\Lambda_{q^m}(T_{\mathbf{C}}M)}{\Lambda_{q^m}(\mathbf{C}^{24})}\right) \otimes \left(\bigotimes_{r=1}^{\infty} \frac{\Lambda_{q^{r-\frac{1}{2}}}(T_{\mathbf{C}}M)}{\Lambda_{q^{r-\frac{1}{2}}}(\mathbf{C}^{24})}\right) \\ &= [1 + (T_{\mathbf{C}}M - 24)q] \otimes [1 + (T_{\mathbf{C}}M - 24)q] \otimes \frac{1 + T_{\mathbf{C}}Mq^{\frac{1}{2}} + \Lambda^2 T_{\mathbf{C}}Mq}{1 + 24q^{\frac{1}{2}} + 276q} + O(q^{3/2}) \\ &= [1 + (2T_{\mathbf{C}}M - 48)q] \otimes (1 + T_{\mathbf{C}}Mq^{\frac{1}{2}} + \Lambda^2 T_{\mathbf{C}}Mq) \otimes (1 - 24q^{\frac{1}{2}} + 300q) + O(q^{3/2}) \\ &= 1 + (T_{\mathbf{C}}M - 24)q^{\frac{1}{2}} + (\Lambda^2 T_{\mathbf{C}}M - 22T_{\mathbf{C}}M + 252)q + O(q^{3/2}). \end{aligned}$$

So we have

$$(2.23) \quad B_0 = 1,$$

$$(2.24) \quad B_1 = T_{\mathbf{C}}M - 24,$$

$$(2.25) \quad B_2 = \Lambda^2 T_{\mathbf{C}}M - 22T_{\mathbf{C}}M + 252.$$

Then by (2.19)-(2.22), we get

$$(2.26) \quad h_0 = \int_M \widehat{L}(TM),$$

$$(2.27) \quad h_1 = \int_M \widehat{L}(TM)\text{ch}(T_{\mathbf{C}}M - 168),$$

$$(2.28) \quad h_2 = \int_M \widehat{L}(TM)\text{ch}(\Lambda^2 T_{\mathbf{C}}M - 126T_{\mathbf{C}}M + 8940).$$

Also by Theorem 1.3, the following identity holds:

$$(2.29) \quad R_1\left(-\frac{1}{\tau}\right) = 2^{-12}\tau^{12}R_2(\tau).$$

Therefore, by (1.16) and (2.18), we have

$$(2.30) \quad R_1(\tau) = 2^{-12}[h_0(8\delta_1)^6 + h_1(8\delta_1)^4\varepsilon_1 + h_2(8\delta_1)^2\varepsilon_1^2 + h_3\varepsilon_1^3].$$

Note that

$$\begin{aligned}
 & (8\delta_1)^{6-2r} \varepsilon_1^r \\
 (2.31) \quad & = (2 + 48q)^{6-2r} \left(\frac{1}{16} - q\right)^r O(q^2) \\
 & = 2^{6-6r} [1 + 24(6 - 2r)q][1 - 16rq] + O(q^2) \\
 & = 2^{6-6r} [1 + (144 - 64r)q] + O(q^2).
 \end{aligned}$$

Comparing the coefficient of q , we have

$$(2.32) \quad \int_M \widehat{A}(TM) \text{ch}(A_2) = 2^{-12} \sum_{r=0}^3 2^{6-6r} (144 - 64r) h_r.$$

By (1.19) and (2.10), we can explicitly expand $\Theta_1(T_{\mathbf{C}}M, T_{\mathbf{C}}M, 0, 1)$ as follows:

$$\begin{aligned}
 (2.33) \quad & \Theta_1(T_{\mathbf{C}}M, T_{\mathbf{C}}M, 0, 1) \\
 & = \bigotimes_{n=1}^{\infty} \frac{\Lambda_{-q^n}(\mathbf{C}^{24})}{\Lambda_{-q^n}(T_{\mathbf{C}}M)} \otimes \bigotimes_{r=1}^{\infty} \frac{\Lambda_{q^{r-\frac{1}{2}}}(T_{\mathbf{C}}M)}{\Lambda_{q^{r-\frac{1}{2}}}(\mathbf{C}^{24})} \otimes \bigotimes_{s=1}^{\infty} \frac{\Lambda_{-q^{s-\frac{1}{2}}}(T_{\mathbf{C}}M)}{\Lambda_{-q^{s-\frac{1}{2}}}(\mathbf{C}^{24})} \\
 & = \frac{1 - 24q}{1 - T_{\mathbf{C}}Mq} \frac{1 + T_{\mathbf{C}}Mq^{\frac{1}{2}} + \Lambda^2 T_{\mathbf{C}}Mq}{1 + 24q^{\frac{1}{2}} + 276q} \frac{1 - T_{\mathbf{C}}Mq^{\frac{1}{2}} + \Lambda^2 T_{\mathbf{C}}Mq}{1 - 24q^{\frac{1}{2}} + 276q} + O(q^{3/2}) \\
 & = \frac{1 - 24q}{1 - T_{\mathbf{C}}Mq} \frac{1 + (2\Lambda^2 T_{\mathbf{C}}M - T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M)q}{1 - 24q} + O(q^{3/2}) \\
 & = (1 - 24q)(1 + T_{\mathbf{C}}Mq)[1 + (2\Lambda^2 T_{\mathbf{C}}M - T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M)q](1 + 24q) + O(q^{3/2}) \\
 & = 1 + (\Lambda^2 T_{\mathbf{C}}M - S^2 T_{\mathbf{C}}M + T_{\mathbf{C}}M)q + O(q^{3/2}).
 \end{aligned}$$

So one has

$$(2.34) \quad A_2 = \Lambda^2 T_{\mathbf{C}}M - S^2 T_{\mathbf{C}}M + T_{\mathbf{C}}M.$$

By (2.32) and (2.34), we have

$$\begin{aligned}
 (2.35) \quad & \int_M \widehat{A}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M - S^2 T_{\mathbf{C}}M + T_{\mathbf{C}}M) \\
 & = 2^{-12} \sum_{r=0}^3 2^{6-6r} (144 - 64r) h_r \\
 & = 2^{-20} (2^{18} \cdot 9h_0 + 2^{12} \cdot 5h_1 + 2^6 h_2 - 3h_3).
 \end{aligned}$$

Hence

$$\begin{aligned}
 (2.36) \quad & 2^{20} \int_M \widehat{A}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M - S^2 T_{\mathbf{C}}M + T_{\mathbf{C}}M) \\
 & = 2^{18} \cdot 9h_0 + 2^{12} \cdot 5h_1 + 2^6 h_2 - 3h_3 \\
 & \equiv h_2 - h_1 \pmod{3\mathbf{Z}} \\
 & = \int_M \widehat{L}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M - 127T_{\mathbf{C}}M + 9108) \\
 & \equiv \int_M \widehat{L}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M - T_{\mathbf{C}}M) \pmod{3\mathbf{Z}}.
 \end{aligned}$$

Noting that $2^{20} \equiv 1 \pmod{3\mathbf{Z}}$, we get Lemma 2.2. □

Putting $a = 1, b = 0$ and $V = TM$ in Liu-Wang’s construction, one obtains another pair of modular forms (cf. [16]). Applying the modularity of this pair, we have

Lemma 2.3.

$$(2.37) \quad \int_M \widehat{L}(TM)\text{ch}(T_{\mathbf{C}}M) = 2^{11} \int_M \widehat{A}(TM)\text{ch}(\Lambda^2 T_{\mathbf{C}}M - 47T_{\mathbf{C}}M + 900),$$

and therefore

$$(2.38) \quad \int_M \widehat{L}(TM)\text{ch}(T_{\mathbf{C}}M) \equiv \int_M \widehat{A}(TM)\text{ch}(-\Lambda^2 T_{\mathbf{C}}M - T_{\mathbf{C}}M) \pmod{3\mathbf{Z}}.$$

Proof. When $a = 1, b = 0$ and $V = TM$, we have

$$\begin{aligned} & \int_M Q_1(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau) \\ &= \int_M \widehat{L}(TM)\text{ch} \left(\bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{V_{\mathbf{C}}}) \right), \\ & \int_M Q_2(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau) + [p_1(TM) - (a + 2b)p_1(V)]\overline{Q_2}(T_{\mathbf{C}}M, V_{\mathbf{C}}, a, b, \tau) \\ &= \int_M \widehat{A}(M)\text{ch} \left(\bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}M}) \right). \end{aligned}$$

