

On the Witten Rigidity Theorem for Odd Dimensional Manifolds

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Abstract

We establish several Witten type rigidity and vanishing theorems for twisted Toeplitz operators on odd dimensional manifolds. We obtain our results by combining the modular method, modular transgression and some careful analysis of odd Chern classes for cocycles in odd K -theory. Moreover we discover that in odd dimensions, the fundamental group of manifolds plays an important role in the rigidity.

Contents

1	Introduction	1
2	Elliptic genera on odd dimensional manifolds	6
2.1	Some Characteristic Forms	6
2.2	Modularities of odd Chern character forms	9
2.3	Elliptic genera in odd dimensions	12
3	Witten rigidity on odd dimensional manifolds	14
3.1	An S^1 -equivariant index theorem for Toeplitz operators . . .	14
3.2	Witten rigidity in odd dimensions	16
3.3	A proof of Theorem 3.3	24

1 Introduction

Let M be a closed smooth manifold and P be a Fredholm operator on M . We assume that a compact connected Lie group G acts on M nontrivially and

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that P is G -equivariant, by which we mean it commutes with the G action. Then the kernel and cokernel of P are finite dimensional representations of G . The equivariant index of P is the virtual character of G defined by

$$\text{Ind}(h, P) = \text{Tr} [h|_{\ker P}] - \text{Tr} [h|_{\text{coker } P}], \quad h \in G. \quad (1.1)$$

P is said to be *rigid* for this G action if $\text{Ind}(h, P)$ does not depend on $h \in G$. Furthermore, we say that P has *vanishing property* if $\text{Ind}(h, P)$ is identically zero. To study rigidity and vanishing properties of Fredholm operators, we only need to restrict to the case when $G = S^1$.

It is well known that classical operators: the signature operator for oriented manifolds, the Dolbeault operator for almost complex manifolds and the Dirac operator for spin manifolds are rigid [2]. In [30], Witten derived a series of twisted Dirac operators on the free loop space LM of a spin manifold M . The elliptic genus constructed by Landweber-Stong [17] and Ochanine [27] in a topological way turns out to be the index of one of these operators. Motivated by physics, Witten conjectured that these elliptic operators should be rigid. In particular, as a highly nontrivial consequence, the twisted operator

$$D \otimes T_{\mathbb{C}}M$$

which is known as the *Rarita-Schwinger operator* [29] is rigid. We refer to [16] for a brief early history of the subject.

The Witten conjecture was first proved by Taubes [28] and Bott-Taubes [4]. Hirzebruch [13] and Krichever [15] proved Witten's conjecture for almost complex manifold case. In [20, 21], using the modular invariance property, Liu presented a simple and unified proof as well as various vast generalizations of the Witten conjecture. Several new vanishing theorems were also established in [20, 21]. In [7], Dessai established the rigidity and vanishing theorems for spin^c case. Liu-Ma [22, 23] and Liu-Ma-Zhang [24, 25] generalized the rigidity and vanishing theorems to the family case on the levels of equivariant Chern character and of equivariant K -theory. However, since Dirac operators on odd dimensional manifolds are self adjoint and therefore have index zero, the rigidity and vanishing properties for twisted Dirac operators make sense only for even dimensional manifolds.

Now let M be an odd dimensional closed smooth spin Riemannian manifold. The appropriate index to consider on M is that of twisted Toeplitz operators ([3, 11]), which gives the natural pairing between odd K -homology and odd K -group. Thus it fits with the interpretation of the index of twisted Dirac operator on even dimensional manifolds as a pairing between even K -homology and even K -group. An element of $K^{-1}(M)$ can be represented

by a differentiable map from M into the general linear group

$$g : M \longrightarrow \mathrm{GL}(N, \mathbb{C}),$$

where N is a positive integer. We recall the definition of Toeplitz operators as follows.

Let $\Delta(TM)$ be the Hermitian bundle of spinors and \mathcal{E} be a complex Hermitian vector bundle carrying a Hermitian connection $\nabla^{\mathcal{E}}$ over M . The twisted Dirac operator $D \otimes \mathcal{E}$ induces a splitting of $L^2(M, \Delta(TM) \otimes \mathcal{E})$, the L^2 -completion of the space $\Gamma(M, \Delta(TM) \otimes \mathcal{E})$ of smooth sections of $\Delta(TM) \otimes \mathcal{E}$ over M , into an orthogonal direct sum as

$$L^2(M, \Delta(TM) \otimes \mathcal{E}) = \bigoplus_{\lambda \in \mathrm{Spec}(D \otimes \mathcal{E})} \mathcal{E}_\lambda,$$

where \mathcal{E}_λ is the eigenspace associated to the eigenvalue λ of $D \otimes \mathcal{E}$. Set

$$L^2_+(M, \Delta(TM) \otimes \mathcal{E}) = \bigoplus_{\lambda \geq 0} \mathcal{E}_\lambda,$$

and denote by P_+ the orthogonal projection from $L^2(M, \Delta(TM) \otimes \mathcal{E})$ to $L^2_+(M, \Delta(TM) \otimes \mathcal{E})$.

Now consider the trivial vector bundle $\mathbb{C}^N|_M$ of rank N over M . We equip $\mathbb{C}^N|_M$ with the canonical trivial metric and connection. Then P_+ extends naturally to an orthogonal projection

$$L^2(M, \Delta(TM) \otimes \mathcal{E} \otimes \mathbb{C}^N|_M) \longrightarrow L^2_+(M, \Delta(TM) \otimes \mathcal{E} \otimes \mathbb{C}^N|_M)$$

by acting as identity on $\mathbb{C}^N|_M$. We still denote this extension by P_+ .

The map g can be interpreted as an automorphism of $\mathbb{C}^N|_M$. Moreover, g extends naturally to a bounded map from

$$L^2(M, \Delta(TM) \otimes \mathcal{E} \otimes \mathbb{C}^N|_M)$$

to itself by acting as identity on $L^2(M, \Delta(TM) \otimes \mathcal{E})$. We still denote this extended map by g .

With the above data given, the twisted Toeplitz operator associated to $D \otimes \mathcal{E}$ and g can be defined as

$$\begin{aligned} \mathcal{T} \otimes \mathcal{E} \otimes (\mathbb{C}^N|_M, g) &:= P_+ g P_+ : \\ L^2_+(M, \Delta(TM) \otimes \mathcal{E} \otimes \mathbb{C}^N|_M) &\longrightarrow L^2_+(M, \Delta(TM) \otimes \mathcal{E} \otimes \mathbb{C}^N|_M). \end{aligned} \tag{1.2}$$

The important fact is that $\mathcal{T} \otimes \mathcal{E} \otimes (\mathbb{C}^N|_M, g)$ is a Fredholm operator. Moreover, we can compute its index as follows (see [3, 11]),

$$\begin{aligned} & \text{Ind}(\mathcal{T} \otimes \mathcal{E} \otimes (\mathbb{C}^N|_M, g)) \\ &= -\left\langle \widehat{A}(TM, \nabla^{TM}) \text{ch}(\mathcal{E}, \nabla^{\mathcal{E}}) \text{ch}(\mathbb{C}^N|_M, g, d), [M] \right\rangle, \end{aligned} \quad (1.3)$$

where $[M]$ denotes the fundamental class of M . See Section 2.1 for the relevant characteristic forms.

In [26], Liu-Wang for the first time study the rigidity and vanishing properties of Toeplitz operators. They obtained a very interesting result [26, Theorem 2.4] which states that

$$\mathcal{T} \otimes (\mathbb{C}^N|_M, g)$$

is rigid and this can be thought of as an odd analogue of Atiyah-Hirzebruch theorem [2]. Furthermore, they established the rigidity for twisted Toeplitz operators associated to the Witten bundles by Liu's approach [20, 21] under the assumption that the fixed point sets of the group action are 1-dimensional.

The purpose of the present paper is to extend their rigidity and vanishing properties for twisted Toeplitz operators to the cases of fixed points of general dimensions. Let

$$\Theta_2(TM) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(T_{\mathbb{C}}M), \quad (1.4)$$

$$\Theta_3(TM) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(T_{\mathbb{C}}M). \quad (1.5)$$

We obtain the following result (see the proof after Corollary 3.7).

Theorem 1.1. *Let M be an odd dimensional smooth closed spin Riemannian manifold with a nontrivial circle action. Let $g : M \rightarrow \text{SO}(N)$ be an S^1 -invariant cocycle in the odd real K -theory of M . Suppose M is simply connected and $H^3(M, \mathbb{R}) = 0$. Then the Toeplitz-Witten operators*

$$\mathcal{T} \otimes \Theta_2(TM) \otimes (\mathbb{C}^N|_M, g), \quad \mathcal{T} \otimes \Theta_3(TM) \otimes (\mathbb{C}^N|_M, g)$$

are rigid. In particular, the Toeplitz-Rarita-Schwinger operator

$$\mathcal{T} \otimes T_{\mathbb{C}}M \otimes (\mathbb{C}^N|_M, g)$$

is rigid.

We would like to point out that if we extend $D \otimes \mathcal{E}$ as an operator from

$$\Gamma(M, \Delta(TM) \otimes \mathcal{E} \otimes \mathbb{C}^N|_M)$$

to itself by acting as identity on $\mathbb{C}^N|_M$, the equivariant index

$$\text{Ind}(h, \mathcal{T} \otimes \mathcal{E} \otimes (\mathbb{C}^N|_M, g)), \quad h \in S^1$$

computes the *equivariant spectral flow* (see [10, Theorem 2.7])

$$\text{sf}(h, D \otimes \mathcal{E}, g(D \otimes \mathcal{E})g^{-1})$$

for the path

$$(1 - u)D \otimes \mathcal{E} + ug(D \otimes \mathcal{E})g^{-1}, \quad u \in [0, 1].$$

Hence when M is simply connected and $H^3(M, \mathbb{R}) = 0$, the equivariant spectral flows

$$\text{sf}\left(h, D \otimes \Theta_2(TM), g(D \otimes \Theta_2(TM))g^{-1}\right)$$

and

$$\text{sf}\left(h, D \otimes \Theta_3(TM), g(D \otimes \Theta_3(TM))g^{-1}\right)$$

do not depend on $h \in S^1$. In particular, the equivariant spectral flow

$$\text{sf}\left(h, D \otimes T_{\mathbb{C}}M, g(D \otimes T_{\mathbb{C}}M)g^{-1}\right)$$

for the Rarita-Schwinger operators does not depend on $h \in S^1$.

