

Elliptic genera, transgression and loop space Chern–Simons forms

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We compute the Chern–Simons transgressed forms of some characteristic forms with modular properties, which are related to elliptic genera. We study the modularity properties of these secondary characteristic forms and the relations among them. We also compute the Chern–Simons forms of some vector bundles over free loop space.

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1. Introduction

Connections in vector bundles play a very important role in differential geometry. The famous Chern–Weil theory relates connections to the theory of characteristic classes, which provides a geometric way to understand characteristic classes and therefore play a fundamental role in global differential geometry. Let ∇ be a connection on an n -dimensional real or complex vector bundle $E \rightarrow M$, where M is a compact smooth manifold. The basic

local invariant of ∇ is its curvature, which is a closed 2-form with values in the $\mathfrak{gl}(n)$ -bundle associated to E . Invariant polynomials of the curvature give us Chern–Weil characteristic forms, which are closed and thus represent cohomology classes in the de Rham cohomology of M [8]. These classes do not depend on the choice of the connections and are actually topological invariants of E . Particular invariant polynomials give us the famous Chern classes and Pontryagin classes.

The Chern-Simons theory studies the dependence of characteristic forms on ∇ , which, by transgression method, leads to secondary geometric invariants, called the Chern–Simons forms. Chern and Simons [10] are led to this theory by concrete geometric questions in combinatorial and conformal geometry. Cheeger and Simons go further along this clue to define some refined secondary invariants called the Cheeger-Simons differential characters [9]. It turns out that these secondary invariants are very useful in many areas of mathematics and physics. For example, Witten [32] uses the secondary invariant associated to a particular characteristic form to construct a topological quantum field theory in three dimension and obtains some quantum invariants including the Jones polynomial of knots as well as new invariants of 3-manifolds. The Chern–Simons forms or more generally the geometric invariants of connections have special advantages to study flat vector bundles. A flat bundle is the vector bundle equipped with a connection whose curvature vanishes identically. Hence, by the Chern–Weil theory, all characteristic forms vanish. Thus we cannot read off any information from this theory. However, based on the idea of Chern–Simons and the method of transgression, one is able to construct certain cohomology classes by using flat connections, which turn out to be useful tools to study flat vector bundles. In Section 2 of this paper, we briefly review the method of the Chern–Simons transgression and give Theorem 2.2, which serves as a convenient tool to do transgressions on the characteristic forms that we are going to use.

A lesson we have learned from the above stories is that one may apply the Chern–Simons transgression to particularly chosen characteristic forms and obtain interesting secondary characteristic forms, which might have potential applications. In this paper, we apply the Chern–Simons transgression to some characteristic forms related to elliptic genera to derive secondary characteristic forms with modular properties. The characteristic forms that we use here are related to the theory of elliptic genera. Ochanine [28] first introduced the notion of elliptic genera from the topological point of view. He was motivated by a study of S^1 -action on Spin manifolds by Landweber and Stong [19], who attempted to answer a question

raised by Witten [31] on rigidity of certain twisted Dirac operators on Spin manifold. In [33] Witten reinterpreted the Landweber–Stong elliptic genus as the index of the formal signature operator on loop space as well as introduced the formal index of the Dirac operator on loop space (called the Witten operator), which is the Witten genus. In Section 3 of this paper, we apply the Chern–Weil theory to express the local indices of some formal elliptic operators on loop spaces by connections (or curvatures), which are characteristic forms with modular properties. Then in Section 4, we apply the Chern–Simons transgression to these characteristic forms to obtain some interesting secondary characteristic forms. Most of these secondary characteristic forms have modular properties except for the one associated to the Witten operator. However, it turns out that this specific secondary characteristic form is a modular form over $SL_2(\mathbb{Z})$ when it is associated to two flat connections on a flat manifold. Moreover, this secondary characteristic form cannot be of weight 2 over $SL_2(\mathbb{Z})$, which agrees with the standard fact in the theory of modular forms. As the original characteristic forms, the secondary characteristic forms we obtain are also modularly related. We would like to point out that the modularities of these secondary characteristic forms are not a direct consequence of the modularities of the original characteristic forms. See details in Section 4. We hope that these new geometric invariants of connections with modularity properties obtained here could be applied somewhere.

Motivated by string theory, people have been attempting to generalize many things like vector bundles, Dirac operators, the Atiyah–Singer index theory and so on to loop spaces. The relevant vector bundles over loop space are constructed from ordinary finite-rank vector bundles $V \rightarrow M$. For example, in Section 5, we will consider vector bundles \mathcal{V} and \mathcal{V}' over loop space:

$$\mathcal{V} := \bigotimes_{j=1}^{\infty} \Lambda_{-q^{j-1/2}}(V_{\mathbb{C}}), \quad \mathcal{V}' := \bigotimes_{j=1}^{\infty} \Lambda_{q^{j-1/2}}(V_{\mathbb{C}}).$$

As Witten remarked in his lecture notes [34] that physically, this analogue is very important because it arises in heterotic string theory. Following [21], in our discussion section (Section 5), we describe the Atiyah–Singer index theory on loop spaces. Then for flat bundles on loop space, we compute their Chern–Simons forms, which turn out to have modular properties. This could be understood as certain generalization of the Chern–Simons forms for flat bundles on finite-dimensional smooth manifolds to free loop spaces.

2. Construction of the Chern–Simons transgressed forms

Let us briefly review the construction of the Chern–Simons transgressed forms in this section.

Let M be a compact smooth manifold and T^*M denote the cotangent bundle of M . We denote by $\Lambda^*(T^*M)$ the (complex) exterior algebra bundle of T^*M , and

$$\Omega^*(M, \mathbb{C}) \triangleq \Gamma(\Lambda^*(T^*M))$$

the space of smooth sections of $\Lambda^*(T^*M)$. In particular, for any integer p such that $0 \leq p \leq \dim M$, we denote by

$$\Omega^p(M, \mathbb{C}) \triangleq \Gamma(\Lambda^p(T^*M))$$

the space of smooth p -forms over M . Let E be a smooth complex vector bundle over M . We denote by $\Omega^*(M; E)$ the space of smooth sections of the tensor product vector bundle $\Lambda^*(T^*M) \otimes E$ obtained from $\Lambda^*(T^*M)$ and E ,

$$\Omega^*(M; E) \triangleq \Gamma(\Lambda^*(T^*M) \otimes E).$$

Let $\text{End}(E)$ be the bundle of endomorphisms of E . On $\Omega^*(M; \text{End}(E))$, one can define a super Lie bracket (cf. [36]) by extending the Lie bracket operation on $\text{End}(E)$ as follows: if $\omega, \eta \in \Omega^*(M)$ and $A, B \in \Gamma(\text{End}(E))$, then we use the convention that

$$[\omega A, \eta B] = (\omega A)(\eta B) - (-1)^{(\deg \omega)(\deg \eta)}(\eta B)(\omega A).$$

It is not hard to see that: for any $A, B \in \Omega^*(M; \text{End}(E))$, the trace of $[A, B]$ vanishes (cf. [32, Lemma 1.7]).

Let ∇^E be a connection on E and $A \in \Omega^*(M; \text{End}(E))$. One has the following result.

Lemma 2.1 (cf. [36, Lemma 1.8]). *The following identity holds,*

$$(2.1) \quad d \text{tr}[A] = \text{tr}[[\nabla^E, A]].$$

Let

$$f(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots$$

be a power series in one variable. Let $R^E = (\nabla^E)^2$ be the curvature of ∇^E on E . The trace of $f(R^E)$ is an element in $\Omega^*(M, \mathbb{C})$. A form of the Chern–Weil theorem (cf. [32, Theorem 1.9]) can be stated as follows.

Theorem 2.1. (1) *The form $\text{tr}[f(R^E)]$ is closed. That is,*

$$d \text{tr}[f(R^E)] = 0;$$

(2) *If $\widetilde{\nabla}^E$ is another connection on E and \widetilde{R}^E is its curvature. For any $t \in [0, 1]$, let ∇_t^E be the deformed connection on E given by $\nabla_t^E = (1 - t)\nabla^E + t\widetilde{\nabla}^E$. Let $R_t^E, t \in [0, 1]$, denote the curvature of ∇_t^E . Let $f'(x)$ be the power series obtained from the derivative of $f(x)$ with respect to x . Then the following identity holds,*

$$(2.2) \quad \text{tr}[f(\widetilde{R}^E)] - \text{tr}[f(R^E)] = d \int_0^1 \text{tr} \left[\frac{d\nabla_t^E}{dt} f'(R_t^E) \right] dt.$$

Proof. (1) From Lemma 2.1, one verifies directly that

$$\begin{aligned} d \text{tr}[f(R^E)] &= \text{tr}[[\nabla^E, f(R^E)]] \\ &= \text{tr}[a_1[\nabla^E, R^E] + \cdots + a_n[\nabla^E, (R^E)^n] + \cdots] = 0, \end{aligned}$$

as for any integer $k \geq 0$ one has the obvious Bianchi identity

$$(2.3) \quad [\nabla^E, (R^E)^k] = [\nabla^E, (\nabla^E)^{2k}] = 0.$$

(2) Note that ∇_t^E is a connection on E such that $\nabla_0^E = \nabla^E$ and $\nabla_1^E = \widetilde{\nabla}^E$. Moreover,

$$\frac{d\nabla_t^E}{dt} = \widetilde{\nabla}^E - \nabla^E \in \Omega^1(M, \text{End}(E)).$$

We deduce that

$$\begin{aligned} \frac{d}{dt} \text{tr} [f(R_t^E)] &= \text{tr} \left[\frac{dR_t^E}{dt} f'(R_t^E) \right] = \text{tr} \left[\frac{d(\nabla_t^E)^2}{dt} f'(R_t^E) \right] \\ &= \text{tr} \left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt} \right] f'(R_t^E) \right] = \text{tr} \left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt} f'(R_t^E) \right] \right], \end{aligned}$$

where the last equality follows from the Bianchi identity (2.3).