By applying the modularity of this pair of modular forms, we can use Theorem 2.3 in [5], which asserts that if M is an $8m$ dimensional smooth closed oriented manifold,

$$(2.39) \quad \int_M \widehat{L}(TM)\text{ch}(T_{\mathbf{C}}M) = 2^{11} \left[\sum_{r=0}^{m-1} (m - r)2^{6(m-r-1)}h_r \right],$$

where the h_r ’s are determined by

$$\begin{aligned} & \int_M Q_2(T_{\mathbf{C}}M, T_{\mathbf{C}}M, 1, 0, \tau) \\ &= \int_M \widehat{A}(M)\text{ch} \left(\bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}M}) \right) \\ &= \sum_{r=0}^m h_r (8\delta_2)^{2m-r} \varepsilon_2^r. \end{aligned}$$

When M is 24 dimensional,

$$(2.40) \quad \int_M \widehat{L}(TM)\text{ch}(T_{\mathbf{C}}M) = 2^{11}(3 \times 2^{12}h_0 + 2^7h_1 + h_2).$$

To determine h_0, h_1 and h_2 , we expand the q -series

$$\begin{aligned}
 (2.41) \quad & \Theta_2(T_{\mathbf{C}}M, T_{\mathbf{C}}M, 1, 0) \\
 &= B_0 + B_1q^{1/2} + B_2q + \dots \\
 &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}M}) \\
 &= \bigotimes_{n=1}^{\infty} \frac{\Lambda_{-q^n}(\mathbf{C}^{24})}{\Lambda_{-q^n}(T_{\mathbf{C}}M)} \otimes \bigotimes_{s=1}^{\infty} \frac{\Lambda_{-q^{s-\frac{1}{2}}}(T_{\mathbf{C}}M)}{\Lambda_{-q^{s-\frac{1}{2}}}(\mathbf{C}^{24})} \\
 &= \frac{1 - 24q}{1 - T_{\mathbf{C}}Mq} \frac{1 - T_{\mathbf{C}}Mq^{\frac{1}{2}} + \Lambda^2 T_{\mathbf{C}}Mq}{1 - 24q^{\frac{1}{2}} + 276q} + O(q^{3/2}) \\
 &= (1 - 24q)(1 + T_{\mathbf{C}}Mq)(1 - T_{\mathbf{C}}Mq^{\frac{1}{2}} + \Lambda^2 T_{\mathbf{C}}Mq)(1 + 24q^{\frac{1}{2}} + 300q) + O(q^{3/2}) \\
 &= 1 + (24 - T_{\mathbf{C}}M)q^{\frac{1}{2}} + (\Lambda^2 T_{\mathbf{C}}M - 23T_{\mathbf{C}}M + 276)q + O(q^{3/2})
 \end{aligned}$$

and note that the h_i 's (similar to (2.19)-(2.21)) satisfy

$$(2.42) \quad \int_M \widehat{A}(TM) \text{ch}(B_0) = h_0,$$

$$(2.43) \quad \int_M \widehat{A}(TM) \text{ch}(B_1) = 144z = h_0 + h_1,$$

$$(2.44) \quad \int_M \widehat{A}(TM) \text{ch}(B_2) = 8784h_0 + 104h_1 + h_2.$$

So

$$\begin{aligned}
 h_0 &= \int_M \widehat{A}(TM), \\
 h_1 &= - \int_M \widehat{A}(TM) \text{ch}(T_{\mathbf{C}}M + 120), \\
 h_2 &= \int_M \widehat{A}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M + 81T_{\mathbf{C}}M + 3972).
 \end{aligned}$$

(2.37) then easily follows from (2.40). □

Remark 2.2. To derive Lemma 2.3, the string condition is not necessary. However, as we have seen, to obtain Lemma 2.2, the string condition is indispensable.

Remark 2.3. The strategy of the proof of Lemma 2.3 is essentially the same as that of Lemma 2.2. In each case, one constructs a pair of modular forms, the modularity of which gives us the desired result. The application of Theorem 2.3 from [5] in the above proof is only to simplify the process to derive (2.40), which is similar to the process to derive (2.32) in the proof of Lemma 2.2.