Actually we have obtained more general results; see Theorems 3.3, 3.4 and Corollaries 3.7, 3.8. Our approach is to combine Liu's modularity methods [20, 21], the modular transgression in [6] and some careful analysis of Chern classes for cocycles in odd K -theory. Also, parallel to [26, Theorem 2.4], we are able to show the rigidity of the *Toeplitz-Signature operator*

$$\mathcal{T} \otimes \Delta(TM) \otimes (\mathbb{C}^N|_M, g)$$

in Theorem 3.2 without assuming the simply connectedness of M and the vanishing of $H^3(M, \mathbb{R})$. Our results should have applications to the study of topology of odd dimensional manifolds.

A brief outline of the paper is as follows. In Section 2, we first review some important characteristic forms and then study modularities of odd Chern character forms. We then introduce elliptic genera and Witten genera

for the pair $(M, [g])$ on odd dimensional manifolds as well as relate them to indices of Toeplitz operators. Section 3 is devoted to the study of rigidity of the twisted Toeplitz operators.

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2 Elliptic genera on odd dimensional manifolds

2.1 Some Characteristic Forms

Let M be a closed smooth Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM} . Let $\widehat{A}(TM, \nabla^{TM})$ and $\widehat{L}(TM, \nabla^{TM})$ be the characteristic forms defined, respectively, by (c.f. [31, Section 1.6], [18, Section III.11])

$$\begin{aligned}\widehat{A}(TM, \nabla^{TM}) &= \det^{1/2} \left(\frac{\sqrt{-1}R^{TM}/(4\pi)}{\sinh(\sqrt{-1}R^{TM}/(4\pi))} \right), \\ \widehat{L}(TM, \nabla^{TM}) &= \det^{1/2} \left(\frac{\sqrt{-1}R^{TM}/(4\pi)}{\tanh(\sqrt{-1}R^{TM}/(4\pi))} \right).\end{aligned}\tag{2.1}$$

Let W, W' be two complex vector bundles over M carrying connections $\nabla^W, \nabla^{W'}$, respectively. Then the formal difference $W - W'$ carries a naturally induced connection which we denote by ∇^\ominus . We recall that the Chern character form associated to $(W - W', \nabla^\ominus)$ is (see [31, Section 1.6])

$$\text{ch}(W - W', \nabla^\ominus) = \text{Tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^W \right) \right] - \text{Tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^{W'} \right) \right], \tag{2.2}$$

where R^W and $R^{W'}$ denote the curvatures of ∇^W and $\nabla^{W'}$ respectively.

For any $t \in \mathbb{C}$, let

$$\Lambda_t(W) = \sum_{i=0}^{\infty} t^i \Lambda^i(W), \quad S_t(W) = \sum_{i=0}^{\infty} t^i S^i(W) \tag{2.3}$$

denote, respectively, the total exterior and symmetric powers of W , which live in $K(M)[[t]]$. The following relations between these two operations hold

(see [1, Chapter 3]),

$$S_t(W) = \frac{1}{\Lambda_{-t}(W)}, \quad \Lambda_t(W - W') = \frac{\Lambda_t(W)}{\Lambda_t(W')}. \quad (2.4)$$

For a real Euclidean vector bundle V over M , we denote by $V_{\mathbb{C}}$ the complexification of V , which carries a naturally induced Hermitian metric. Set

$$\widetilde{V}_{\mathbb{C}} = V_{\mathbb{C}} - \mathbb{C}^{\dim V} \in K(M).$$

If V carries a spin structure, we denote by $\Delta(V)$ the Hermitian bundle of spinors associated to V .

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$, the upper half complex plane. Set

$$\begin{aligned} Q_1(V)_v &= \Delta(V) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{V}_{\mathbb{C}}), \\ Q_2(V)_v &= \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(\widetilde{V}_{\mathbb{C}}), \quad Q_3(V)_v = \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(\widetilde{V}_{\mathbb{C}}). \end{aligned} \quad (2.5)$$

Let ∇^V be an Euclidean connection on V , which canonically induces Hermitian connections on the coefficients of the formal Fourier expansions of $Q_j(V)_v$, $j = 1, 2, 3$, in $q^{1/2}$. Thus we get induced connections $\nabla^{Q_j(V)_v}$ with $q^{1/2}$ -coefficients on $Q_j(V)_v$, $j = 1, 2, 3$.

To express the Chern character forms of $(Q_j(V)_v, \nabla^{Q_j(V)_v})$ explicitly, we recall the four Jacobi theta functions as follows (see [5]),

$$\theta(v, \tau) = 2c(q)q^{1/8} \sin(\pi v) \prod_{n=1}^{\infty} (1 - q^n e^{2\pi\sqrt{-1}v}) \prod_{n=1}^{\infty} (1 - q^n e^{-2\pi\sqrt{-1}v}), \quad (2.6)$$

$$\theta_1(v, \tau) = 2c(q)q^{1/8} \cos(\pi v) \prod_{n=1}^{\infty} (1 + q^n e^{2\pi\sqrt{-1}v}) \prod_{n=1}^{\infty} (1 + q^n e^{-2\pi\sqrt{-1}v}), \quad (2.7)$$

$$\theta_2(v, \tau) = c(q) \prod_{n=1}^{\infty} (1 - q^{n-1/2} e^{2\pi\sqrt{-1}v}) \prod_{n=1}^{\infty} (1 - q^{n-1/2} e^{-2\pi\sqrt{-1}v}), \quad (2.8)$$

$$\theta_3(v, \tau) = c(q) \prod_{n=1}^{\infty} (1 + q^{n-1/2} e^{2\pi\sqrt{-1}v}) \prod_{n=1}^{\infty} (1 + q^{n-1/2} e^{-2\pi\sqrt{-1}v}), \quad (2.9)$$

where $c(q) = \prod_{n=1}^{\infty} (1 - q^n)$. By using the Chern roots algorithm as in [6,

Section 3], we obtain the following formulas,

$$\begin{aligned} \text{ch}(Q_1(V)_v, \nabla^{Q_1(V)_v}, \tau) &= \det^{1/2} \left(\frac{2\theta_1(R^V/(4\pi^2), \tau)}{\theta_1(0, \tau)} \right), \\ \text{ch}(Q_j(V)_v, \nabla^{Q_j(V)_v}, \tau) &= \det^{1/2} \left(\frac{\theta_j(R^V/(4\pi^2), \tau)}{\theta_j(0, \tau)} \right), \quad j = 2, 3, \end{aligned} \quad (2.10)$$

where R^V denotes the curvature of ∇^V .

We recall characteristic forms for odd K -theory ([11], cf. [31]). Let

$$g : M \longrightarrow \text{GL}(N, \mathbb{C})$$

be a smooth map from M to the general linear group $\text{GL}(N, \mathbb{C})$ with N a positive integer, and let $\mathbb{C}^N|_M$ denote the trivial complex vector bundle of rank N over M . Then g can be viewed as an automorphism of $\mathbb{C}^N|_M$.

Let d denote a trivial connection on $\mathbb{C}^N|_M$, and we associate with g a natural element

$$g^{-1}dg = g^{-1} \cdot d \cdot g - d \in \Omega^1(M, \text{End}(\mathbb{C}^N|_M)).$$

Then $\text{Tr} [(g^{-1}dg)^n]$ is closed for any positive odd integer n . Moreover, the cohomology class determined by $\text{Tr} [(g^{-1}dg)^n]$ in $H^{\text{odd}}(M, \mathbb{C})$ depends only on the homotopy class $[g]$ of g . We will denote by $c_n(M, [g])$ the cohomology class associated to the closed n -form

$$c_n(\mathbb{C}^N|_M, g, d) = \left(\frac{1}{2\pi\sqrt{-1}} \right)^{(n+1)/2} \text{Tr} [(g^{-1}dg)^n]. \quad (2.11)$$

The odd Chern character form $\text{ch}(\mathbb{C}^N|_M, g, d)$ associated to g and d by definition is

$$\text{ch}(\mathbb{C}^N|_M, g, d) = \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!} c_{2n+1}(\mathbb{C}^N|_M, g, d). \quad (2.12)$$

Alternatively, $\text{ch}(\mathbb{C}^N|_M, g, d)$ is exactly the Chern-Simons form associated to the curve

$$\nabla_u = (1-u)d + u g^{-1} \cdot d \cdot g = d + u g^{-1}dg, \quad u \in [0, 1] \quad (2.13)$$

of connections on $\mathbb{C}^N|_M$, which is such that (see [11, Section 1])

$$d \text{ch}(\mathbb{C}^N|_M, g, d) = \text{ch}(\mathbb{C}^N|_M, d) - \text{ch}(\mathbb{C}^N|_M, g^{-1} \cdot d \cdot g). \quad (2.14)$$

2.2 Modularities of odd Chern character forms

Let $g : M \rightarrow \mathrm{SO}(N)$ be a smooth map from M to the special orthogonal group $\mathrm{SO}(N)$ with N a positive integer. We assume that N is even and large enough. Otherwise, we replace g by $\begin{pmatrix} g & 0 \\ 0 & I \end{pmatrix}$ with I a certain identity matrix of large rank.