Combining with Lemma 2.1, we have

$$(2.4) \quad \frac{d}{dt} \text{tr} [f(R_t^E)] = d \text{tr} \left[\frac{d\nabla_t^E}{dt} f'(R_t^E) \right],$$

from which one obtains

$$\mathrm{tr}[f(\tilde{R}^E)] - \mathrm{tr}[f(R^E)] = d \int_0^1 \mathrm{tr} \left[\frac{d\nabla_t^E}{dt} f'(R_t^E) \right] dt. \quad \square$$

The transgressed term

$$(2.5) \quad \int_0^1 \mathrm{tr} \left[\frac{d\nabla_t^E}{dt} f'(R_t^E) \right] dt$$

is usually called a *Chern–Simons term*.

In particular, let M be a 3-dimensional oriented compact smooth manifold. It is known that TM is trivial. If we take ∇_0^{TM} to be the trivial connection d^{TM} associated to some global basis of $\Gamma(TM)$ and $\nabla_1^{TM} = d^{TM} + A$, (2.5) gives us the well-known Chern–Simons form (cf. [10, 36])

$$(2.6) \quad \mathrm{CS}(A) \triangleq \mathrm{tr}[A \wedge d^{TM} A + \frac{2}{3} A \wedge A \wedge A],$$

which plays a very important role in quantum field theory and low-dimensional topology [32].

In this paper, we are going to use the following theorem, which we obtain by modifying Theorem 2.1.

Theorem 2.2. *Assume $a_0 \neq 0$. The following identity holds,*

$$(2.7) \quad \begin{aligned} & \det^{1/2}(f(\tilde{R}^E)) - \det^{1/2}(f(R^E)) \\ &= d \int_0^1 \frac{1}{2} \det^{1/2}(f(R_t^E)) \mathrm{tr} \left[\frac{d\nabla_t^E}{dt} \frac{f'(R_t^E)}{f(R_t^E)} \right] dt. \end{aligned}$$

Proof. Observe that $\det^{1/2}(f(R_t^E)) = e^{(1/2) \mathrm{tr}[\ln f(R_t^E)]}$. Hence by (2.4),

$$(2.8) \quad \begin{aligned} \frac{d}{dt} \det^{1/2}(f(R_t^E)) &= \frac{d}{dt} e^{(1/2) \mathrm{tr}[\ln f(R_t^E)]} \\ &= \frac{1}{2} e^{(1/2) \mathrm{tr}[\ln f(R_t^E)]} \frac{d}{dt} \mathrm{tr}[\ln f(R_t^E)] \\ &= \frac{1}{2} e^{(1/2) \mathrm{tr}[\ln f(R_t^E)]} d \mathrm{tr} \left[\frac{d\nabla_t^E}{dt} \frac{f'(R_t^E)}{f(R_t^E)} \right] \\ &= d \left\{ \frac{1}{2} e^{(1/2) \mathrm{tr}[\ln f(R_t^E)]} \mathrm{tr} \left[\frac{d\nabla_t^E}{dt} \frac{f'(R_t^E)}{f(R_t^E)} \right] \right\} \\ &= d \left\{ \frac{1}{2} \det^{1/2}(f(R_t^E)) \mathrm{tr} \left[\frac{d\nabla_t^E}{dt} \frac{f'(R_t^E)}{f(R_t^E)} \right] \right\}. \end{aligned}$$

Therefore (2.7) follows. □

Remark 2.1. In the above Theorem 2.1(2) and Theorem 2.2, one can actually choose different paths connecting the two connections ∇_0 and ∇_1 . The corresponding Chern–Simons terms then differ by an exact form. A good reference is Theorem B.5.4 in [26]. The path $(1-t)\nabla^E + t\tilde{\nabla}^E, 0 \leq t \leq 1$ is the canonical path.

3. The Chern–Weil forms for elliptic genera

In this section, we identify the Chern–Weil forms of some important elliptic genera.

Let M be a Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM} . Let $\hat{A}(TM, \nabla^{TM})$ and $L(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined, respectively, by (cf. [36])

$$(3.1) \quad \begin{aligned} \hat{A}(TM, \nabla^{TM}) &= \det^{1/2} \left(\frac{(\sqrt{-1}/4\pi)R^{TM}}{\sinh((\sqrt{-1}/4\pi)R^{TM})} \right), \\ L(TM, \nabla^{TM}) &= \det^{1/2} \left(\frac{(\sqrt{-1}/2\pi)R^{TM}}{\tanh((\sqrt{-1}/2\pi)R^{TM})} \right). \end{aligned}$$

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^E, ∇^F , respectively. Let $R^E = (\nabla^E)^2$ (resp. $R^F = (\nabla^F)^2$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as (cf. [36])

$$(3.2) \quad \text{ch}(G, \nabla^G) = \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^E \right) \right] - \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^F \right) \right].$$

Sometimes we also need to use the following modified Chern character (cf. [11])

$$(3.3) \quad \tilde{\text{ch}}(G, \nabla^G) = \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{\pi} R^E \right) \right] - \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{\pi} R^F \right) \right].$$

For any complex number t , let

$$\Lambda_t(E) = \mathbb{C}|_M + tE + t^2\Lambda^2(E) + \cdots, \quad S_t(E) = \mathbb{C}|_M + tE + t^2S^2(E) + \cdots$$

denote, respectively, the total exterior and symmetric powers of E , which live in $K(M)[[t]]$. The following relations between these two operations [3, Chapter 3] hold,

$$(3.4) \quad S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \quad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}.$$

The connections ∇^E, ∇^F naturally induce connections on $S_t E, \Lambda_t E$, etc. Moreover, if $\{\omega_i\}, \{\omega_j'\}$ are formal Chern roots for Hermitian vector bundles E, F , respectively, then [15, Chapter 1]

$$(3.5) \quad \text{ch}\left(\Lambda_t(E), \nabla^{\Lambda_t(E)}\right) = \prod_i (1 + e^{\omega_i t}).$$

Therefore, we have the following formulas for Chern character forms:

$$(3.6) \quad \text{ch}(S_t(E), \nabla^{S_t(E)}) = \frac{1}{\text{ch}(\Lambda_{-t}(E), \nabla^{\Lambda_{-t}(E)})} = \frac{1}{\prod_i (1 - e^{\omega_i t})},$$

$$(3.7) \quad \text{ch}(\Lambda_t(E - F), \nabla^{\Lambda_t(E - F)}) = \frac{\text{ch}(\Lambda_t(E), \nabla^{\Lambda_t(E)})}{\text{ch}(\Lambda_t(F), \nabla^{\Lambda_t(F)})} = \frac{\prod_i (1 + e^{\omega_i t})}{\prod_j (1 + e^{\omega_j' t})}.$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W , then its complexification $W_{\mathbb{C}} = W \otimes \mathbb{C}$ is a complex vector bundle over M carrying a canonically induced Hermitian metric from that of W , as well as a Hermitian connection $\nabla^{W_{\mathbb{C}}}$ induced from ∇^W . If E is a vector bundle (complex or real) over M , set $\tilde{E} = E - \dim E$ in $K(M)$ or $KO(M)$.

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$, the upper half complex plane. Set (cf. [20, 33])

$$(3.8) \quad \Theta_1(T_{\mathbb{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbb{C}}M}),$$

$$(3.9) \quad \Theta_2(T_{\mathbb{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-1/2}}(\widetilde{T_{\mathbb{C}}M}),$$

$$(3.10) \quad \Theta_3(T_{\mathbb{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^{m-1/2}}(\widetilde{T_{\mathbb{C}}M}),$$

$$(3.11) \quad \Theta(T_{\mathbb{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}).$$

$\Theta_1(T_{\mathbb{C}}M)$, $\Theta_2(T_{\mathbb{C}}M)$, $\Theta_3(T_{\mathbb{C}}M)$ and $\Theta(T_{\mathbb{C}}M)$ admit formal Fourier expansion in $q^{1/2}$ as

$$(3.12) \quad \Theta_1(T_{\mathbb{C}}M) = A_0(T_{\mathbb{C}}M) + A_1(T_{\mathbb{C}}M)q^{1/2} + \dots,$$

$$(3.13) \quad \Theta_2(T_{\mathbb{C}}M) = B_0(T_{\mathbb{C}}M) + B_1(T_{\mathbb{C}}M)q^{1/2} + \dots,$$

$$(3.14) \quad \Theta_3(T_{\mathbb{C}}M) = C_0(T_{\mathbb{C}}M) + C_1(T_{\mathbb{C}}M)q^{1/2} + \dots,$$

$$(3.15) \quad \Theta(T_{\mathbb{C}}M) = D_0(T_{\mathbb{C}}M) + D_1(T_{\mathbb{C}}M)q^{1/2} + \dots,$$

where the A_j 's, B_j 's, C_j 's and D_j 's are elements in the semi-group formally generated by complex vector bundles over M . Moreover, they carry canonically induced connections denoted by ∇^{A_j} , ∇^{B_j} , ∇^{C_j} and ∇^{D_j} , respectively, and let $\nabla^{\Theta_i(T_{\mathbb{C}}M)}$, $\nabla^{\Theta(T_{\mathbb{C}}M)}$ be the induced connections with $q^{1/2}$ -coefficients on Θ_i , Θ from the ∇^{A_j} , ∇^{B_j} , ∇^{C_j} and ∇^{D_j} .

If ω is a differential form on M , we denote by $\omega^{(i)}$ its degree i component.