Combining Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
 (2.45) \quad & \int_M \widehat{L}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M) \\
 & \equiv \int_M \widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M) + \int_M \widehat{A}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M - S^2 T_{\mathbf{C}}M + T_{\mathbf{C}}M) \pmod{3\mathbf{Z}} \\
 & \equiv \int_M \widehat{A}(TM) \text{ch}(-\Lambda^2 T_{\mathbf{C}}M - T_{\mathbf{C}}M) \\
 & \quad + \int_M \widehat{A}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M - S^2 T_{\mathbf{C}}M + T_{\mathbf{C}}M) \pmod{3\mathbf{Z}} \\
 & = \int_M \widehat{A}(TM) \text{ch}(-S^2 T_{\mathbf{C}}M) \\
 & \equiv \int_M \widehat{A}(TM) \text{ch}(T_{\mathbf{C}}M) \pmod{3\mathbf{Z}},
 \end{aligned}$$

as desired.

This finishes the proof of Theorem 0.1.

3. THE EXAMPLES AND COMPUTATION

In this section, we do computations on the two examples $B^8 \times \mathbf{H}P^2 \times \mathbf{H}P^2$ and $M_0^8 \times M_0^8 \times M_0^8$.

Recall that the twisted signature

$$\text{Sig}(M, \Lambda^2 \mathcal{T}) := \text{Ind}(D_{\text{Sig}} \otimes \Lambda^2 T_{\mathbf{C}}M)^+,$$

and so by the Atiyah-Singer index theorem,

$$\text{Sig}(M, \Lambda^2 \mathcal{T}) = \int_M \widehat{L}(TM) \text{ch}(\Lambda^2 T_{\mathbf{C}}M).$$

First we have a lemma about $\text{Sig}(M, \Lambda^2 \mathcal{T})$ when the manifold M is a product of several manifolds.

Lemma 3.1. *If $M = \prod_{i=1}^s N_i$, then*

$$\begin{aligned}
 (3.1) \quad & \text{Sig}(M, \Lambda^2 \mathcal{T}) \\
 & = \sum_{i=1}^s \text{Sig}(N_i, \Lambda^2 \mathcal{T}) \prod_{j \neq i} \text{Sig}(N_j) + \sum_{1 \leq i < j \leq s} \text{Sig}(N_i, \mathcal{T}) \text{Sig}(N_j, \mathcal{T}) \prod_{p \neq i, j} \text{Sig}(N_p).
 \end{aligned}$$

Proof. It is not hard to see that the lemma follows from the multiplicity of the Hirzebruch \widehat{L} -class

$$\widehat{L}(M) = \prod_{i=1}^s \widehat{L}(N_i)$$

and the following property of the exterior square:

$$\Lambda^2 \left(\bigoplus_{i=1}^s V_i \right) = \bigoplus_{i=1}^s \Lambda^2(V_i) \oplus \bigoplus_{i < j} V_i \otimes V_j,$$

where the V_i 's are vector spaces. □

Assume N is an 8 dimensional smooth closed oriented manifold. Let p_1, p_2 be the first and second Pontryagin classes of N . Let $[N]$ be the fundamental class. By direct computations, we have

$$\begin{aligned}
 (3.2) \quad \text{Sig}(N) &= \frac{7p_2 - p_1^2}{45}[N], \\
 \text{Sig}(N, \mathcal{T}) &= \frac{112p_1^2 - 64p_2}{45}[N], \\
 \text{Sig}(N, \Lambda^2\mathcal{T}) &= \frac{692p_1^2 + 196p_2}{45}[N], \\
 \widehat{A}(N) &= \frac{7p_1^2 - 4p_2}{5760}[N].
 \end{aligned}$$

The first three equalities can be derived from the following formulas about the Chern character and \widehat{L} -class for a real vector bundle V :

$$\text{ch}(V_{\mathbf{C}}) = \dim(V) + p_1(V) + \frac{p_1(V)^2 - 2p_2(V)}{12} + \dots,$$

and when $\dim V = 8$,

$$\widehat{L}(V) = 16 + \frac{4}{3}p_1(V) + \frac{7p_2(V) - p_1(V)^2}{45} + \dots.$$

Let B^8 be the Bott manifold, which is 8 dimensional and spin with $\widehat{A}(B^8) = 1$, $\text{Sig}(B^8) = 0$ ([15]). By (3.2), it is easy to see that $p_1^2[B^8] = 7 \times 128$, $p_2[B^8] = 128$, and therefore

$$\text{Sig}(B^8, \mathcal{T}) = 2048, \text{Sig}(B^8, \Lambda^2\mathcal{T}) = 14336.$$