Let E denote the trivial real vector bundle of rank N over M . We equip E with the canonical trivial metric and trivial connection d which is clearly Euclidean. Set (cf. (2.13))

$$\nabla_u = d + u g^{-1} dg, \quad u \in [0, 1], \quad (2.15)$$

then ∇_u , $u \in [0, 1]$ defines a curve of Euclidean connections on E with $\nabla_0 = d$ and $\nabla_1 = g^{-1} \cdot d \cdot g$. Let R_u be the curvature of ∇_u , then

$$R_u = (u^2 - u)(g^{-1} dg)^2, \quad u \in [0, 1]. \quad (2.16)$$

By complexification, the metric and the trivial connection on E induce naturally a Hermitian metric and a trivial Hermitian connection on $E_{\mathbb{C}}$. Also, g extends to a unitary automorphism of $E_{\mathbb{C}}$, due to the obvious embedding $\mathrm{SO}(N) \subset \mathrm{U}(N)$, the unitary group. Hence, (2.15) extends naturally to Hermitian connections on $E_{\mathbb{C}}$ with curvatures still given by (2.16).

Let $\Delta(E)$ be the spinor bundle of E , which is a trivial Hermitian bundle of rank $2^{N/2}$.

Let $\pi_1(\cdot)$ denote the fundamental group of a topological space. We assume that the induced map

$$g_* : \pi_1(M) \longrightarrow \pi_1(\mathrm{SO}(N)) = \mathbb{Z}_2$$

is trivial. Then by [12, Chapter 1], there exists a unique lift (depending on the choice of the base point) to the Spin group $\mathrm{Spin}(N)$,

$$g^\Delta : M \longrightarrow \mathrm{Spin}(N). \quad (2.17)$$

g^Δ can be viewed as an automorphism of $\Delta(E)$ preserving the Hermitian metric. If we lift d on E to be a trivial Hermitian connection d^Δ on $\Delta(E)$, then

$$\nabla_u^\Delta = (1 - u) d^\Delta + u (g^\Delta)^{-1} \cdot d^\Delta \cdot g^\Delta, \quad u \in [0, 1] \quad (2.18)$$

lift the connections in (2.15) on E to $\Delta(E)$.

Let $Q_j(E)_v$, $j = 1, 2, 3$ be the virtual bundles defined as in (2.5). Then the action of g on E naturally induces actions $g^{Q_j(E)_v}$ on $Q_j(E)_v$. As explained below (2.5), let $\nabla_0^{Q_j(E)_v}$ and $\nabla_1^{Q_j(E)_v}$ denote, respectively, the connections on $Q_j(E)_v$ induced by ∇_0 and ∇_1 . From (2.10) and [6, Theorem 2.2], we get that

$$\begin{aligned} & \text{ch}(Q_1(E)_v, \nabla_0^{Q_1(E)_v}, \tau) - \text{ch}(Q_1(E)_v, \nabla_1^{Q_1(E)_v}, \tau) \\ &= -d \int_0^1 \frac{1}{8\pi^2} \det^{1/2} \left(\frac{2\theta_1(R_u/(4\pi^2), \tau)}{\theta_1(0, \tau)} \right) \text{Tr} \left[g^{-1} dg \frac{\theta'_1(R_u/(4\pi^2), \tau)}{\theta_1(R_u/(4\pi^2), \tau)} \right] du \end{aligned}$$

and that for $j = 2, 3$ (compare with [6, (5.4)-(5.5)]),

$$\begin{aligned} & \text{ch}(Q_j(E)_v, \nabla_0^{Q_j(E)_v}, \tau) - \text{ch}(Q_j(E)_v, \nabla_1^{Q_j(E)_v}, \tau) \\ &= -d \int_0^1 \frac{1}{8\pi^2} \det^{1/2} \left(\frac{\theta_j(R_u/(4\pi^2), \tau)}{\theta_j(0, \tau)} \right) \text{Tr} \left[g^{-1} dg \frac{\theta'_j(R_u/(4\pi^2), \tau)}{\theta_j(R_u/(4\pi^2), \tau)} \right] du. \end{aligned}$$

However, since $\text{Tr} [(g^{-1}dg)^n]$ vanishes for any positive even integer n (see [31, (1.40)]), we deduce from (2.16) that for $j = 1, 2, 3$,

$$\det^{1/2} \left(\frac{\theta_j(R_u/(4\pi^2), \tau)}{\theta_j(0, \tau)} \right) = \exp \left(\frac{1}{2} \text{Tr} \log \frac{\theta_j(R_u/(4\pi^2), \tau)}{\theta_j(0, \tau)} \right) = 1. \quad (2.19)$$

Therefore we have for $j = 1, 2, 3$,

$$\begin{aligned} & \text{ch}(Q_j(E)_v, \nabla_0^{Q_j(E)_v}, \tau) - \text{ch}(Q_j(E)_v, \nabla_1^{Q_j(E)_v}, \tau) \\ &= d \text{ch}(Q_j(E)_v, g^{Q_j(E)_v}, d, \tau), \end{aligned}$$

where

$$\begin{aligned} & \text{ch}(Q_1(E)_v, g^{Q_1(E)_v}, d, \tau) \\ &= -\frac{2^{N/2}}{8\pi^2} \int_0^1 \text{Tr} \left[g^{-1} dg \frac{\theta'_1(R_u/(4\pi^2), \tau)}{\theta_1(R_u/(4\pi^2), \tau)} \right] du. \end{aligned} \quad (2.20)$$

and

$$\begin{aligned} & \text{ch}(Q_j(E)_v, g^{Q_j(E)_v}, d, \tau) \\ &= -\frac{1}{8\pi^2} \int_0^1 \text{Tr} \left[g^{-1} dg \frac{\theta'_j(R_u/(4\pi^2), \tau)}{\theta_j(R_u/(4\pi^2), \tau)} \right] du, \quad j = 2, 3. \end{aligned} \quad (2.21)$$

As explained in [31, Section 1.8],

$$\text{ch}(Q_j(E)_v, g^{Q_j(E)_v}, d, \tau), \quad j = 1, 2, 3$$

are closed, and each determines an element in $H^{4i-1}(M, \mathbb{C})[[q^{1/2}]]$ depending only on the homotopy class $[g]$.

Let

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

as usual be the modular group, and

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{2} \right\}, \quad (2.22)$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod{2} \right\}, \quad (2.23)$$

$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2} \right\}. \quad (2.24)$$

be the three modular subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbb{H} \cup \{\infty\}$ such that for any

$$\mathcal{G} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

the following property holds,

$$f(\mathcal{G}\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(\mathcal{G})(c\tau + d)^k f(\tau),$$

where $\chi : \Gamma \rightarrow \mathbb{C}^*$ is a character of Γ and k is called the weight of f .

If ω is a differential form on M , we denote by $\omega^{(i)}$ the degree i component of ω .

Proposition 2.2 (Compare with [6, Theorem 5.1]). *For any integer $i \geq 2$,*

$$\{\mathrm{ch}(Q_j(E)_v, g^{Q_j(E)_v}, d, \tau)\}^{(4i-1)}, \quad j = 1, 2, 3$$

are modular forms of weight $2i$ over $\Gamma_0(2)$, $\Gamma^0(2)$ and Γ_θ , respectively.

Proof. Using the transformation formulas [6, (3.29)-(3.31),(4.6)], we deduce directly from (2.20) and (2.21) that

$$\begin{aligned} \mathrm{ch}(Q_1(E)_v, g^{Q_1(E)_v}, d, \tau + 1) &= \mathrm{ch}(Q_1(E)_v, g^{Q_1(E)_v}, d, \tau), \\ \mathrm{ch}(Q_2(E)_v, g^{Q_2(E)_v}, d, \tau + 1) &= \mathrm{ch}(Q_3(E)_v, g^{Q_3(E)_v}, d, \tau). \end{aligned} \quad (2.25)$$

and that for any integer $i \geq 1$,

$$\begin{aligned} & \left\{ \text{ch} \left(Q_1(E)_v, g^{Q_1(E)_v}, d, -\frac{1}{\tau} \right) \right\}^{(4i-1)} \\ &= 2^{N/2} \left\{ \tau^{2i} \text{ch}(Q_2(E)_v, g^{Q_2(E)_v}, d, \tau) - \frac{\tau\sqrt{-1}}{24\pi} c_3(E_{\mathbb{C}}, g, d) \right\}^{(4i-1)}, \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} & \left\{ \text{ch} \left(Q_3(E)_v, g^{Q_3(E)_v}, d, -\frac{1}{\tau} \right) \right\}^{(4i-1)} \\ &= \left\{ \tau^{2i} \text{ch}(Q_3(E)_v, g^{Q_3(E)_v}, d, \tau) - \frac{\tau\sqrt{-1}}{24\pi} c_3(E_{\mathbb{C}}, g, d) \right\}^{(4i-1)}. \end{aligned} \quad (2.27)$$

Recall that the generators of $\Gamma_0(2)$ are T, ST^2ST , the generators of $\Gamma^0(2)$ are STS, T^2STS and the generators of Γ_θ are S, T^2 (see [5]), where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.28)$$

Due to the above fact, the proposition now follows from (2.25)-(2.27) in a standard way (see, e.g., [6, (4.13)]). \square

2.3 Elliptic genera in odd dimensions

Set (see [30, 19])

$$\Theta_1(TM)_v = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{T_{\mathbb{C}}M}), \quad (2.29)$$

$$\Theta_2(TM)_v = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(\widetilde{T_{\mathbb{C}}M}), \quad (2.30)$$

$$\Theta_3(TM)_v = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(\widetilde{T_{\mathbb{C}}M}), \quad (2.31)$$

$$\Theta(TM)_v = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}). \quad (2.32)$$

We introduce the odd analogues of the Landweber-Stong forms and the Witten forms as follows.