Definition 3.1.

$$(3.16) \quad \Phi_L(\nabla^{TM}, \tau) \triangleq L(TM, \nabla^{TM}) \widetilde{\text{ch}}(\Theta_1(T_{\mathbb{C}}M), \nabla^{\Theta_1(T_{\mathbb{C}}M)})$$

is called the *Landweber–Stong form* of M with respect to ∇^{TM} ;

$$(3.17) \quad \Phi_W(\nabla^{TM}, \tau) \triangleq \widehat{A}(TM, \nabla^{TM}) \text{ch}(\Theta_2(T_{\mathbb{C}}M), \nabla^{\Theta_2(T_{\mathbb{C}}M)}),$$

$$(3.18) \quad \Phi'_W(\nabla^{TM}, \tau) \triangleq \widehat{A}(TM, \nabla^{TM}) \text{ch}(\Theta_3(T_{\mathbb{C}}M), \nabla^{\Theta_3(T_{\mathbb{C}}M)}),$$

$$(3.19) \quad \Psi_W(\nabla^{TM}, \tau) \triangleq \widehat{A}(TM, \nabla^{TM}) \text{ch}(\Theta(T_{\mathbb{C}}M), \nabla^{\Theta(T_{\mathbb{C}}M)})$$

are called the *Witten forms* of M with respect to ∇^{TM} .

The four Jacobi theta functions are defined as follows (cf. [7]):

$$(3.20)$$

$$\theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^j)(1 - e^{-2\pi\sqrt{-1}v} q^j)],$$

$$(3.21)$$

$$\theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^j)(1 + e^{-2\pi\sqrt{-1}v} q^j)],$$

(3.22)

$$\theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 - e^{-2\pi\sqrt{-1}v} q^{j-1/2})],$$

(3.23)

$$\theta_3(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 + e^{-2\pi\sqrt{-1}v} q^{j-1/2})].$$

They are all holomorphic functions for $(v, \tau) \in \mathbb{C} \times \mathbb{H}$, where \mathbb{C} is the complex plane and \mathbb{H} is the upper half plane.

Let $\theta'(0, \tau) = \frac{\partial}{\partial v} \theta(v, \tau)|_{v=0}$. The *Jacobi identity* [7],

$$\theta'(0, \tau) = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)$$

holds.

Applying the Chern–Weil theory, we can express the Landweber–Stong forms and the Witten forms in terms of theta functions and curvatures, which look new in the literature (cf. [16, 20, 23, 36]).

Proposition 3.1. *The following identities hold:*

$$(3.24) \quad \Phi_L(\nabla^{TM}, \tau) = \det^{1/2} \left(\frac{R^{TM}}{2\pi^2} \frac{\theta'(0, \tau)}{\theta(R^{TM}/2\pi^2, \tau)} \frac{\theta_1(R^{TM}/2\pi^2, \tau)}{\theta_1(0, \tau)} \right),$$

$$(3.25) \quad \Phi_W(\nabla^{TM}, \tau) = \det^{1/2} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta(R^{TM}/4\pi^2, \tau)} \frac{\theta_2(R^{TM}/4\pi^2, \tau)}{\theta_2(0, \tau)} \right),$$

$$(3.26) \quad \Phi'_W(\nabla^{TM}, \tau) = \det^{1/2} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta(R^{TM}/4\pi^2, \tau)} \frac{\theta_3(R^{TM}/4\pi^2, \tau)}{\theta_3(0, \tau)} \right),$$

$$(3.27) \quad \Psi_W(\nabla^{TM}, \tau) = \det^{1/2} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta(R^{TM}/4\pi^2, \tau)} \right).$$

Let

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

as usual be the modular group. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the two generators of $\mathrm{SL}_2(\mathbb{Z})$. Their actions on \mathbb{H} are given by

$$S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1.$$

Let

$$\begin{aligned}\Gamma_0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{2} \right\}, \\ \Gamma^0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod{2} \right\}, \\ \Gamma_\theta &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2} \right\}\end{aligned}$$

be the three modular subgroups of $\mathrm{SL}_2(\mathbb{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST , the generators of $\Gamma^0(2)$ are STS, T^2STS and the generators of Γ_θ are S, T^2 . (cf. [7]).

If we act theta-functions by S and T , the theta functions obey the following transformation laws (cf. [7]):

$$\begin{aligned}\theta(v, \tau + 1) &= e^{\pi\sqrt{-1}/4}\theta(v, \tau), \\ (3.28)\end{aligned}$$

$$\theta(v, -1/\tau) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau);$$

$$\begin{aligned}\theta_1(v, \tau + 1) &= e^{\pi\sqrt{-1}/4}\theta_1(v, \tau), \\ (3.29)\end{aligned}$$

$$\begin{aligned}\theta_1(v, -1/\tau) &= \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_2(\tau v, \tau); \\ (3.30)\end{aligned}$$

$$\begin{aligned}\theta_2(v, \tau + 1) &= \theta_3(v, \tau), \quad \theta_2(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_1(\tau v, \tau); \\ (3.31)\end{aligned}$$

$$\theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_3(\tau v, \tau).$$

Definition 3.2. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbb{H} \cup \{\infty\}$ such that for any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

the following property holds

$$f(g\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^k f(\tau),$$

where $\chi : \Gamma \rightarrow \mathbf{C}^*$ is a character of Γ and k is called the weight of f .

If Γ is a modular subgroup, let $\mathcal{M}_{\mathbb{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients. Writing simply $\theta_j = \theta_j(0, \tau)$, $1 \leq j \leq 3$, we introduce six explicit modular forms (cf. [18, 20]),

$$\begin{aligned}\delta_1(\tau) &= \frac{1}{8}(\theta_2^4 + \theta_3^4), & \varepsilon_1(\tau) &= \frac{1}{16}\theta_2^4\theta_3^4, \\ \delta_2(\tau) &= -\frac{1}{8}(\theta_1^4 + \theta_3^4), & \varepsilon_2(\tau) &= \frac{1}{16}\theta_1^4\theta_3^4, \\ \delta_3(\tau) &= \frac{1}{8}(\theta_1^4 - \theta_2^4), & \varepsilon_3(\tau) &= -\frac{1}{16}\theta_1^4\theta_2^4.\end{aligned}$$

They have the following Fourier expansions in $q^{1/2}$:

$$\begin{aligned}\delta_1(\tau) &= \frac{1}{4} + 6q + 6q^2 + \cdots, & \varepsilon_1(\tau) &= \frac{1}{16} - q + 7q^2 + \cdots, \\ \delta_2(\tau) &= -\frac{1}{8} - 3q^{1/2} - 3q + \cdots, & \varepsilon_2(\tau) &= q^{1/2} + 8q + \cdots, \\ \delta_3(\tau) &= -\frac{1}{8} + 3q^{1/2} - 3q + \cdots, & \varepsilon_3(\tau) &= -q^{1/2} + 8q + \cdots,\end{aligned}$$

where the “ \cdots ” terms are the higher degree terms, all of which have integral coefficients. They also satisfy the transformation laws (cf. [18, 20, 21]),

$$(3.32) \quad \delta_2\left(-\frac{1}{\tau}\right) = \tau^2\delta_1(\tau), \quad \varepsilon_2\left(-\frac{1}{\tau}\right) = \tau^4\varepsilon_1(\tau).$$

$$(3.33) \quad \delta_2(\tau + 1) = \delta_3(\tau), \quad \varepsilon_2(\tau + 1) = \varepsilon_3(\tau).$$

Lemma 3.1 (cf. [20]). *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, while $\delta_3(\tau)$ (resp. $\varepsilon_3(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_\theta(2)$ and, moreover, $\mathcal{M}_{\mathbb{R}}(\Gamma^0(2)) = \mathbb{R}[\delta_2(\tau), \varepsilon_2(\tau)]$.*

Acting the transformations S, T on the Landweber–Stong forms and the Witten forms, we have (cf. [14, 20, 21])

Proposition 3.2. *For any integer $i \geq 0$, one has that*

- (1) $\{\Phi_L(\nabla^{TM}, \tau)\}^{(4i)}$ is a modular form of weight $2i$ over $\Gamma_0(2)$;
 $\{\Phi_W(\nabla^{TM}, \tau)\}^{(4i)}$ is a modular form of weight $2i$ over $\Gamma^0(2)$;
 $\{\Phi'_W(\nabla^{TM}, \tau)\}^{(4i)}$ is a modular form of weight $2i$ over Γ_θ ;
moreover, if the first Pontryagin form $p_1(M, \nabla^{TM}) = 0$, then
 $\{\Psi_W(\nabla^{TM}, \tau)\}^{(4i)}$ is a modular form of weight $2i$ over $\text{SL}_2(\mathbb{Z})$.

(2) *The following equalities hold:*

$$(3.34) \quad \{\Phi_L(\nabla^{TM}, -1/\tau)\}^{(4i)} = (2\tau)^{2i} \{\Phi_W(\nabla^{TM}, \tau)\}^{(4i)},$$

$$(3.35) \quad \Phi_W(\nabla^{TM}, \tau + 1) = \Phi'_W(\nabla^{TM}, \tau).$$

The modularities in Proposition 3.2 have some interesting applications. For example, let M be 12-dimensional and $i = 3$. $\{\Phi_L(\nabla^{TM}, \tau)\}^{(12)}$ is a modular form of weight 6 over $\Gamma_0(2)$, $\{\Phi_W(\nabla^{TM}, \tau)\}^{(12)}$ is a modular form of weight 6 over $\Gamma^0(2)$ and

$$\{\Phi_L(\nabla^{TM}, -1/\tau)\}^{(12)} = (2\tau)^6 \{\Phi_W(\nabla^{TM}, \tau)\}^{(12)}.$$

Then by Lemma 3.1, we have

$$(3.36) \quad \{\Phi_W(\nabla^{TM}, \tau)\}^{(12)} = h_0(8\delta_2)^3 + h_1(8\delta_2)\epsilon_2,$$

and by (3.32) and (3.34),

$$(3.37) \quad \{\Phi_L(\nabla^{TM}, \tau)\}^{(12)} = 2^6[h_0(8\delta_1)^3 + h_1(8\delta_1)\epsilon_1],$$

where by comparing the $q^{1/2}$ -expansion coefficients in (3.36), $h_0 = -\{\widehat{A}(TM, \nabla^{TM})\}^{(12)}$ and $h_1 = \{60\widehat{A}(TM, \nabla^{TM}) + \widehat{A}(TM, \nabla^{TM})\text{ch}(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M})\}^{(12)}$. Then comparing the constant coefficients of the q -expansions of both sides of (3.37), one obtains that $\{L(TM, \nabla^{TM})\}^{(12)} = 2^3(2^6h_0 + h_1)$, consequently

$$(3.38) \quad \begin{aligned} \{L(TM, \nabla^{TM})\}^{(12)} &= \{8\widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M}) \\ &\quad - 32\widehat{A}(TM, \nabla^{TM})\}^{(12)}, \end{aligned}$$

which is just the gravitational anomaly cancellation formula derived by Alvarez-Gaumé and Witten [1] from very nontrivial computations. Liu [20] generalizes the miraculous cancellation formula (3.38) to arbitrary $8k + 4$ -dimensional smooth manifolds by developing modular invariance properties of characteristic forms. Formulas of this type have interesting applications in the study of divisibility and congruence phenomena for characteristic numbers. We refer interested readers to [4, 12, 13, 20, 27, 35].