By a theorem of Hirzebruch [10], for the quaterionic projective plane \mathbf{HP}^2 the total Pontryagin class

$$p(\mathbf{HP}^2) = (1 + u)^6(1 + 4u)^{-1},$$

where $u \in H^4(\mathbf{HP}^2, \mathbf{Z})$ is the generator. So $p_1(\mathbf{HP}^2) = 2u$ and $p_2(\mathbf{HP}^2) = 7u^2$. By (3.2), it is easy to get

$$\text{Sig}(\mathbf{HP}^2) = 1, \text{Sig}(\mathbf{HP}^2, \mathcal{T}) = 0, \text{Sig}(\mathbf{HP}^2, \Lambda^2\mathcal{T}) = 92.$$

A manifold is called *almost-parallelizable* if its tangent bundle is trivial on the complement of a point ([18]). For a $4k$ dimensional almost-parallelizable manifold M^{4k} , all the Pontryagin classes $p_i = 0$ for $i < k$. By the Cauchy lemma (cf. [10]), each genus is a multiple of the \widehat{A} -genus; actually one has

$$(3.3) \quad W(M^{4k}) = E_{2k}(\tau) \int_M \widehat{A}(M),$$

$$(3.4) \quad \text{Sig}(M^{4k}) = -2^{2k+1}(2^{2k-1} - 1) \int_M \widehat{A}(M).$$

Put $a_k = 1$ if k is even and $a_k = 2$ if k is odd. By the plumbing method, Milnor and Kervaire have constructed an almost-parallelizable manifold M_0^{4k} such that

$$(3.5) \quad \text{Sig}(M_0^{4k}) = a_k 2^{2k+1}(2^{2k-1} - 1) \cdot \text{numerator} \left(\frac{B_{2k}}{4k} \right),$$

where B_{2k} is the Bernoulli number. Since $B_4 = -\frac{1}{30}$, the numerator $(\frac{B_4}{8}) = 1$. One sees from (3.5) that $\text{Sig}(M_0^8) = 224$ and therefore from (3.4) and (3.3) that

$\int_M \widehat{A}(M_0^8) = -1$ and $W(M_0^8) = -E_4(\tau)$. We would like to point out that $M_0^8 \times M_0^8 \times M_0^8$ is an interesting 24 dimensional string manifold whose Witten genus

$$W(M_0^8 \times M_0^8 \times M_0^8) = -E_4(\tau)^3.$$

Plugging $Sig(M_0^8) = 224$ into the first equality in (3.2) and using $p_1(M_0^8) = 0$, we have $p_2(M_0^8) = 1440$. Then by the second and the third equalities in (3.2), we get

$$Sig(M_0^8, \mathcal{T}) = -2048, \quad Sig(M_0^8, \Lambda^2 \mathcal{T}) = 6272.$$

By Lemma 3.1 and the above computations of the signature and twisted signatures of B^8 , \mathbf{HP}^2 and M_0^8 , we have

$$\begin{aligned} & Sig(B^8 \times \mathbf{HP}^2 \times \mathbf{HP}^2, \Lambda^2 \mathcal{T}) \\ &= Sig(B^8, \Lambda^2 \mathcal{T}) Sig(\mathbf{HP}^2)^2 + 2 Sig(\mathbf{HP}^2, \Lambda^2 \mathcal{T}) Sig(B^8) Sig(\mathbf{HP}^2) \\ (3.6) \quad &+ 2 Sig(B^8, \mathcal{T}) Sig(\mathbf{HP}^2, \mathcal{T}) Sig(\mathbf{HP}^2) + Sig(B^8) Sig(\mathbf{HP}^2, \mathcal{T})^2 \\ &= 14336 \\ &\equiv 2 \pmod{3\mathbf{Z}} \end{aligned}$$

and

$$\begin{aligned} & Sig(M_0^8 \times M_0^8 \times M_0^8, \Lambda^2 \mathcal{T}) \\ (3.7) \quad &= 3 Sig(M_0^8, \Lambda^2 \mathcal{T}) Sig(M_0^8)^2 + 3 Sig(M_0^8, \mathcal{T})^2 Sig(M_0^8) \\ &= 3 \times 6272 \times 224^2 + 3 \times (-2048)^2 \times 224 \\ &\equiv 3 \pmod{9\mathbf{Z}}. \end{aligned}$$

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