Definition 2.3 (Compare with [6, Definition 3.1]). We call

$$\begin{aligned} \Phi_L(\nabla^{TM}, g, d, \tau) &= 2^{(\dim M - 1)/2} \widehat{L}(TM, \nabla^{TM}) \\ &\cdot \text{ch}(\Theta_1(TM)_v, \nabla^{\Theta_1(TM)_v}, \tau) \cdot \text{ch}(Q_1(E)_v, g^{Q_1(E)_v}, d, \tau) \end{aligned} \quad (2.33)$$

the Landweber-Stong type form of M associated to ∇^{TM} , d and g . Also, we call

$$\begin{aligned} \Phi_{\mathbb{W}}(\nabla^{TM}, g, d, \tau) &= \widehat{A}(TM, \nabla^{TM}) \text{ch}(\Theta_2(TM)_v, \nabla^{\Theta_2(TM)_v}, \tau) \\ &\quad \cdot \text{ch}(Q_2(E)_v, g^{Q_2(E)_v}, d, \tau), \end{aligned} \quad (2.34)$$

$$\begin{aligned} \Phi'_{\mathbb{W}}(\nabla^{TM}, g, d, \tau) &= \widehat{A}(TM, \nabla^{TM}) \text{ch}(\Theta_3(TM)_v, \nabla^{\Theta_3(TM)_v}, \tau) \\ &\quad \cdot \text{ch}(Q_3(E)_v, g^{Q_3(E)_v}, d, \tau), \end{aligned} \quad (2.35)$$

$$\begin{aligned} \Psi_{\mathbb{W},j}(\nabla^{TM}, g, d, \tau) &= \widehat{A}(TM, \nabla^{TM}) \text{ch}(\Theta(TM)_v, \nabla^{\Theta(TM)_v}, \tau) \\ &\quad \cdot \text{ch}(Q_j(E)_v, g^{Q_j(E)_v}, d, \tau), \quad j = 1, 2, 3 \end{aligned} \quad (2.36)$$

the Witten type forms of M associated to ∇^{TM} , d and g .

Applying the Chern-Weil theory, we can express the Landweber-Stong type forms and the Witten type forms in terms of theta functions and curvatures as in [6, Proposition 3.1] (see also [21, 19, 20]).

Combining Proposition 2.2 and [6, Proposition 3.2], we establish the following modularities.

Proposition 2.4 (Compare with [6, Proposition 3.2]).

(i) If $c_3(E_{\mathbb{C}}, g, d) = 0$, then for any integer $i \geq 1$,

$$\{\Phi_{\mathbb{L}}(\nabla^{TM}, g, d, \tau)\}^{(4i-1)}, \quad \{\Phi_{\mathbb{W}}(\nabla^{TM}, g, d, \tau)\}^{(4i-1)}$$

and $\{\Phi'_{\mathbb{W}}(\nabla^{TM}, g, d, \tau)\}^{(4i-1)}$ are modular forms of weight $2i$ over $\Gamma_0(2)$, $\Gamma^0(2)$ and Γ_{θ} respectively.

(ii) If $c_3(E_{\mathbb{C}}, g, d) = 0$ and the first Pontryagin form $p_1(TM, \nabla^{TM}) = 0$, then for any integer $i \geq 1$,

$$\{\Psi_{\mathbb{W},j}(\nabla^{TM}, g, d, \tau)\}^{(4i-1)}, \quad j = 1, 2, 3$$

are modular forms of weight $2i$ over $\Gamma_0(2)$, $\Gamma^0(2)$ and Γ_{θ} respectively.

We now assume that M is a $(4k-1)$ -dimensional closed oriented smooth manifold. Let $[M]$ be the fundamental class of M . Set

$$\begin{aligned} \phi_{\mathbb{L}}(M, [g], \tau) &= \langle \Phi_{\mathbb{L}}(\nabla^{TM}, g, d, \tau), [M] \rangle, \\ \phi_{\mathbb{W}}(M, [g], \tau) &= \langle \Phi_{\mathbb{W}}(\nabla^{TM}, g, d, \tau), [M] \rangle, \\ \phi'_{\mathbb{W}}(M, [g], \tau) &= \langle \Phi'_{\mathbb{W}}(\nabla^{TM}, g, d, \tau), [M] \rangle, \\ \psi_{\mathbb{W},j}(M, [g], \tau) &= \langle \Psi_{\mathbb{W},j}(\nabla^{TM}, g, d, \tau), [M] \rangle, \quad j = 1, 2, 3. \end{aligned}$$

Definition 2.5. We call $\phi_L(M, [g], \tau)$, $\phi_W(M, [g], \tau)$ and $\phi'_W(M, [g], \tau)$ the *elliptic genera of the pair* $(M, [g])$, and call $\phi_W(M, [g], \tau)$, $\phi'_W(M, [g], \tau)$, $\psi_{W,j}(M, [g], \tau)$, $j = 1, 2, 3$ the *Witten genera of the pair* $(M, [g])$.

Theorem 2.6. Assume $c_3(M, [g]) = 0$. We have
(i) the elliptic genera of the pair $(M, [g])$

$$\phi_L(M, [g], \tau), \quad \phi_W(M, [g], \tau) \quad \text{and} \quad \phi'_W(M, [g], \tau)$$

are modular forms of weight $(\dim M + 1)/2$ over $\Gamma_0(2)$, $\Gamma^0(2)$, and Γ_θ , respectively;

(ii) if the first Pontryagin class $p_1(M) = 0$, then the Witten genera of the pair $(M, [g])$

$$\psi_{W,j}(M, [g], \tau), \quad j = 1, 2, 3$$

are modular forms of weight $(\dim M + 1)/2$ over $\Gamma_0(2)$, $\Gamma^0(2)$, and Γ_θ , respectively.

Assume that M is spin. Then by the index formula (1.3), we can interpret the above elliptic genera and Witten genera of the pair $(M, [g])$ on $(4k - 1)$ -dimensional manifolds analytically as the indices of the twisted Toeplitz operators as follows,

$$\begin{aligned} \phi_L(M, [g], \tau) &= -\text{Ind}(\mathcal{T} \otimes \Delta(TM) \otimes \Theta_1(TM)_v \otimes (Q_1(E)_v, g^{Q_1(E)_v})), \\ \phi_W(M, [g], \tau) &= -\text{Ind}(\mathcal{T} \otimes \Theta_2(TM)_v \otimes (Q_2(E)_v, g^{Q_2(E)_v})), \\ \phi'_W(M, [g], \tau) &= -\text{Ind}(\mathcal{T} \otimes \Theta_3(TM)_v \otimes (Q_3(E)_v, g^{Q_3(E)_v})), \\ \psi_{W,j}(M, [g], \tau) &= -\text{Ind}(\mathcal{T} \otimes \Theta(TM)_v \otimes (Q_j(E)_v, g^{Q_j(E)_v})), \quad j = 1, 2, 3. \end{aligned}$$

3 Witten rigidity on odd dimensional manifolds

3.1 An S^1 -equivariant index theorem for Toeplitz operators

Let M be an odd dimensional closed smooth spin Riemannian manifold which admits a circle action. Without loss of generality, we may assume that S^1 acts on M isometrically and preserves the spin structure of M .

Let \mathcal{E} be an S^1 -equivariant complex vector bundle over M carrying an S^1 -invariant Hermitian connection. Then the associated twisted Dirac operator $D \otimes \mathcal{E}$ is S^1 -equivariant, which implies that the corresponding orthogonal projection P_+ is also S^1 -equivariant.

In addition, we assume $g : M \rightarrow \text{GL}(N, \mathbb{C})$ is S^1 -invariant, i.e.,

$$g(hx) = g(x), \quad \text{for any } h \in S^1 \text{ and } x \in M. \quad (3.1)$$

Thus the twisted Toeplitz operator $\mathcal{T} \otimes \mathcal{E} \otimes (\mathbb{C}^N|_M, g)$ is S^1 -equivariant.

Let M^{S^1} denote the fixed submanifold of the circle action on M . In general, M^{S^1} is not connected. We fix a connected component $M_\alpha^{S^1}$ of M^{S^1} , and omit the subscript α if there is no confusion.

Let N denote the normal bundle to M^{S^1} in M , which can be identified as the orthogonal complement of TM^{S^1} in $TM|_{M^{S^1}}$. Then we have the following S^1 -equivariant decomposition when restricted upon M^{S^1} ,

$$TM|_{M^{S^1}} = N_{m_1} \oplus \cdots \oplus N_{m_l} \oplus TM^{S^1}, \quad (3.2)$$

where each N_γ , $\gamma = m_1, \dots, m_l$, is a complex vector bundle such that $h \in S^1$ acts on it by h^γ . To simplify the notation, we will write that

$$TM|_{M^{S^1}} = \bigoplus_{\gamma \neq 0} N_\gamma \oplus TM^{S^1}, \quad (3.3)$$

where N_γ is a complex vector bundle such that $h \in S^1$ acts on it by h^γ with $\gamma \in \mathbb{Z} \setminus \{0\}$. Clearly, $N = \bigoplus_{\gamma \neq 0} N_\gamma$. From now on, we will regard N as a complex vector bundle. Let

$$2\pi\sqrt{-1}x_\gamma^j, \quad j = 1, \dots, \dim N_\gamma$$

be the Chern roots of N_γ . Let

$$\pm 2\pi\sqrt{-1}y_j, \quad j = 1, \dots, (\dim M^{S^1} - 1)/2$$

be the Chern roots of $TM^{S^1} \otimes \mathbb{C}$.

Similarly, let

$$\mathcal{E}|_{M^{S^1}} = \bigoplus_{\nu} \mathcal{E}_\nu \quad (3.4)$$

be the S^1 -equivariant decomposition of the restrictions of \mathcal{E} over M^{S^1} , where \mathcal{E}_ν is a complex vector bundle such that $h \in S^1$ acts on it by h^ν with $\nu \in \mathbb{Z}$. We denote by

$$2\pi\sqrt{-1}w_\nu^j, \quad j = 1, \dots, \dim \mathcal{E}_\nu$$

the Chern roots of \mathcal{E}_ν .

For $f(\cdot)$ a holomorphic function, we denote by $f(y)(TM^{S^1}) = \prod_j f(y_j)$ the symmetric polynomial that gives characteristic class of TM^{S^1} , and we use the same notation for N_γ .

The following equivariant index formula is an immediate consequence of the odd equivariant index theorem for Toeplitz operators of Fang [10, Theorem 4.3] and Liu-Wang [26, Theorem 2.3].