Let M be a $4k$ -dimensional closed oriented smooth manifold and $[M]$ is the fundamental class. $\phi_L(M, \tau) \triangleq \langle \Phi_L(\nabla^{TM}, \tau), [M] \rangle$ is a modular form of weight $2k$ over $\Gamma_0(2)$ with integral q -expansion coefficients and called the *Landweber–Stong genus* of M . $\phi_W(M, \tau) \triangleq \langle \Phi_W(\nabla^{TM}, \tau), [M] \rangle$ and $\phi'_W(M, \tau) \triangleq \langle \Phi'_W(\nabla^{TM}, \tau), [M] \rangle$ are modular forms of weight $2k$ over $\Gamma^0(2)$ and

Γ_θ , respectively. If the first Pontryagin class of M vanishes, then $\psi_W(M, \tau) \triangleq \langle \Psi_W(\nabla^{TM}, \tau), [M] \rangle$ is a modular form over $\mathrm{SL}_2(\mathbb{Z})$. They are modular forms with rational q -expansion coefficients and called the *Witten genera* of M . These are all examples of the *elliptic genera*, which were first defined by Ochanine [28].

Let d_s be the signature operator of M . According to the Atiyah–Singer index theorem, we can express the Landweber–Stong genus analytically by the index of the twisted signature operator as

$$\phi_L(M, \tau) = \mathrm{Ind}(d_s \otimes \Theta_1(T_{\mathbb{C}}M)).$$

Moreover, let M be spin and D be the Atiyah–Singer Dirac operator over M . The Witten genera can also be analytically expressed by the indices of the twisted Dirac operators as follows:

$$\begin{aligned} \phi_W(M, \tau) &= \mathrm{Ind}(D \otimes \Theta_2(T_{\mathbb{C}}M)), \\ \phi'_W(M, \tau) &= \mathrm{Ind}(D \otimes \Theta_3(T_{\mathbb{C}}M)), \\ \psi_W(M, \tau) &= \mathrm{Ind}(D \otimes \Theta(T_{\mathbb{C}}M)). \end{aligned}$$

Heuristically, these twisted operators are viewed as elliptic operators on the smooth loop space LM from the view point of string theory. See details in Section 5. The elliptic operators $d_s \otimes \Theta_1(T_{\mathbb{C}}M)$, $D \otimes \Theta_2(T_{\mathbb{C}}M)$ and $D \otimes \Theta_3(T_{\mathbb{C}}M)$ are all rigid according to the Witten rigidity theorem, which is proved by Taubes [30], Bott and Taubes [6] and Liu [22].

4. Modular transgressions

Consider the following functions defined on $\mathbb{C} \times \mathbb{H}$:

$$\begin{aligned} f_{\Phi_L}(z, \tau) &= 2z \frac{\theta'(0, \tau)}{\theta(2z, \tau)} \frac{\theta_1(2z, \tau)}{\theta_1(0, \tau)}, \\ f_{\Phi_W}(z, \tau) &= z \frac{\theta'(0, \tau)}{\theta(z, \tau)} \frac{\theta_2(z, \tau)}{\theta_2(0, \tau)}, \\ f_{\Phi'_W}(z, \tau) &= z \frac{\theta'(0, \tau)}{\theta(z, \tau)} \frac{\theta_3(z, \tau)}{\theta_3(0, \tau)}, \\ f_{\Psi_W}(z, \tau) &= z \frac{\theta'(0, \tau)}{\theta(z, \tau)}. \end{aligned}$$

For the $4k$ -dimensional manifold M , by Proposition 3.1, we have

$$\begin{aligned}\Phi_{\mathbb{L}}(\nabla^{TM}, \tau) &= \det^{1/2} \left(f_{\Phi_{\mathbb{L}}} \left(\frac{R^{TM}}{4\pi^2}, \tau \right) \right), \\ \Phi_{\mathbb{W}}(\nabla^{TM}, \tau) &= \det^{1/2} \left(f_{\Phi_{\mathbb{W}}} \left(\frac{R^{TM}}{4\pi^2}, \tau \right) \right), \\ \Phi'_{\mathbb{W}}(\nabla^{TM}, \tau) &= \det^{1/2} \left(f_{\Phi'_{\mathbb{W}}} \left(\frac{R^{TM}}{4\pi^2}, \tau \right) \right), \\ \Psi_{\mathbb{W}}(\nabla^{TM}, \tau) &= \det^{1/2} \left(f_{\Psi_{\mathbb{W}}} \left(\frac{R^{TM}}{4\pi^2}, \tau \right) \right).\end{aligned}$$

From now on, let M be a $4k - 1$ -dimensional smooth manifold. Let $\nabla_i^{TM}, i = 0, 1$ be two connections on TM and $R_i^{TM}, i = 0, 1$ be their curvatures, respectively. Let $\nabla_t^{TM} = (1 - t)\nabla_0^{TM} + t\nabla_1^{TM}$ and R_t^{TM} be the corresponding curvature. Let $A = \nabla_1 - \nabla_0 \in \Omega^1(M, \text{End}(TM))$.

By Theorem 2.2, one has

$$\begin{aligned}(4.1) \quad & \det^{1/2} \left(f_{\Phi_{\mathbb{L}}} \left(\frac{R_1^{TM}}{4\pi^2}, \tau \right) \right) - \det^{1/2} \left(f_{\Phi_{\mathbb{L}}} \left(\frac{R_0^{TM}}{4\pi^2}, \tau \right) \right) \\ &= d \int_0^1 \frac{1}{8\pi^2} \det^{1/2} \left(f_{\Phi_{\mathbb{L}}} \left(\frac{R_t^{TM}}{4\pi^2}, \tau \right) \right) \text{tr} \left[A \frac{f'_{\Phi_{\mathbb{L}}} (R_t^{TM}/4\pi^2, \tau)}{f_{\Phi_{\mathbb{L}}} (R_t^{TM}/4\pi^2, \tau)} \right] dt.\end{aligned}$$

We define

$$\begin{aligned}(4.2) \quad & \text{CS } \Phi_{\mathbb{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \\ & \triangleq \frac{1}{4\pi^2} \int_0^1 \Phi_{\mathbb{L}}(\nabla_t^{TM}, \tau) \text{tr} \\ & \times \left[A \left(\frac{1}{R_t^{TM}/2\pi^2} - \frac{\theta'(R_t^{TM}/2\pi^2, \tau)}{\theta(R_t^{TM}/2\pi^2, \tau)} + \frac{\theta'_1(R_t^{TM}/2\pi^2, \tau)}{\theta_1(R_t^{TM}/2\pi^2, \tau)} \right) \right] dt,\end{aligned}$$

which is in $\Omega^{\text{odd}}(M, \mathbb{C})[[q]]$. Since M is $4k - 1$ -dimensional, $\{\text{CS } \Phi_{\mathbb{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4k-1)}$ represents an element in $H^{4k-1}(M, \mathbb{C})[[q]]$.

Similarly, we can compute the transgressed forms for Φ_W, Φ'_W and Ψ_W , respectively, and define

$$(4.3) \quad \begin{aligned} & \text{CS } \Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \\ & \triangleq \frac{1}{8\pi^2} \int_0^1 \Phi_W(\nabla_t^{TM}, \tau) \\ & \quad \times \text{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} - \frac{\theta'(R_t^{TM}/4\pi^2, \tau)}{\theta(R_t^{TM}/4\pi^2, \tau)} + \frac{\theta'_2(R_t^{TM}/4\pi^2, \tau)}{\theta_2(R_t^{TM}/4\pi^2, \tau)} \right) \right] dt, \end{aligned}$$

$$(4.4) \quad \begin{aligned} & \text{CS } \Phi'_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \\ & \triangleq \frac{1}{8\pi^2} \int_0^1 \Phi'_W(\nabla_t^{TM}, \tau) \\ & \quad \times \text{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} - \frac{\theta'(R_t^{TM}/4\pi^2, \tau)}{\theta(R_t^{TM}/4\pi^2, \tau)} + \frac{\theta'_3(R_t^{TM}/4\pi^2, \tau)}{\theta_3(R_t^{TM}/4\pi^2, \tau)} \right) \right] dt, \end{aligned}$$

and

$$(4.5) \quad \begin{aligned} & \text{CS } \Psi_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \\ & \triangleq \frac{1}{8\pi^2} \int_0^1 \Psi_W(\nabla_t^{TM}, \tau) \\ & \quad \times \text{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} - \frac{\theta'(R_t^{TM}/4\pi^2, \tau)}{\theta(R_t^{TM}/4\pi^2, \tau)} \right) \right] dt, \end{aligned}$$

which lie in $\Omega^{\text{odd}}(M, \mathbb{C})[[q^{1/2}]]$ and their top degree components represent elements in $H^{4k-1}(M, \mathbb{C})[[q^{1/2}]]$.