Proposition 3.1 ([26, (2.5)]). *Let $h = e^{2\pi\sqrt{-1}t}$, $t \in [0, 1] \setminus \mathbb{Q}$ be a topological generator of S^1 . Then*

$$\begin{aligned} \text{Ind}(h, \mathcal{T} \otimes \mathcal{E} \otimes (\mathbb{C}^N|_M, g)) &= - \left\langle \text{ch}(\mathbb{C}^N|_M, g, d) \frac{\pi y}{\sin(\pi y)} (TM^{S^1}) \right. \\ &\cdot \prod_{\gamma} \frac{1}{2\sqrt{-1} \sin \pi(x_{\gamma} + \gamma t)} (N_{\gamma}) \cdot \sum_{\nu} \sum_j e^{2\pi\sqrt{-1}(w_{\nu}^j + \nu t)}, [M^{S^1}] \left. \right\rangle, \end{aligned} \quad (3.5)$$

where $[M^{S^1}]$ is the fundamental class of M^{S^1} which carries the orientation compatible with that of M and N .

As a direct application of Proposition 3.1, we deduce an odd analogue of the rigidity of the signature operator.

Theorem 3.2. $\mathcal{T} \otimes \Delta(TM) \otimes (\mathbb{C}^N|_M, g)$ is rigid.

Proof. For $z \in \mathbb{C}$, set

$$\begin{aligned} f(z) &= -2^{(\dim M^{S^1} - 1)/2} \left\langle \text{ch}(\mathbb{C}^N|_M, g, d) \frac{\pi y}{\tan(\pi y)} (TM^{S^1}) \right. \\ &\cdot \prod_{\gamma} \frac{z^{\gamma} e^{\pi\sqrt{-1}x_{\gamma}} + e^{-\pi\sqrt{-1}x_{\gamma}}}{z^{\gamma} e^{\pi\sqrt{-1}x_{\gamma}} - e^{-\pi\sqrt{-1}x_{\gamma}}} (N_{\gamma}), [M^{S^1}] \left. \right\rangle. \end{aligned} \quad (3.6)$$

Then $f(z)$ is a rational function and has no poles on $\mathbb{C} \setminus S^1$.

But by (3.3) and (3.5), we see that f coincides with the continuous function $\text{Ind}(z, \mathcal{T} \otimes \mathcal{E} \otimes (\mathbb{C}^N|_M, g))$ on the dense subset which consists of the topological generators of S^1 . Thus f must be bounded on S^1 .

Now $f(z)$ is constant on \mathbb{C} due to the fact that $\lim_{z \rightarrow \infty} f(z)$ exists. \square

3.2 Witten rigidity in odd dimensions

Let V be an S^1 -equivariant real spin vector bundle over M . Let $\Delta(V)$ be the corresponding spinor bundle.

Let $g : M \rightarrow \text{SO}(N)$ be an S^1 -invariant smooth map from M to $\text{SO}(N)$ with N a positive even integer large enough. Let E denote the trivial real vector bundle of rank N over M , which is equipped with the canonical trivial

metric and trivial connection d . Set (cf. (2.5))

$$\begin{aligned} Q_1(E) &= \Delta(E) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(E_{\mathbb{C}}), \\ Q_2(E) &= \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(E_{\mathbb{C}}), \quad Q_3(E) = \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(E_{\mathbb{C}}). \end{aligned} \quad (3.7)$$

Let $g^{Q_j(E)}$, $j = 1, 2, 3$ be the actions on $Q_j(E)$ respectively induced from the action of g on E .

Following [21], we introduce the following elements in $K(M)[[q^{1/2}]]$,

$$\Theta_1(TM|V) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(V_{\mathbb{C}}), \quad (3.8)$$

$$\Theta_2(TM|V) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(V_{\mathbb{C}}), \quad (3.9)$$

$$\Theta_3(TM|V) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(V_{\mathbb{C}}). \quad (3.10)$$

For simplicity, denote

$$\Theta_1(TM|TM), \quad \Theta_2(TM|TM) \quad \text{and} \quad \Theta_3(TM|TM)$$

by

$$\Theta_1(TM), \quad \Theta_2(TM) \quad \text{and} \quad \Theta_3(TM),$$

respectively.

If V is even dimensional, let $\Delta(V) = \Delta_+(V) \oplus \Delta_-(V)$ be the natural \mathbb{Z}_2 -grading. Set

$$\Theta(TM|V) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(V_{\mathbb{C}}). \quad (3.11)$$

Let $H_{S^1}^*(M, \mathbb{Z}) = H^*(M \times_{S^1} ES^1, \mathbb{Z})$ denote the S^1 -equivariant cohomology group of M , where ES^1 is the universal S^1 -principal bundle over the classifying space BS^1 of S^1 . So $H_{S^1}^*(M, \mathbb{Z})$ is a module over $H^*(BS^1, \mathbb{Z})$ induced by the projection $\pi : M \times_{S^1} ES^1 \rightarrow BS^1$. Recall that

$$H^*(BS^1, \mathbb{Z}) = \mathbb{Z}[[u]] \quad (3.12)$$

with u being a generator of degree 2.

The S^1 -equivariant characteristic class of an S^1 -bundle W over M by definition is the usual characteristic class of the bundle $W \times_{S^1} ES^1$ over $M \times_{S^1} ES^1$. Let $p_1(\cdot)_{S^1}$ denote the first S^1 -equivariant pontrjagin class.

We suppose that there exists some integer $n \in \mathbb{Z}$ such that

$$p_1(V)_{S^1} - p_1(TM)_{S^1} = n \cdot \pi^* u^2. \quad (3.13)$$

Following [20], we call n the anomaly to rigidity.

The following theorems generalize the Witten rigidity theorems and vanishing theorems [20, Corollary 3.1] to the case of odd dimensional manifolds. They are similar to [26, Theorem 2.5 and Corollary 3.6], while without putting restriction on the dimension of fixed point set. Instead we put some topological conditions on g .

Theorem 3.3. *Suppose $g_* = 1$, $c_3(M, [g]) = 0$ and (3.13) holds. Then we have*

(i) *If $n = 0$, then following S^1 -equivariant Toeplitz operators are rigid,*

$$\begin{aligned} & \mathcal{T} \otimes \Delta(V) \otimes \Theta_1(TM|V) \otimes (Q_1(E), g^{Q_1(E)}), \\ & \mathcal{T} \otimes \Theta_2(TM|V) \otimes (Q_2(E), g^{Q_2(E)}), \\ & \mathcal{T} \otimes \Theta_3(TM|V) \otimes (Q_3(E), g^{Q_3(E)}), \\ & \mathcal{T} \otimes (\Delta_+(V) - \Delta_-(V)) \otimes \Theta(TM|V) \otimes (Q_j(E), g^{Q_j(E)}), \quad j = 1, 2, 3. \end{aligned}$$

(ii) *If $n < 0$, then the equivariant indices of the above operators all vanish.*

Proof. Combining [9, Theorem 1.2] and Theorem 3.9, which will be given in the next subsection, we obtain the virtual version of the theorem. Since the equivariant index of the virtual case and that of the non-virtual one differ by a constant (depending on q), we complete the proof of Theorem 3.3. \square

Theorem 3.4. *Suppose $g_* = 1$, $c_3(M, [g]) = 0$ and (3.13) holds. Then we have*

(i) *If $n = 0$, then following S^1 -equivariant Toeplitz operators are rigid,*

$$\mathcal{T} \otimes \Theta_2(TM|V) \otimes (E_{\mathbb{C}}, g), \quad \mathcal{T} \otimes \Theta_3(TM|V) \otimes (E_{\mathbb{C}}, g).$$

(ii) *If $n < 0$, then the equivariant indices of the above operators vanish.*

Proof. Suppose $\Theta_2(TM|V)$ and $Q_2(E)$ admit formal Fourier expansion in $q^{1/2}$ as

$$\Theta_2(TM|V) = \sum_{j=0}^{\infty} A_j q^{j/2}, \quad Q_2(E) = \sum_{j=0}^{\infty} B_j q^{j/2}, \quad (3.14)$$

with A_j 's, B_j 's being elements in $K(M)$. In particular, we verify that

$$\begin{aligned} A_0 &= \mathbb{C}|_M, & A_1 &= -V_{\mathbb{C}}, & A_2 &= T_{\mathbb{C}}M \oplus \Lambda^2(V_{\mathbb{C}}), \\ B_0 &= \mathbb{C}|_M, & B_1 &= -E_{\mathbb{C}}, & B_2 &= \Lambda^2(E_{\mathbb{C}}). \end{aligned} \quad (3.15)$$

We will use the following convention for the sake of simplicity.

Convention We will say an S^1 -equivariant twisted Toeplitz operator P has *good property*, if P is rigid when $n = 0$, and if P has vanishing property when $n < 0$.

Applying Theorem 3.3 and picking up the $q^{\frac{j+1}{2}}$ -coefficient in the expansion of the operator

$$\mathcal{T} \otimes \Theta_2(TM|V) \otimes (Q_2(E), g^{Q_2(E)}),$$

we see that the equivariant Toeplitz operator

$$\mathcal{T} \otimes A_0 \otimes (B_{j+1}, g^{B_{j+1}}) \cdots + \mathcal{T} \otimes A_j \otimes (B_1, g^{B_1}) \quad (3.16)$$

has the good property.

First note that the operator $\mathcal{T} \otimes A_0 \otimes (B_j, g^{B_j})$ has the good property for all $j \geq 1$ due to the theorem of Liu-Wang [26, Theorem 2.4]. Here no conditions about Chern classes of (B_j, g^{B_j}) are needed.

From Theorem 3.3, we see that under the condition

$$g_* = 1, \quad c_3(M, [g]) = 0,$$

taking $j = 1$, the operator

$$\mathcal{T} \otimes A_0 \otimes (B_2, g^{B_2}) + \mathcal{T} \otimes A_1 \otimes (B_1, g^{B_1})$$

has the good property. Therefore, under the condition $g_* = 1, c_3(M, [g]) = 0$, the equivariant Toeplitz operator

$$\mathcal{T} \otimes A_1 \otimes (B_1, g^{B_1})$$

has the good property. We will show that the operator

$$\mathcal{T} \otimes A_1 \otimes (B_j, g^{B_j})$$

has the good property for all $j \geq 1$. It suffices to show that

$$(g^{B_j})_* = 1, \quad c_3(M, [g^{B_j}]) = 0, \quad \text{for } j \geq 2. \quad (3.17)$$

Since $c_3(M, [g]) = 0$, taking the degree 3 component of (2.21), we get

$$\sum_{j=1}^{\infty} c_3(M, [g^{B_j}]) q^{j/2} = 0, \quad (3.18)$$

which implies $c_3(M, [g^{B_j}]) = 0$ for all $j \geq 2$.

Observe that B_j is the sum of bundles of the form

$$\Lambda^{n_1} E_{\mathbb{C}} \otimes \Lambda^{n_2} E_{\mathbb{C}} \otimes \cdots \otimes \Lambda^{n_k} E_{\mathbb{C}}, \quad 1 \leq n_1, \dots, n_k \leq \dim E.$$

We have the following lemma.

Lemma 3.5. *Let V_1, \dots, V_n, V be complex vector spaces. Let*

$$h_i : M \longrightarrow \text{Aut}(V_i), \quad i = 1, \dots, n \quad \text{and} \quad h : M \longrightarrow \text{Aut}(V)$$

be cocycles in $K^{-1}(M)$. If the induced maps on the fundamental groups satisfy $(h_i)_ = 1$, $i = 1, \dots, n$ and $h_* = 1$, then we have*

$$(h_1 \otimes \cdots \otimes h_n)_* = 1 \quad \text{and} \quad \left(\overbrace{h \wedge \cdots \wedge h}^m \right)_* = 1,$$

where $h_1 \otimes \cdots \otimes h_n \in \text{Aut}(V_1 \otimes \cdots \otimes V_n)$ is defined by

$$(h_1 \otimes \cdots \otimes h_n)(v_1 \otimes \cdots \otimes v_n) = h_1(v_1) \otimes \cdots \otimes h_n(v_n), \quad v_i \in V_i,$$

and $\overbrace{h \wedge \cdots \wedge h}^m \in \text{Aut}(\Lambda^m(V))$ is defined by

$$\left(\overbrace{h \wedge \cdots \wedge h}^m \right)(u_1 \wedge \cdots \wedge u_m) = h(u_1) \wedge \cdots \wedge h(u_m), \quad u_i \in V.$$

Proof. For a loop γ representing an element in $\pi_1(M)$, since $(h_i)_* = 1$, $i = 1, \dots, n$, and $h_* = 1$, there exist homotopies $\eta_i(t)$, $\eta(t)$, $t \in [0, 1]$ such that

$$\eta_i(0) = h_i \circ \gamma, \quad \eta_i(1) = \text{const}; \quad \eta(0) = h \circ \gamma, \quad \eta(1) = \text{const}.$$

Therefore, $\eta_1(t) \otimes \cdots \otimes \eta_n(t)$ connects the loop

$$(h_1 \otimes \cdots \otimes h_n) \circ \gamma = (h_1 \circ \gamma) \otimes \cdots \otimes (h_n \circ \gamma)$$

to a constant loop in $\text{Aut}(V_1 \otimes \cdots \otimes V_n)$. Similarly $\overbrace{\eta(t) \wedge \cdots \wedge \eta(t)}^m$ connects the loop

$$\left(\overbrace{h \wedge \cdots \wedge h}^m \right) \circ \gamma = \overbrace{(h \circ \gamma) \wedge \cdots \wedge (h_n \circ \gamma)}^m$$

to a constant loop in $\text{Aut}(\Lambda^m(V))$. \square

As we have assumed $g_* = 1$, the above lemma tells us that $(g^{B_j})_* = 1$. Therefore (3.17) holds and the operator

$$\mathcal{T} \otimes A_1 \otimes (B_j, g^{B_j})$$

has the good property for all $j \geq 1$.

Now taking $j = 2$ in (3.16), we see that the operator

$$\mathcal{T} \otimes A_0 \otimes (B_3, g^{B_3}) + \mathcal{T} \otimes A_1 \otimes (B_2, g^{B_2}) + \mathcal{T} \otimes A_2 \otimes (B_1, g^{B_1}).$$

has the good property. Since from the above discussion the operators

$$\mathcal{T} \otimes A_0 \otimes (B_3, g^{B_3}) \text{ and } \mathcal{T} \otimes A_1 \otimes (B_2, g^{B_2})$$

both have the good property, we obtain that the operator

$$\mathcal{T} \otimes A_2 \otimes (B_1, g^{B_1})$$

has the good property. Therefore, due to (3.17), the operator

$$\mathcal{T} \otimes A_2 \otimes (B_j, g^{B_j})$$

has the good property for all $j \geq 1$.

A standard induction procedure shows that $\mathcal{T} \otimes A_i \otimes B_j$ has the good property for any $i, j \geq 1$. In particular, we see $\mathcal{T} \otimes A_i \otimes B_1$ has the good property for all $i \geq 0$. We finish the proof for $\mathcal{T} \otimes \Theta_2(TM|V) \otimes (E_{\mathbb{C}}, g)$.

It is easy to see that with little modification, the above deduction still applies to the operator $\mathcal{T} \otimes \Theta_3(TM|V) \otimes (E_{\mathbb{C}}, g)$. \square

We would like to point out that we deduce $c_3(M, [g^{B_j}]) = 0$ in (3.17) from (2.21) by taking the advantage of the special positive energy representation that $Q_2(E)$ is constructed on. An alternative deduction is by using the following proposition and the observation above Lemma 3.5.

Proposition 3.6. *Let V_1, \dots, V_n, V be complex vector spaces. Let*

$$h_i : M \longrightarrow \text{Aut}(V_i), \quad i = 1, \dots, n \quad \text{and} \quad h : M \longrightarrow \text{Aut}(V)$$

be cocycles in $K^{-1}(M)$.

(i) *The following equality holds,*

$$\text{ch}(V_1 \otimes \dots \otimes V_n, h_1 \otimes \dots \otimes h_n, d) = \sum_{i=1}^n \frac{\dim V_1 \cdots \dim V_n}{\dim V_i} \text{ch}(V_i, h_i, d). \quad (3.19)$$

(ii) *For any $j, m \geq 1$,*

$$c_{2j-1} \left(\Lambda^m(V), \overbrace{h \wedge \cdots \wedge h}^m, d \right)$$

is a constant multiples of $c_{2j-1}(V, h, d)$.

Proof. (i) Let (cf. (2.13))

$$\nabla_u^{V_i} = d + u h_i^{-1} dh_i, \quad u \in [0, 1]$$

be the curves of connections on $V_i, i = 1, 2$, respectively. Then $V_1 \otimes V_2$ naturally carries a curve of tensor connections

$$\nabla_u^{V_1 \otimes V_2} = d + u(h_1^{-1} dh_1 \otimes \text{id} + \text{id} \otimes h_2^{-1} dh_2), \quad u \in [0, 1].$$

We verify directly that the curvature $R_u^{V_1 \otimes V_2}$ of $\nabla_u^{V_1 \otimes V_2}$ is given by

$$R_u^{V_1 \otimes V_2} = (u^2 - u)((h_1^{-1} dh_1)^2 \otimes \text{id} + \text{id} \otimes (h_2^{-1} dh_2)^2). \quad (3.20)$$

Using the explicit formula of the Chern-Simons form [11, (1.25)], we get

$$\begin{aligned} \text{ch}(V_1 \otimes V_2, h_1 \otimes h_2, d) &= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \text{Tr} \left[(h_1^{-1} dh_1 \otimes \text{id} + \text{id} \otimes h_2^{-1} dh_2) \right. \\ &\quad \left. \cdot \exp(\sqrt{-1} R_u^{V_1 \otimes V_2} / (2\pi)) \right] du. \end{aligned} \quad (3.21)$$

Since $\text{Tr} [(h^{-1} dh)^k]$ vanishes for any positive even integer k (see [31, (1.40)]), by (3.20) and (3.21), we obtain (3.19) for the case $n = 2$. The proof of (3.19) for the general case follows in a similar way.

(ii) We consider the curve of connections

$$\nabla_u^V = d + u h^{-1} dh, \quad u \in [0, 1]$$

on V . Let $R_u^V = (u^2 - u)(h^{-1}dh)^2$ be the curvature of ∇_u^V .

∇_u^V canonically induces a connection $\nabla^{\Lambda_t(V)}$ on $\Lambda_t(V)$ for any $t \in \mathbb{C}$. Furthermore, we can compute the Chern character form of $(\Lambda_t(V), \nabla^{\Lambda_t(V)})$ as follows,

$$\text{ch}(\Lambda_t(V), \nabla^{\Lambda_t(V)}) = \det\left(1 + t \exp\left(\frac{\sqrt{-1}}{2\pi} R_u^V\right)\right), \quad u \in [0, 1]. \quad (3.22)$$

As in [6, (2.8)], we deduce from (3.22) that for $t \in \mathbb{C} \setminus \{-1\}$,

$$\begin{aligned} & \frac{d}{du} \text{ch}(\Lambda_t(V), \nabla^{\Lambda_t(V)}) \\ &= \frac{\sqrt{-1}}{2\pi} d\left(\text{ch}(\Lambda_t(V), \nabla_u^V) \text{Tr}\left[\frac{d\nabla_u^V}{du} \frac{t e^{\sqrt{-1}R_u^V/(2\pi)}}{1 + t e^{\sqrt{-1}R_u^V/(2\pi)}}\right]\right), \end{aligned} \quad (3.23)$$

Let $h^{\Lambda_t(V)}$ be the actions on $\Lambda_t(V)$ induced from the action of h on V . By [11, (1.25)] and (3.23), we get that for $t \in \mathbb{C} \setminus \{-1\}$,

$$\begin{aligned} & \text{ch}(\Lambda_t(V), h^{\Lambda_t(V)}, d) \\ &= -\frac{\sqrt{-1}}{2\pi} \int_0^1 (1+t)^{\dim V} \text{Tr}\left[h^{-1}dh \frac{t e^{\sqrt{-1}R_u^V/(2\pi)}}{1 + t e^{\sqrt{-1}R_u^V/(2\pi)}}\right] du. \end{aligned} \quad (3.24)$$

Taking the degree $2j - 1$ component of (3.24), we get

$$c_{2j-1}(\Lambda_t(V), h^{\Lambda_t(V)}, d) = \mathcal{P}_j(t) c_{2j-1}(V, h, d), \quad j \geq 1, \quad (3.25)$$

where $\mathcal{P}_j(t)$ is a polynomial in t . Now the second item of the proposition follows by taking the coefficients of t^m of both sides of (3.25). \square

Putting $V = TM$ in Theorem 3.4, we have

Corollary 3.7. *Suppose $g_* = 1$, $c_3(M, [g]) = 0$. Then the operators*

$$\mathcal{T} \otimes \Theta_2(TM) \otimes (E_{\mathbb{C}}, g), \quad \mathcal{T} \otimes \Theta_3(TM) \otimes (E_{\mathbb{C}}, g)$$

are rigid.