Remark 4.1. In (4.2) to (4.5) and in the following, to make sense, $\frac{1}{z} - \frac{\theta'(z, \tau)}{\theta(z, \tau)}$ should be understood as the z -expansion of $\frac{[\theta(z, \tau) - z\theta'(z, \tau)]/z^2}{\theta(z, \tau)/z}$, where both the numerator and the denominator have nonzero constant terms in their z -expansions.

Equality (4.1) and the modular invariance properties of $\det^{1/2} \left(f_{\Phi_L} \left(\frac{R_1^{TM}}{4\pi^2}, \tau \right) \right), \det^{1/2} \left(f_{\Phi_L} \left(\frac{R_0^{TM}}{4\pi^2}, \tau \right) \right)$ are not enough to guarantee that $\text{CS } \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \tau)$ is a modular form. Actually $(\text{CS } \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \tau) + \text{any closed form})$ will also satisfy (4.1). This is also true for other transgressed forms (4.3)–(4.5). However, we do have the following results.

Theorem 4.1. *Let M be a $4k - 1$ -dimensional smooth manifold and ∇_0^{TM} , ∇_1^{TM} be two connections on TM , then for integer i , $1 \leq i \leq k$, we have*

(1) $\{\text{CS } \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over $\Gamma_0(2)$;

$\{\text{CS } \Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over $\Gamma^0(2)$;

$\{\text{CS } \Phi'_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over Γ_θ .

(2) The following equalities hold:

$$\begin{aligned} \{\text{CS } \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, -1/\tau)\}^{(4i-1)} &= (2\tau)^{2i} \{\text{CS } \Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}, \\ \text{CS } \Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau + 1) &= \text{CS } \Phi'_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau). \end{aligned}$$

Proof. Differentiating the transformation formulas (3.28) to (3.31), we obtain that

(4.6)

$$\begin{aligned} \theta'(v, \tau + 1) &= e^{\pi\sqrt{-1}/4} \theta'(v, \tau), \\ \theta'(v, -1/\tau) &= \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} (2\pi\sqrt{-1}\tau v \theta(\tau v, \tau) + \tau \theta'(\tau v, \tau)); \\ \theta'_1(v, \tau + 1) &= e^{\pi\sqrt{-1}/4} \theta'_1(v, \tau), \\ \theta'_1(v, -1/\tau) &= \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} (2\pi\sqrt{-1}\tau v \theta_2(\tau v, \tau) + \tau \theta'_2(\tau v, \tau)); \\ \theta'_2(v, \tau + 1) &= \theta'_3(v, \tau), \\ \theta'_2(v, -1/\tau) &= \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} (2\pi\sqrt{-1}\tau v \theta_1(\tau v, \tau) + \tau \theta'_1(\tau v, \tau)); \\ \theta'_3(v, \tau + 1) &= \theta'_2(v, \tau), \\ \theta'_3(v, -1/\tau) &= \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} (2\pi\sqrt{-1}\tau v \theta_3(\tau v, \tau) + \tau \theta'_3(\tau v, \tau)). \end{aligned}$$

Therefore

$$(4.7) \quad \theta'(0, -1/\tau) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} \tau \theta'(0, \tau).$$

Note that the $(4i - 1)$ component of the right-hand side of (4.10) consists of terms like

$$\int_0^1 \operatorname{tr} \left[\left(\frac{2\tau R_t^{TM}}{4\pi^2} \right)^{p_1} \right] \cdots \operatorname{tr} \left[\left(\frac{2\tau R_t^{TM}}{4\pi^2} \right)^{p_s} \right] \operatorname{tr} \left[A \left(\frac{2\tau R_t^{TM}}{4\pi^2} \right)^q \right] dt,$$

where $2p_1 + \cdots + 2p_s + 2q + 1 = 4i - 1$, i.e., $p_1 + \cdots + p_s + q = 2i - 1$ due to the fact that $A \in \Omega^1(M, \operatorname{End}(TM))$ and $R_t^{TM} \in \Omega^2(M, \operatorname{End}(TM))$. Hence we have

$$\begin{aligned} & \{ \operatorname{CS} \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, -1/\tau) \}^{(4i-1)} \\ &= \left\{ \frac{2\tau}{8\pi^2} \int_0^1 \det^{1/2} \left(f_{\Phi_W} \left(\frac{2\tau R_t^{TM}}{4\pi^2}, \tau \right) \right) \operatorname{tr} \left[A \left(\frac{1}{2\tau R_t^{TM}/4\pi^2} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\theta'(2\tau R_t^{TM}/4\pi^2, \tau)}{\theta(2\tau R_t^{TM}/4\pi^2, \tau)} + \frac{\theta'_2(2\tau R_t^{TM}/4\pi^2, \tau)}{\theta_2(2\tau R_t^{TM}/4\pi^2, \tau)} \right) \right] dt \right\}^{(4i-1)} \\ (4.11) \quad &= (2\tau)^{1+2i-1} \left\{ \frac{1}{8\pi^2} \int_0^1 \Phi_W(\nabla_t^{TM}, \tau) \operatorname{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} \right. \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\theta'(R_t^{TM}/4\pi^2, \tau)}{\theta(R_t^{TM}/4\pi^2, \tau)} + \frac{\theta'_2(R_t^{TM}/4\pi^2, \tau)}{\theta_2(R_t^{TM}/4\pi^2, \tau)} \right) \right] dt \right\}^{(4i-1)} \\ &= (2\tau)^{2i} \{ \operatorname{CS} \Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \}^{(4i-1)}. \end{aligned}$$

Similarly, applying the transformation laws (3.28) to (3.31) and (4.6), we can show that

$$\begin{aligned} (4.12) \quad & \operatorname{CS} \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \tau + 1) = \operatorname{CS} \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \tau), \\ & \{ \operatorname{CS} \Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, -1/\tau) \}^{(4i-1)} = \left(\frac{\tau}{2} \right)^{2i} \{ \operatorname{CS} \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \}^{(4i-1)}, \\ & \operatorname{CS} \Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau + 1) = \operatorname{CS} \Phi'_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau), \\ & \{ \operatorname{CS} \Phi'_W(\nabla_0^{TM}, \nabla_1^{TM}, -1/\tau) \}^{(4i-1)} = \tau^{2i} \{ \operatorname{CS} \Phi'_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \}^{(4i-1)}, \\ & \operatorname{CS} \Phi'_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau + 1) = \operatorname{CS} \Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \tau). \end{aligned}$$

Acting ST^2ST to $\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)$, we can see from (4.11) and (4.12) that

$$\begin{aligned}
& \{\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, ST^2ST\tau)\}^{(4i-1)} \\
&= \{\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, S(T^2ST\tau))\}^{(4i-1)} \\
&= (2T^2ST\tau)^{2i} \{\text{CS } \Phi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, (T^2ST\tau))\}^{(4i-1)} \\
&= (2T^2ST\tau)^{2i} \{\text{CS } \Phi'_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, (TST\tau))\}^{(4i-1)} \\
(4.13) \quad &= (2T^2ST\tau)^{2i} \{\text{CS } \Phi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, (ST\tau))\}^{(4i-1)} \\
&= (2T^2ST\tau)^{2i} \left(\frac{T\tau}{2}\right)^{2i} \{\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, (T\tau))\}^{(4i-1)} \\
&= \left(2\frac{2\tau+1}{\tau+1}\right)^{2i} \left(\frac{\tau+1}{2}\right)^{2i} \{\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, (T\tau))\}^{(4i-1)} \\
&= (2\tau+1)^{2i} \{\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}.
\end{aligned}$$

Note that $ST^2ST\tau = -\frac{\tau+1}{2\tau+1}$. By the first equality in (4.12) and (4.13), we see that $\{\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}$ is modular invariance under the actions of T and ST^2ST , which form a basis for $\Gamma_0(2)$. Thus $\{\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over $\Gamma_0(2)$.

We can similarly show that $\{\text{CS } \Phi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over $\Gamma^0(2)$ and $\{\text{CS } \Phi'_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over Γ_θ . \square

Remark 4.2. From the proof of the above theorem, it is not hard to see that if we use another path connecting ∇_0^{TM} and ∇_1^{TM} instead of the canonical path $(1-t)\nabla_0^{TM} + t\nabla_1^{TM}$, $0 \leq t \leq 1$, we still get relevant modular Chern–Simons terms, which differ by exact forms (also modular!) from the above canonical Chern–Simons terms, respectively.

Let us take a look at a concrete example. Let M be a compact oriented smooth 3-dimensional manifold. We have

$$\begin{aligned}
& \text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \\
&= \frac{1}{4\pi^2} \int_0^1 \Phi_{\text{L}}(\nabla_t^{TM}, \tau) \text{tr} \left[A \left(\frac{1}{R_t^{TM}/2\pi^2} - \frac{\theta'(R_t^{TM}/2\pi^2, \tau)}{\theta(R_t^{TM}/2\pi^2, \tau)} \right. \right. \\
&\quad \left. \left. + \frac{\theta'_1(R_t^{TM}/2\pi^2, \tau)}{\theta_1(R_t^{TM}/2\pi^2, \tau)} \right) \right] dt
\end{aligned}$$

$$\begin{aligned}
 (4.14) \quad &= \frac{1}{4\pi^2} \int_0^1 \operatorname{tr} \left[A \left(\frac{1}{R_t^{TM}/2\pi^2} - \frac{\theta'(R_t^{TM}/2\pi^2, \tau)}{\theta(R_t^{TM}/2\pi^2, \tau)} \right. \right. \\
 &\quad \left. \left. + \frac{\theta'_1(R_t^{TM}/2\pi^2, \tau)}{\theta_1(R_t^{TM}/2\pi^2, \tau)} \right) \right] dt \\
 &= \frac{1}{8\pi^4} \frac{\partial}{\partial z} \left(\frac{1}{z} - \frac{\theta'(z, \tau)}{\theta(z, \tau)} + \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)} \right) \Big|_{z=0} \int_0^1 \operatorname{tr}[AR_t^{TM}] dt,
 \end{aligned}$$

where the second equality holds because the dimension of M is only 3 while

$$\int_0^1 \operatorname{tr} \left[A \left(\frac{1}{R_t^{TM}/2\pi^2} - \frac{\theta'(R_t^{TM}/2\pi^2, \tau)}{\theta(R_t^{TM}/2\pi^2, \tau)} + \frac{\theta'_1(R_t^{TM}/2\pi^2, \tau)}{\theta_1(R_t^{TM}/2\pi^2, \tau)} \right) \right] dt$$

gives differential forms of degree greater than or equal to 3.