Proof of Theorem 1.1. Since M is simply connected, g_* is automatically trivial. Also $c_3(M, [g]) \in H^3(M, \mathbb{R})$ is zero. Therefore the conditions of Corollary 3.7 is verified. \square

Corollary 3.8. *Assume M is connected and the circle action is nontrivial. If $g_* = 1$, $c_3(M, [g]) = 0$ and $p_1(TM)_{S^1} = -n \cdot \pi^* u^2$ for some integer n , then the equivariant index of the Toeplitz-Witten operator*

$$\mathcal{T} \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M) \otimes (E_{\mathbb{C}}, g)$$

is identically zero.

Proof. Taking $V = 0$ in the third equality in (3.41), we see that

$$-\sum_{\gamma} \sum_j \gamma^2 = n,$$

from which we know the $n > 0$ case can never happen. If $n = 0$, then all the numbers $\dim N_{\gamma}$ are zero, so that the fixed point set of the circle action is empty. From Proposition 3.1, we know $\mathcal{T} \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M) \otimes (E_{\mathbb{C}}, g)$ has vanishing equivariant index. For $n < 0$, we may take $V = 0$ in Theorem 3.4 to derive the result. \square

3.3 A proof of Theorem 3.3

We continue in the notations of the previous subsection.

Similarly to (3.3), let

$$V|_{MS^1} = \bigoplus_{\nu \neq 0} V_{\nu} \oplus V_0^{\mathbb{R}} \tag{3.26}$$

be the S^1 -equivariant decomposition of the restrictions of V over M^{S^1} , where V_{ν} is a complex vector bundle such that $h \in S^1$ acts on it by h^{ν} with $\nu \in \mathbb{Z} \setminus \{0\}$, and $V_0^{\mathbb{R}}$ is the real subbundle of $V|_{MS^1}$ such that S^1 acts as identity. Set $V_0 = V_0^{\mathbb{R}} \otimes \mathbb{C}$. We denote by

$$2\pi\sqrt{-1}u_{\nu}^j, \quad j = 1, \dots, \dim V_{\nu}$$

the Chern roots of V_{ν} with $\nu \neq 0$, and by

$$\pm 2\pi\sqrt{-1}u_0^j, \quad j = 1, \dots, \lceil \dim V_0^{\mathbb{R}}/2 \rceil,$$

the Chern roots of V_0 . Here we use the notation that for $s \in \mathbb{R}$, $[s]$ denotes the greatest integer which is less than or equal to s .

We use the virtual version of the operators in (3.8)-(3.11). Set

$$\Theta_1(TM|V)_v = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{V_{\mathbb{C}}}), \quad (3.27)$$

$$\Theta_2(TM|V)_v = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(\widetilde{V_{\mathbb{C}}}), \quad (3.28)$$

$$\Theta_3(TM|V)_v = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(\widetilde{V_{\mathbb{C}}}). \quad (3.29)$$

If V is even dimensional, set

$$\Theta(TM|V)_v = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(\widetilde{V_{\mathbb{C}}}). \quad (3.30)$$

We keep the notation explained above Proposition 3.1, and define some functions on $\mathbb{C} \times \mathbb{H}$,

$$\begin{aligned} F_L^V(t, \tau) &= -2^{[\dim V/2]} \left(\frac{-\sqrt{-1}}{2\pi} \right)^{\dim N} \left\langle \text{ch}(Q_1(E)_v, g^{Q_1(E)_v}, d, \tau) \right. \\ &\quad \cdot \left(y \frac{\theta'(0, \tau)}{\theta(y, \tau)} \right) (TM^{S^1}) \cdot \prod_{\gamma} \left(\frac{\theta'(0, \tau)}{\theta(x_{\gamma} + \gamma t, \tau)} \right) (N_{\gamma}) \\ &\quad \cdot \prod_{\nu} \left(\frac{\theta_1(u_{\nu} + \nu t, \tau)}{\theta_1(0, \tau)} \right) (V_{\nu}), [M^{S^1}] \Big\rangle, \end{aligned} \quad (3.31)$$

$$\begin{aligned} F_W^V(t, \tau) &= - \left(\frac{-\sqrt{-1}}{2\pi} \right)^{\dim N} \left\langle \text{ch}(Q_2(E)_v, g^{Q_2(E)_v}, d, \tau) \right. \\ &\quad \cdot \left(y \frac{\theta'(0, \tau)}{\theta(y, \tau)} \right) (TM^{S^1}) \cdot \prod_{\gamma} \left(\frac{\theta'(0, \tau)}{\theta(x_{\gamma} + \gamma t, \tau)} \right) (N_{\gamma}) \\ &\quad \cdot \prod_{\nu} \left(\frac{\theta_2(u_{\nu} + \nu t, \tau)}{\theta_2(0, \tau)} \right) (V_{\nu}), [M^{S^1}] \Big\rangle, \end{aligned} \quad (3.32)$$

$$\begin{aligned} F'_W{}^V(t, \tau) &= - \left(\frac{-\sqrt{-1}}{2\pi} \right)^{\dim N} \left\langle \text{ch}(Q_3(E)_v, g^{Q_3(E)_v}, d, \tau) \right. \\ &\quad \cdot \left(y \frac{\theta'(0, \tau)}{\theta(y, \tau)} \right) (TM^{S^1}) \cdot \prod_{\gamma} \left(\frac{\theta'(0, \tau)}{\theta(x_{\gamma} + \gamma t, \tau)} \right) (N_{\gamma}) \\ &\quad \cdot \prod_{\nu} \left(\frac{\theta_3(u_{\nu} + \nu t, \tau)}{\theta_3(0, \tau)} \right) (V_{\nu}), [M^{S^1}] \Big\rangle, \end{aligned} \quad (3.33)$$

$$\begin{aligned}
F_{\text{dR},j}^V(t, \tau) &= -\frac{(-\sqrt{-1})^{\dim N + \dim V/2}}{(2\pi)^{\dim N - \dim V/2}} \left\langle \text{ch} (Q_j(E)_v, g^{Q_j(E)_v}, d, \tau) \right. \\
&\quad \cdot \left(y \frac{\theta'(0, \tau)}{\theta(y, \tau)} \right) (TM^{S^1}) \prod_{\gamma} \left(\frac{\theta'(0, \tau)}{\theta(x_{\gamma} + \gamma t, \tau)} \right) (N_{\gamma}) \\
&\quad \cdot \prod_{\nu} \left(\frac{\theta(u_{\nu} + \nu t, \tau)}{\theta'(0, \tau)} \right) (V_{\nu}, [M^{S^1}]) \left. \right\rangle. \quad j = 1, 2, 3. \quad (3.34)
\end{aligned}$$

By Proposition 3.1, we get, for $t \in [0, 1] \setminus \mathbb{Q}$ and $h = e^{2\pi\sqrt{-1}t}$,

$$F_{\text{L}}^V(t, \tau) = \text{Ind} \left(h, \mathcal{T} \otimes \Delta(V) \otimes \Theta_1(TM|V)_v \otimes (Q_1(E)_v, g^{Q_1(E)_v}) \right), \quad (3.35)$$

$$F_{\text{W}}^V(t, \tau) = \text{Ind} \left(h, \mathcal{T} \otimes \Theta_2(TM|V)_v \otimes (Q_2(E)_v, g^{Q_2(E)_v}) \right), \quad (3.36)$$

$$F'_{\text{W}}^V(t, \tau) = \text{Ind} \left(h, \mathcal{T} \otimes \Theta_3(TM|V)_v \otimes (Q_3(E)_v, g^{Q_3(E)_v}) \right), \quad (3.37)$$

$$\begin{aligned}
F_{\text{dR},j}^V(t, \tau) &= \text{Ind} \left(h, \mathcal{T} \otimes (\Delta_+(V) - \Delta_-(V)) \right. \\
&\quad \left. \otimes \Theta(TM|V)_v \otimes (Q_j(E)_v, g^{Q_j(E)_v}) \right), \quad j = 1, 2, 3. \quad (3.38)
\end{aligned}$$

Recall that a (meromorphic) Jacobi form of index m and weight l over $L \rtimes \Gamma$, where L is an integral lattice in the complex plane \mathbb{C} preserved by the modular subgroup $\Gamma \subset SL_2(\mathbb{Z})$, is a (meromorphic) function $F(t, \tau)$ over $\mathbb{C} \times \mathbb{H}$ such that

$$\begin{aligned}
F\left(\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^l e^{2\pi\sqrt{-1}m(ct^2/(c\tau+d))} F(t, \tau), \\
F(t + \lambda\tau + \mu, \tau) &= e^{-2\pi\sqrt{-1}m(\lambda^2\tau + 2\lambda t)} F(t, \tau),
\end{aligned} \quad (3.39)$$

where $(\lambda, \mu) \in L$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If F is holomorphic over $\mathbb{C} \times \mathbb{H}$, we say that F is a holomorphic Jacobi form.

The following theorem can be thought of as an odd analogue of [20, Theorem 3] (compare with [26, Theorem 3.1]).