But

$$\begin{aligned}
 (4.15) \quad &\int_0^1 \operatorname{tr}[AR_t^{TM}] dt = \int_0^1 \operatorname{tr}[A((1-t)\nabla_0^{TM} + t\nabla_1^{TM})^2] dt \\
 &= \int_0^1 \operatorname{tr}[A((1-t)^2(\nabla_0^{TM})^2 + (1-t)t[\nabla_0^{TM}, \nabla_1^{TM}] \\
 &\quad + t^2(\nabla_1^{TM})^2)] dt \\
 &= \operatorname{tr} \left[A \left(\frac{1}{3}(\nabla_0^{TM})^2 + \frac{1}{6}[\nabla_0^{TM}, \nabla_1^{TM}] + \frac{1}{3}(\nabla_1^{TM})^2 \right) \right] \\
 &= \frac{1}{3} \operatorname{tr} \left[A \left((\nabla_0^{TM} - \nabla_1^{TM})^2 + \frac{3}{2}[\nabla_0^{TM}, \nabla_1^{TM}] \right) \right] \\
 &= \frac{1}{2} \operatorname{tr} \left[A[\nabla_0^{TM}, \nabla_1^{TM}] + \frac{2}{3}A \wedge A \wedge A \right].
 \end{aligned}$$

Note that $\frac{\partial}{\partial z} \left(\frac{1}{z} - \frac{\theta'(z, \tau)}{\theta(z, \tau)} + \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)} \right) \Big|_{z=0}$ is a modular form of weight 2 over $\Gamma_0(2)$. Then by Lemma 3.1, it should be a scalar multiple of $\delta_1(\tau)$. By direct computations, we can see that $\frac{\partial}{\partial z} \left(\frac{1}{z} - \frac{\theta'(z, \tau)}{\theta(z, \tau)} + \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)} \right) \Big|_{z=0} = -\frac{2}{3}\pi^2 + O(q)$. So

$$\frac{\partial}{\partial z} \left(\frac{1}{z} - \frac{\theta'(z, \tau)}{\theta(z, \tau)} + \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)} \right) \Big|_{z=0} = -\frac{8}{3}\pi^2 \delta_1(\tau).$$

Thus we have

$$(4.16) \quad \text{CS } \Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \tau) = -\frac{1}{6\pi^2} \delta_1(\tau) \operatorname{tr} \left[A[\nabla_0^{TM}, \nabla_1^{TM}] + \frac{2}{3}A \wedge A \wedge A \right].$$

Similarly, we obtain that

$$(4.17) \quad \text{CS } \Phi_{\mathbb{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau) = -\frac{1}{24\pi^2} \delta_2(\tau) \text{tr} \left[A[\nabla_0^{TM}, \nabla_1^{TM}] + \frac{2}{3} A \wedge A \wedge A \right],$$

and

$$(4.18) \quad \text{CS } \Phi'_{\mathbb{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau) = -\frac{1}{24\pi^2} \delta_3(\tau) \text{tr} \left[A[\nabla_0^{TM}, \nabla_1^{TM}] + \frac{2}{3} A \wedge A \wedge A \right].$$

Note that TM is trivial. Let's take ∇_0^{TM} to be the trivial connection d^{TM} associated to some global basis of $\Gamma(TM)$ and $\nabla_1^{TM} = d^{TM} + A$. We therefore have the following proposition.

Proposition 4.1. *When M is a compact oriented smooth 3-dimensional manifold, the following identities hold,*

$$(4.19) \quad \text{CS } \Phi_{\mathbb{L}}(d^{TM}, d^{TM} + A, \tau) = -\frac{1}{6\pi^2} \delta_1(\tau) \text{CS}(A),$$

$$(4.20) \quad \text{CS } \Phi_{\mathbb{W}}(d^{TM}, d^{TM} + A, \tau) = -\frac{1}{24\pi^2} \delta_2(\tau) \text{CS}(A),$$

$$(4.21) \quad \text{CS } \Phi'_{\mathbb{W}}(d^{TM}, d^{TM} + A, \tau) = -\frac{1}{24\pi^2} \delta_3(\tau) \text{CS}(A).$$

Very similar to the application after Proposition 3.2, the modularities in Theorem 4.1 also imply some relations among transgressed forms, which might be viewed as anomaly cancellation formulas for odd-dimensional manifolds. For example, let M be 11-dimensional and $i = 3$. We also similarly have that $\{\text{CS } \Phi_{\mathbb{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(11)}$ is a modular form of weight 6 over $\Gamma_0(2)$, $\{\text{CS } \Phi_{\mathbb{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(11)}$ is a modular form of weight 6 over $\Gamma^0(2)$ and

$$\{\text{CS } \Phi_{\mathbb{L}}(\nabla_0^{TM}, \nabla_1^{TM}, -1/\tau)\}^{(11)} = (2\tau)^6 \{\text{CS } \Phi_{\mathbb{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(11)}.$$

Then still by Lemma 3.1, we have

$$(4.22) \quad \{\text{CS } \Phi_{\mathbb{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(11)} = z_0(8\delta_2)^3 + z_1(8\delta_2)\epsilon_2,$$

and by (3.32) and Theorem 4.1,

$$(4.23) \quad \{\text{CS } \Phi_{\mathbb{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(11)} = 2^6[z_0(8\delta_1)^3 + z_1(8\delta_1)\epsilon_1],$$

where, by comparing the $q^{1/2}$ -expansion coefficients in (4.22),
 (4.24)

$$z_0 = - \left\{ \int_0^1 \widehat{A}(TM, \nabla_t^{TM}) \operatorname{tr} \left[A \left(\frac{1}{2R_t^{TM}} - \frac{1}{8\pi \tan(R_t^{TM}/4\pi)} \right) \right] dt \right\}^{(11)},$$

$$(4.25) \quad \begin{aligned} z_1 = & \left\{ \int_0^1 \widehat{A}(TM, \nabla_t^{TM}) \operatorname{ch} \left(T_{\mathbb{C}}M, \nabla_t^{T_{\mathbb{C}}M} \right) \right. \\ & \times \operatorname{tr} \left[A \left(\frac{1}{2R_t^{TM}} - \frac{1}{8\pi \tan(R_t^{TM}/4\pi)} \right) \right] dt \\ & + \int_0^1 \widehat{A}(TM, \nabla_t^{TM}) \operatorname{tr} \left[A \left(-\frac{1}{2\pi} \sin \left(\frac{R_t^{TM}}{2\pi} \right) \right. \right. \\ & \left. \left. + 61 \left(\frac{1}{2R_t^{TM}} - \frac{1}{8\pi \tan(R_t^{TM}/4\pi)} \right) \right) \right] dt \right\}^{(11)}. \end{aligned}$$

Plugging (4.24) and (4.25) into (4.23) and comparing the constant terms of both sides, we obtain that

$$\begin{aligned} & \left\{ \int_0^1 L(TM, \nabla_t^{TM}) \operatorname{tr} \left[A \left(\frac{1}{2R_t^{TM}} - \frac{1}{2\pi \sin(R_t^{TM}/\pi)} \right) \right] dt \right\}^{(11)} \\ & = 2^3(2^6 z_0 + z_1), \end{aligned}$$

consequently

$$(4.26) \quad \begin{aligned} & \left\{ \int_0^1 L(TM, \nabla_t^{TM}) \operatorname{tr} \left[A \left(\frac{1}{2R_t^{TM}} - \frac{1}{2\pi \sin(R_t^{TM}/\pi)} \right) \right] dt \right\}^{(11)} \\ & = \left\{ \int_0^1 \widehat{A}(TM, \nabla_t^{TM}) \operatorname{ch} \left(T_{\mathbb{C}}M, \nabla_t^{T_{\mathbb{C}}M} \right) \right. \\ & \times \operatorname{tr} \left[A \left(\frac{1}{2R_t^{TM}} - \frac{1}{8\pi \tan(R_t^{TM}/4\pi)} \right) \right] dt \\ & + \int_0^1 \widehat{A}(TM, \nabla_t^{TM}) \operatorname{tr} \left[A \left(-\frac{1}{2\pi} \sin \left(\frac{R_t^{TM}}{2\pi} \right) \right. \right. \\ & \left. \left. - 3 \left(\frac{1}{2R_t^{TM}} - \frac{1}{8\pi \tan(R_t^{TM}/4\pi)} \right) \right) \right] dt \right\}^{(11)}. \end{aligned}$$

We can view (4.26) as a 11-dimensional analogue of the miraculous cancellation formula (3.38). We would like to remark that (4.26) cannot be

obtained directly from (3.38) just by applying Chern–Simons transgression to both sides as we pointed out right before Theorem 4.1. Hopefully the odd-dimensional cancellation formula (4.26) could make sense in physics.

As for $\text{CS } \Psi_{\mathbb{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)$, we have the following results.