Theorem 3.9. *Assume (3.13) holds.*

- (i) $F_{\text{L}}^V(t, \tau)$, $F_{\text{W}}^V(t, \tau)$ and $F'_{\text{W}}^V(t, \tau)$ are holomorphic Jacobi forms of index $n/2$ and weight $(\dim M + 1)/2$ over $(2\mathbb{Z})^2 \rtimes \Gamma$ with Γ equal to $\Gamma_0(2)$, $\Gamma^0(2)$ and Γ_{θ} , respectively.

- (ii) $F_{\text{dR},1}^V(t, \tau)$, $F_{\text{dR},2}^V(t, \tau)$ and $F_{\text{dR},3}^V(t, \tau)$ are holomorphic Jacobi forms of index $n/2$ and weight $(\dim M - \dim V + 1)/2$ over $(2\mathbb{Z})^2 \rtimes \Gamma$ with Γ equal to $\Gamma_0(2)$, $\Gamma^0(2)$ and Γ_θ , respectively.

The remaining part is devoted to a proof of Theorem 3.9. First, the condition (3.13) implies that

$$\sum_{\nu} \sum_j (u_{\nu}^j + \nu t)^2 - \sum_j y_j^2 - \sum_{\gamma} \sum_j (x_{\gamma}^j + \gamma t)^2 = nt^2, \quad (3.40)$$

which gives the equalities

$$\begin{aligned} \sum_{\nu} \sum_j (u_{\nu}^j)^2 - \sum_j y_j^2 - \sum_{\gamma} \sum_j (x_{\gamma}^j)^2 &= 0, \\ \sum_{\nu} \sum_j \nu u_{\nu}^j - \sum_{\gamma} \sum_j \gamma x_{\gamma}^j &= 0, \quad \sum_{\nu} \sum_j \nu^2 - \sum_{\gamma} \sum_j \gamma^2 = n. \end{aligned} \quad (3.41)$$

Under the action $t \mapsto t + \lambda\tau + \mu$ with $\lambda, \mu \in 2\mathbb{Z}$, we have

$$\begin{aligned} \theta(x_{\gamma} + \gamma(t + \lambda\tau + \mu)) &= e^{-\pi\sqrt{-1}(\gamma^2(\lambda^2\tau+2\lambda t)+2\gamma\lambda x_{\gamma})} \theta(x + \gamma t, \tau), \\ \theta_1(u_{\nu} + \nu(t + \lambda\tau + \mu)) &= e^{-\pi\sqrt{-1}(\nu^2(\lambda^2\tau+2\lambda t)+2\nu\lambda u_{\nu})} \theta_1(u_{\nu} + \nu t, \tau). \end{aligned} \quad (3.42)$$

From (3.41) and (3.42), we see that $F_{\text{L}}^V(t, \tau)$ verifies the second line of (3.39) with $m = \frac{n}{2}$. In a very similar way, we can show that for

$$F \in \{F_{\text{W}}^V, F'_{\text{W}}^V, F_{\text{dR},1}^V, F_{\text{dR},2}^V, F_{\text{dR},3}^V\},$$

F also verifies the second line of (3.39) with index $m = \frac{n}{2}$.

Similar to [20, Lemma 3.2], we have the following transformation formulas (compare with [26, Lemma 3.2]).

Lemma 3.10. (i) *The following equalities hold,*

$$F_{\text{L}}^V(t, \tau + 1) = F_{\text{L}}^V(t, \tau), \quad F'_{\text{W}}^V(t, \tau + 1) = F'_{\text{W}}^V(t, \tau), \quad (3.43)$$

$$F_{\text{dR},1}^V(t, \tau + 1) = F_{\text{dR},1}^V(t, \tau), \quad F_{\text{dR},2}^V(t, \tau + 1) = F_{\text{dR},3}^V(t, \tau). \quad (3.44)$$

(ii) *Assume (3.13) holds. Then*

$$\begin{aligned} F_{\text{L}}^V\left(\frac{t}{\tau}, -\frac{1}{\tau}\right) &= 2^{[(N+\dim V)/2]} \tau^{(\dim M+1)/2} e^{\pi\sqrt{-1}nt^2/\tau} F_{\text{W}}^V(t, \tau), \\ F'_{\text{W}}^V\left(\frac{t}{\tau}, -\frac{1}{\tau}\right) &= \tau^{(\dim M+1)/2} e^{\pi\sqrt{-1}nt^2/\tau} F'_{\text{W}}^V(t, \tau), \end{aligned} \quad (3.45)$$

$$\begin{aligned}
F_{\text{dR},1}^V\left(\frac{t}{\tau}, -\frac{1}{\tau}\right) &= 2^{N/2} \tau^{(\dim M - \dim V + 1)/2} e^{\pi\sqrt{-1}nt^2/\tau} F_{\text{dR},2}^V(t, \tau), \\
F_{\text{dR},3}^V\left(\frac{t}{\tau}, -\frac{1}{\tau}\right) &= \tau^{(\dim M - \dim V + 1)/2} e^{\pi\sqrt{-1}nt^2/\tau} F_{\text{dR},3}^V(t, \tau).
\end{aligned} \tag{3.46}$$

Proof. (3.43) and (3.44) follow from [6, (3.28)-(3.31), (4.6)], (2.25) and (3.31)-(3.34) straightforwardly.

From [6, (3.28)-(3.31), (4.7)], we obtain the following transformation formulas,

$$\begin{aligned}
y \frac{\theta'(0, -1/\tau)}{\theta(y, -1/\tau)} &= e^{-\pi\sqrt{-1}\tau y^2} \tau y \frac{\theta'(0, \tau)}{\theta(\tau y, \tau)}, \\
\frac{\theta'(0, -1/\tau)}{\theta(x_\gamma + \gamma t/\tau, -1/\tau)} &= e^{-\pi\sqrt{-1}\tau(x_\gamma + \gamma t/\tau)^2} \frac{\tau\theta'(0, \tau)}{\theta(\tau x_\gamma + \gamma t, \tau)}, \\
\frac{\theta_1(u_\nu + \nu t/\tau, -1/\tau)}{\theta_1(0, -1/\tau)} &= e^{\pi\sqrt{-1}\tau(u_\nu + \nu t/\tau)^2} \frac{\theta_2(\tau u_\nu + \nu t, \tau)}{\theta_2(0, \tau)}.
\end{aligned} \tag{3.47}$$

Combining (2.26), (3.31), (3.41) and (3.47), we obtain the first line in (3.45). The other lines in (3.45) and (3.46) can be verified in a similar way. \square

Since the generators of $\Gamma_0(2)$ are T, ST^2ST , from Lemma 3.10, we can check directly that the first line of (3.39) holds for $F_L^V(t, \tau)$ with $\Gamma = \Gamma_0(2)$, $m = \frac{n}{2}$, $l = \frac{\dim M + 1}{2}$. Thus, $F_L^V(t, \tau)$ is a meromorphic Jacobi form of index $\frac{n}{2}$ and weight $\frac{\dim M + 1}{2}$ over $(2\mathbb{Z})^2 \rtimes \Gamma_0(2)$. In a similar way, we can prove the assertions in Theorem 3.9 except the holomorphic property.

For $\mathcal{G} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and a Jacobi form F of index m and weight l , we write

$$F(\mathcal{G}(t, \tau)) = (c\tau + d)^{-l} e^{-2\pi\sqrt{-1}m(ct^2/(c\tau+d))} F\left(\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right). \tag{3.48}$$

Lemma 3.10 tells us that, if

$$F \in \{F_L^V, F_W^V, F_W^{\prime V}\} \text{ (resp. } \{F_{\text{dR},1}^V, F_{\text{dR},2}^V, F_{\text{dR},3}^V\}),$$

its modular transformation $F(\mathcal{G}(t, \tau))$ which may need a multiplication by a constant is still in

$$\{F_L^V, F_W^V, F_W^{\prime V}\} \text{ (resp. } \{F_{\text{dR},1}^V, F_{\text{dR},2}^V, F_{\text{dR},3}^V\}).$$

Lemma 3.11. *For any function*

$$F \in \{F_L^V, F_W^V, F_W^{\prime V}, F_{dR,1}^V, F_{dR,2}^V, F_{dR,3}^V\},$$

its modular transformation is holomorphic in $(t, \tau) \in \mathbb{R} \times \mathbb{H}$.

Proof. The proof is almost the same as the proof of [20, Lemma 2.3] except that we use Proposition 3.1 instead of the Lefschetz fixed point formula therein. \square

It is crucial that F and its modular transformation are the Lefschetz numbers of certain Toeplitz operators. This is also the place where the spin conditions on M and V as well as the assumptions on g come in. Thus, one can use index theory to cancel part of the poles of these functions.

We now prove that

$$F \in \{F_L^V, F_W^V, F_W^{\prime V}, F_{dR,1}^V, F_{dR,2}^V, F_{dR,3}^V\}$$

is actually holomorphic on $\mathbb{C} \times \mathbb{H}$.

The proof essentially makes use of Liu's key techniques [20, Lemma 3.4]. We give the details here for completeness. By (3.31) and (3.34), we see that the possible poles of $F(t, \tau)$ can be written in the form $t = k(c\tau + d)/r$ for integers k, r, c, d with $(c, d) = 1$.

Suppose $t = k(c\tau + d)/r$ is a pole for $F(t, \tau)$. Find integers a, b such that $ad - bc = 1$. Take $\mathcal{G} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. From (3.48), it is easy to see that $F(\mathcal{G}(t, \tau))$ and $F\left(\frac{t}{-c\tau+a}, \frac{d\tau-b}{-c\tau+a}\right)$ have the same poles. Now that $t = k(c\tau + d)/r$ is a pole of $F(t, \tau)$, a pole of $F(\mathcal{G}(t, \tau))$ is given by solving the equation

$$\frac{t}{-c\tau + a} = \frac{k\left(c\frac{d\tau-b}{-c\tau+a} + d\right)}{r}, \quad (3.49)$$

which exactly gives $t = k/r \in \mathbb{R}$. By Lemma 3.11, we get a contradiction. Therefore, $F(t, \tau)$ is holomorphic on $\mathbb{C} \times \mathbb{H}$.

We complete the proof of Theorem 3.9.

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