Theorem 4.2. *Let M be a $4k - 1$ -dimensional smooth flat manifold, $k \geq 2$, ∇_0^{TM} and ∇_1^{TM} be two flat connections on TM , then for $2 \leq i \leq k$, $\{\text{CS } \Psi_{\mathbb{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over $\text{SL}_2(\mathbb{Z})$.*

Proof. Since both ∇_0^{TM} and ∇_1^{TM} are flat, we have

$$\begin{aligned} (\nabla_0^{TM})^2 &= 0, \\ (\nabla_1^{TM})^2 &= (\nabla_0^{TM} + A)^2 = (\nabla_0^{TM})^2 + [\nabla_0^{TM}, A] + A \wedge A = 0, \end{aligned}$$

which implies

$$(4.27) \quad [\nabla_0^{TM}, A] = -A \wedge A.$$

One also has

$$\begin{aligned} R_t^{TM} &= ((1-t)\nabla_0^{TM} + t\nabla_1^{TM})^2 = (\nabla_0^{TM} + tA)^2 = (\nabla_0^{TM})^2 + t[\nabla_0^{TM}, A] \\ &\quad + t^2 A \wedge A. \end{aligned}$$

Therefore

$$(4.28) \quad R_t^{TM} = (t^2 - t)A \wedge A.$$

Thus we obtain (cf. [32, Lemma 1.7])

$$(4.29) \quad \text{tr}[(R_t^{TM})^n] = (t^2 - t)^n \text{tr}[A^{2n}] = \frac{(t^2 - t)^n}{2} \text{tr}[[A, A^{2n-1}]] = 0 \quad \forall n \in \mathbf{Z}^+.$$

So it is not hard to see that

$$\det^{1/2} \left(f_{\Psi_{\mathbb{W}}} \left(\frac{R_t^{TM}}{4\pi^2}, \tau \right) \right) = e^{(1/2) \text{tr} \ln f_{\Psi_{\mathbb{W}}}(R_t^{TM}/4\pi^2, \tau)} = 1.$$

We therefore have

$$\text{CS } \Psi_{\mathbb{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau) = \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} - \frac{\theta'(R_t^{TM}/4\pi^2, \tau)}{\theta(R_t^{TM}/4\pi^2, \tau)} \right) \right] dt.$$

Then similar to (4.11), we have

$$\begin{aligned}
 (4.30) \quad & \left\{ \text{CS } \Psi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, -1/\tau) \right\}^{(4i-1)} \\
 &= \left\{ \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} - \frac{\theta'(R_t^{TM}/4\pi^2, -1/\tau)}{\theta(R_t^{TM}/4\pi^2, -1/\tau)} \right) \right] dt \right\}^{(4i-1)} \\
 &= \left\{ \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A\tau \left(\frac{1}{\tau R_t^{TM}/4\pi^2} - \frac{\theta'(\tau R_t^{TM}/4\pi^2, \tau)}{\theta(\tau R_t^{TM}/4\pi^2, \tau)} \right) \right. \right. \\
 &\quad \left. \left. - A \left(2\pi\sqrt{-1} \frac{\tau R_t^{TM}}{4\pi^2} \right) \right] dt \right\}^{(4i-1)} \\
 &= \left\{ \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A\tau \left(\frac{1}{\tau R_t^{TM}/4\pi^2} - \frac{\theta'(\tau R_t^{TM}/4\pi^2, \tau)}{\theta(\tau R_t^{TM}/4\pi^2, \tau)} \right) \right] dt \right\}^{(4i-1)},
 \end{aligned}$$

since $4i - 1 \geq 7$ while $\int_0^1 \text{tr} \left[-A \left(2\pi\sqrt{-1} \frac{\tau R_t^{TM}}{4\pi^2} \right) \right]$ is only a degree 3 form.

Thus we obtain that

$$\begin{aligned}
 (4.31) \quad & \left\{ \text{CS } \Psi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, -1/\tau) \right\}^{(4i-1)} \\
 &= \left\{ \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A\tau \left(\frac{1}{\tau R_t^{TM}/4\pi^2} - \frac{\theta'(\tau R_t^{TM}/4\pi^2, \tau)}{\theta(\tau R_t^{TM}/4\pi^2, \tau)} \right) \right] dt \right\}^{(4i-1)}, \\
 &= \tau^{2i} \left\{ \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} - \frac{\theta'(R_t^{TM}/4\pi^2, \tau)}{\theta(R_t^{TM}/4\pi^2, \tau)} \right) \right] dt \right\}^{(4i-1)} \\
 &= \tau^{2i} \text{CS } \Psi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau).
 \end{aligned}$$

It is also not hard to see that

$$(4.32) \quad \text{CS } \Psi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau + 1) = \text{CS } \Psi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau).$$

So $\left\{ \text{CS } \Psi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau) \right\}^{(4i-1)}$ is a modular form of weight $2i$ over $\text{SL}_2(\mathbb{Z})$. \square

Remark 4.3. In Theorem 4.2, i has to be greater than or equal to 2 to get a modular form of weight $2i$ over $\text{SL}_2(\mathbb{Z})$. This agrees with a known fact in number theory that there is no nontrivial modular form of weight 2 over $\text{SL}_2(\mathbb{Z})$.

Remark 4.4. In Theorem 4.2, if we use other paths rather than the canonical path $(1-t)\nabla_0^{TM} + t\nabla_1^{TM}$, $0 \leq t \leq 1$, the resulted Chern–Simons terms

might not be modular forms because some anomaly terms come out when we are doing the modular transformation $\tau \rightarrow -1/\tau$. In other words, some paths may break the modularity of the Chern–Simons term while the canonical one does not (because it avoids the anomaly). We hope to understand this interesting phenomena further. However, on the level of cohomology classes, different paths give the same modular cohomology class.

In particular, by Theorem 4.2, $\{\text{CS } \Psi_{\text{W}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(7)}$ is a weight 4 modular form over $\text{SL}_2(\mathbb{Z})$. Actually,

$$\begin{aligned}
 & \{\text{CS } \Phi_{\text{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(7)} \\
 &= \left\{ \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} - \frac{\theta'(R_t^{TM}/4\pi^2, \tau)}{\theta(R_t^{TM}/4\pi^2, \tau)} \right) \right] dt \right\}^{(7)} \\
 (4.33) \quad &= \left\{ \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A \left(\frac{1}{R_t^{TM}/4\pi^2} - \frac{\theta'(R_t^{TM}/4\pi^2, \tau)}{\theta(R_t^{TM}/4\pi^2, \tau)} \right) \right] dt \right\}^{(7)} \\
 &= \frac{1}{512\pi^8} \frac{1}{3!} \frac{\partial^3}{\partial z^3} \left(\frac{1}{z} - \frac{\theta'(z, \tau)}{\theta(z, \tau)} \right) \Big|_{z=0} \int_0^1 \text{tr}[A(R_t^{TM})^3] dt \\
 &= \frac{1}{512\pi^8} \frac{1}{3!} \frac{\partial^3}{\partial z^3} \left(\frac{1}{z} - \frac{\theta'(z, \tau)}{\theta(z, \tau)} \right) \Big|_{z=0} \left(\int_0^1 (t^2 - t)^3 dt \right) \text{tr}[A^7].
 \end{aligned}$$

By direct computations, we can see that

$$\begin{aligned}
 & \frac{1}{512\pi^8} \frac{1}{3!} \frac{\partial^3}{\partial z^3} \left(\frac{1}{z} - \frac{\theta'(z, \tau)}{\theta(z, \tau)} \right) \Big|_{z=0} \left(\int_0^1 (t^2 - t)^3 dt \right) \\
 &= -\frac{1}{3225600\pi^4} + O(q).
 \end{aligned}$$

Let

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

be the Eisenstein series, which is a weight 4 modular form over $\text{SL}_2(\mathbb{Z})$, where

$$\sigma_k(n) \triangleq \sum_{d|n} d^k.$$

It is a fact in number theory that the space of weight 4 modular forms over $\mathrm{SL}_2(\mathbb{Z})$ has dimension 1. Thus

$$\{\mathrm{CS} \Phi_{\mathrm{L}}(\nabla_0^{TM}, \nabla_1^{TM}, \tau)\}^{(7)} = -\frac{1}{3225600\pi^4} E_4(\tau) \mathrm{tr}[A^7].$$

5. Loop space index theorem and Chern–Simons forms

In this section we first formally describe the loop space version of the Atiyah–Singer index theory (cf. [2, 5]) and its relation with the theory of elliptic genera following [21]. Then we compute the Chern–Simons forms of two formal flat vector bundles on loop space.

Let M be a smooth compact spin manifold of dimension $2k$. Let P be a principle G bundle on M and E an irreducible positive energy representation of $\tilde{L}G$, which is the central extension of the loop group of G [29]. We decompose E according to the rotation action of the loop to get $E = \sum_{\geq 0} E_n$ where each E_n is finite-dimensional representation of G . Constructing associated bundles to P from each E_n , which is still denoted by E_n , we can define an element

$$\psi(P, E) = q^{m_\Lambda} \sum_n E_n q^n,$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$ and m_Λ (called the modular anomaly of the representation E) is a rational number depending on the level and weight of E such that $\mathrm{ch}(\psi(P, E)) = \chi(z, \tau)$, the normalized Kac–Weyl character of E (cf. [17, 24, 25]).

Let us specialize to the case $G = \mathrm{Spin}(2l)$. For any positive integer l , the loop group $\tilde{L}\mathrm{Spin}(2l)$ has four irreducible level 1 positive representations. Denote them by S^+, S^- and S_+, S_- . Let $\{\pm\alpha_j\}$ be the roots of $\mathrm{Spin}(2l)$. Then we have the following normalized Kac–Weyl character formulas:

$$\begin{aligned} \chi_{S^+ - S^-} &= \prod_{j=1}^l \frac{\theta(\alpha_j, \tau)}{\eta(\tau)}, & \chi_{S^+ + S^-} &= \prod_{j=1}^l \frac{\theta_1(\alpha_j, \tau)}{\eta(\tau)}, \\ \chi_{S_+ - S_-} &= \prod_{j=1}^l \frac{\theta_2(\alpha_j, \tau)}{\eta(\tau)}, & \chi_{S_+ + S_-} &= \prod_{j=1}^l \frac{\theta_3(\alpha_j, \tau)}{\eta(\tau)}, \end{aligned}$$

where $\eta(\tau) = q^{1/24} \prod_{l=1}^{\infty} (1 - q^l)$ is the Dedekind eta-function [7].

Let Q be the spin principle bundle associated to the tangent bundle of M . We have the following:

$$\begin{aligned}\psi(Q, S^+ - S^-) &= q^{-k/12}(\Delta^+ - \Delta^-) \otimes \bigotimes_{j=1}^{\infty} \Lambda_{-q^j}(TM), \\ \psi(Q, S^+ + S^-) &= q^{-k/12}(\Delta^+ + \Delta^-) \otimes \bigotimes_{j=1}^{\infty} \Lambda_{q^j}(TM), \\ \psi(Q, S_+ - S_-) &= q^{-k/24} \otimes \bigotimes_{j=1}^{\infty} \Lambda_{-q^{j-1/2}}(TM), \\ \psi(Q, S_+ + S_-) &= q^{-k/24} \otimes \bigotimes_{j=1}^{\infty} \Lambda_{q^{j-1/2}}(TM).\end{aligned}$$

Thus we can view $S^+ \pm S^-$ as the loop group analogues of the finite-dimensional spinor representations $\Delta^+ \pm \Delta^-$, where Δ^+ , Δ^- are the two irreducible spinor representations of $\text{Spin}(2l)$. Similar to the \widehat{A} -class, the loop space \widehat{A} -class is defined as

$$\widehat{\Theta}(M) = \frac{e(M)}{\text{ch}(\psi(Q, S^+) - \psi(Q, S^-))} = \eta(\tau)^k \cdot \prod_{j=1}^k \frac{x_j}{\theta(x_j, \tau)}.$$

Therefore formally one defines the loop space Dirac operator as

$$D^L = q^{\frac{-k}{12}} D \otimes \bigotimes_{j=1}^{\infty} S_{q^j}(TM)$$

and the corresponding index formula

$$\text{Ind}(D^L) = \int_M \widehat{\Theta}(M).$$

The twisted version of this index theorem in this loop group setting is

$$\text{Ind}(D^L \otimes \psi(P, E)) = \int_M \widehat{\Theta}(M) \text{ch}(\psi(P, E)).$$

In this sense we thus formally view $\psi(P, E)$ as a vector bundle over the loop space LM .

To get modular forms instead of Jacobi forms, let us use virtual versions of the above story. Denote by

$$(5.1) \quad \mathcal{D}^L = D \otimes \bigotimes_{j=1}^{\infty} S_{q^j}(\widetilde{T_{\mathbb{C}}M}).$$

Let V be a $2l$ -dimensional spin vector bundle over M . Physically,

$$\mathcal{V} := \bigotimes_{j=1}^{\infty} \Lambda_{-q^{j-1/2}}(\widetilde{V}_{\mathbb{C}}), \quad \mathcal{V}' := \bigotimes_{j=1}^{\infty} \Lambda_{q^{j-1/2}}(\widetilde{V}_{\mathbb{C}})$$

are virtual vector bundles over LM . Let ∇^{TM}, ∇^V be two connections over TM, V and R^{TM}, R^V be their curvatures, respectively. By the Atiyah–Singer index theorem (also cf. [20]), it is not hard to see that

$$\text{Ind}(\mathcal{D}^L \otimes \mathcal{V}) = \int_M \det^{1/2} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta(R^{TM}/4\pi^2, \tau)} \frac{\theta_2(R^V/4\pi^2, \tau)}{\theta_2(0, \tau)} \right)$$

and

$$\text{Ind}(\mathcal{D}^L \otimes \mathcal{V}') = \int_M \det^{1/2} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta(R^{TM}/4\pi^2, \tau)} \frac{\theta_3(R^V/4\pi^2, \tau)}{\theta_3(0, \tau)} \right).$$

When $p_1(TM, \nabla^{TM}) = p_1(V, \nabla^V)$, $\text{Ind}(\mathcal{D}^L \otimes \mathcal{V})$ and $\text{Ind}(\mathcal{D}^L \otimes \mathcal{V}')$ are modular forms over $\Gamma^0(2)$ and Γ_{θ} , respectively (cf. [21]). Especially, taking $V = TM$ and $\nabla^V = \nabla^{TM}$, one has that $\text{Ind}(\mathcal{D}^L \otimes TLM) = \phi_{\text{W}}(M, \tau)$ and $\text{Ind}(\mathcal{D}^L \otimes TLM') = \phi'_{\text{W}}(M, \tau)$.

Now let us assume that V is flat and ∇_0^V, ∇_1^V are two flat connections over V . Let $\nabla_i^V, i = 0, 1$ be two connections on V and $R_i^V, i = 0, 1$ be their curvatures, respectively. Let $\nabla_t^V = (1-t)\nabla_0^V + t\nabla_1^V$ and R_t^V be the corresponding curvature. Let $A = \nabla_1 - \nabla_0 \in \Omega^1(M, \text{End}(V))$. One can lift these two connections to $\mathcal{V}, \mathcal{V}'$ and denote them by $\nabla_0^{\mathcal{V}}, \nabla_1^{\mathcal{V}}$ and $\nabla_0^{\mathcal{V}'}, \nabla_1^{\mathcal{V}'}$. We heuristically view \mathcal{V} and \mathcal{V}' as flat vector bundles on the loop space LM .

Applying Theorem 2.2 similarly as in (4.1) to (4.4), one gets that

$$(5.2) \quad \begin{aligned} & \text{ch}(\mathcal{V}, \nabla_1^{\mathcal{V}}) - \text{ch}(\mathcal{V}, \nabla_0^{\mathcal{V}}) \\ &= \det^{1/2} \left(\frac{\theta_2(R_1^V/4\pi^2, \tau)}{\theta_2(0, \tau)} \right) - \det^{1/2} \left(\frac{\theta_2(R_0^V/4\pi^2, \tau)}{\theta_2(0, \tau)} \right) \\ &= d \int_0^1 \frac{1}{8\pi^2} \det^{1/2} \left(\frac{\theta_2(R_t^V/4\pi^2, \tau)}{\theta_2(0, \tau)} \right) \text{tr} \left[A \left(\frac{\theta_2'(R_t^V/4\pi^2, \tau)}{\theta_2(R_t^V/4\pi^2, \tau)} \right) \right] dt. \end{aligned}$$

However, by similar calculations as in the proof of Theorem 4.2, one has

$$\det^{1/2} \left(\frac{\theta_2(R_t^V/4\pi^2, \tau)}{\theta_2(0, \tau)} \right) = 1.$$

Therefore we have

$$(5.3) \quad \text{ch}(\mathcal{V}, \nabla_1^{\mathcal{V}}) - \text{ch}(\mathcal{V}, \nabla_0^{\mathcal{V}}) = d \text{CS}(\mathcal{V}, \nabla_0^{\mathcal{V}}, \nabla_1^{\mathcal{V}}, \tau),$$

where

$$(5.4) \quad \text{CS}(\mathcal{V}, \nabla_0^{\mathcal{V}}, \nabla_1^{\mathcal{V}}, \tau) \triangleq \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A \left(\frac{\theta_2'(R_t^V/4\pi^2, \tau)}{\theta_2(R_t^V/4\pi^2, \tau)} \right) \right] dt \in \Omega^{\text{odd}}(M)[[q^{1/2}]].$$

Since $\nabla_0^{\mathcal{V}}$ and $\nabla_1^{\mathcal{V}}$ are flat connections, $\text{ch}(\mathcal{V}, \nabla_1^{\mathcal{V}})$ and $\text{ch}(\mathcal{V}, \nabla_0^{\mathcal{V}})$ are both vanishing. Thus $\text{CS}(\mathcal{V}, \nabla_0^{\mathcal{V}}, \nabla_1^{\mathcal{V}}, \tau)$ represents an element in $H^{\text{odd}}(M, \mathbb{C})[[q^{1/2}]]$. One can similarly define

$$(5.5) \quad \begin{aligned} \text{CS}(\mathcal{V}', \nabla_0^{\mathcal{V}'}, \nabla_1^{\mathcal{V}'}, \tau) &\triangleq \frac{1}{8\pi^2} \int_0^1 \text{tr} \left[A \left(\frac{\theta_3'(R_t^V/4\pi^2, \tau)}{\theta_3(R_t^V/4\pi^2, \tau)} \right) \right] dt \\ &\in \Omega^{\text{odd}}(M)[[q^{1/2}]], \end{aligned}$$

which also represents an element in $H^{\text{odd}}(M, \mathbb{C})[[q^{1/2}]]$.

Similarly as Theorem 4.1, we obtain that

Theorem 5.1. *Let V be a $2l$ -dimensional flat vector bundle over M and $\nabla_0^{\mathcal{V}}, \nabla_1^{\mathcal{V}}$ be two flat connections on V , then for any positive integer $i \geq 2$, we have*

- (1) $\{\text{CS}(\mathcal{V}, \nabla_0^{\mathcal{V}}, \nabla_1^{\mathcal{V}}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over $\Gamma^0(2)$;
 $\{\text{CS}(\mathcal{V}', \nabla_0^{\mathcal{V}'}, \nabla_1^{\mathcal{V}'}, \tau)\}^{(4i-1)}$ is a modular form of weight $2i$ over Γ_θ ;
- (2) The following equality holds,

$$\text{CS}(\mathcal{V}, \nabla_0^{\mathcal{V}}, \nabla_1^{\mathcal{V}}, \tau + 1) = \text{CS}(\mathcal{V}', \nabla_0^{\mathcal{V}'}, \nabla_1^{\mathcal{V}'}, \tau).$$

Heuristically, (5.4) and (5.5) can be viewed as the Chern–Simons transgressed forms of flat vector bundles over loop spaces. We hope they could play some roles in the study of loop space vector bundles.

Remark 5.1. In Theorem 5.1, if we use other paths rather than the canonical path $(1-t)\nabla_0^V + t\nabla_1^V, 0 \leq t \leq 1$, the modularity of the resulted Chern–Simons terms might be broken (similar to what we pointed out in Remark 4.4). This indicates that the modularity of secondary characteristic forms involves some subtleties. On the level of cohomology classes, different paths give the same modular cohomology class.

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