

WITTEN GENUS AND STRING COMPLETE INTERSECTIONS

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We prove the vanishing of the Witten genus of generalized string complete intersections in products of complex projective spaces. Our result generalizes a known result of Landweber and Stong.

1. Introduction

Let M be a $4k$ -dimensional closed oriented smooth manifold. Let E be a complex vector bundle over M . For any complex number t , set

$$\begin{aligned}\Lambda_t(E) &= \mathbb{C} \mid M + tE + t^2 \Lambda^2(E) + \cdots, \\ S_t(E) &= \mathbb{C} \mid M + tE + t^2 S^2(E) + \cdots,\end{aligned}$$

where for any integer $j \geq 1$, $\Lambda^j(E)$ is the j -th exterior power of E and $S^j(E)$ is the j -th symmetric power of E ; see [Atiyah 1967]. Set $\widetilde{E} = E - \mathbb{C}^{\text{rk}(E)}$.

Let $q = e^{\pi i \tau}$ with $\tau \in \mathbb{H}$, the upper half plane. Witten [1988] defined

$$\Theta_q(E) = \bigotimes_{n \geq 1} S_{q^{2n}}(E)$$

and then defined $\varphi_W(M)$, now called the Witten genus, for M as

$$\varphi_W(M) = \langle \widehat{A}(M) \text{ch}(\Theta_q(\widetilde{TM} \otimes \mathbb{C})), [M] \rangle,$$

where $\widehat{A}(M)$ is the \widehat{A} -characteristic class of M and $[M]$ is the fundamental class of M . (See [Zhang 2001] for definitions of \widehat{A} and ch by curvature in Chern–Weil theory.) Let $\{\pm 2\pi \sqrt{-1}x_j \text{ for } 1 \leq j \leq 2k\}$ be the formal Chern roots of $TM \otimes \mathbb{C}$. From [Liu 1996; 1995b], the Witten genus can be written using Chern roots as

$$(1-1) \quad \varphi_W(M) = \left\langle \prod_{j=1}^{2k} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)}, [M] \right\rangle,$$

where $\theta(v, \tau)$ is the Jacobi theta function; see page 252 below. The Witten genus, which he introduced for studying two-dimensional supersymmetric quantum field theory, can be viewed as the loop space analogue of the \widehat{A} genus.

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By the Atiyah–Singer index theorem, when M is spin, $\varphi_W(M) \in \mathbb{Z}[[q]]$; see [Hirzebruch et al. 1992]. Moreover, if the spin manifold M is string, that is, $p_1(TM)/2 = 0$ — $p_1(TM)/2$ is a cohomology class in $H^4(M, \mathbb{Z})$, twice of which is the first integral Pontryagin class $p_1(TM)$ — or even weaker, if the first rational Pontryagin class $p_1(M) = 0$, then $\varphi_W(M)$ is a modular form of weight $2k$ with a q -series expansion of integer coefficients; see [Hirzebruch et al. 1992]. (For a study of the Witten genus of spin^c manifolds, see [Zhang 1992].)

Let $V_{(d_{pq})}$ be a nonsingular $4k$ -dimensional generalized complete intersection in the product of complex projective spaces $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s}$, which is dual to $\prod_{p=1}^t (\sum_{q=1}^s d_{pq} x_q) \in H^{2t}(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s}, \mathbb{Z})$, where $x_q \in H^2(\mathbb{C}P^{n_q}, \mathbb{Z})$ for $1 \leq q \leq s$ is the generator of $H^*(\mathbb{C}P^{n_q}, \mathbb{Z})$ and d_{pq} for $1 \leq p \leq t$ and $1 \leq q \leq s$ are integers. Let $P_q : \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_s} \rightarrow \mathbb{C}P^{n_q}$ for $1 \leq q \leq s$ be the q -th projection. $V_{(d_{pq})}$ is the intersection of the zero loci of smooth global sections of line bundles $\otimes_{q=1}^s P_q^*(\mathcal{O}(d_{pq}))$ for $1 \leq p \leq t$, where $\mathcal{O}(d_{pq}) = \mathcal{O}(1)^{d_{pq}}$ is the d_{pq} -th power of the canonical line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^{n_q}$.

Here we have somewhat abused the terminology of complete intersection from algebraic geometry. We don't require that the integers d_{pq} be nonnegative, so $V_{(d_{pq})}$ might not be an algebraic variety. However, by transversality, $V_{(d_{pq})}$ can always be chosen to be smooth. $V_{(d_{pq})}$ is a representative in $\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_s}$ of the virtual submanifold $(\sum_{q=1}^s d_{1q} \chi_q, \dots, \sum_{q=1}^s d_{tq} \chi_q)$, in the sense of [Hirzebruch et al. 1992]. Putting certain conditions (Proposition 3.1 below) on the data n_q for $1 \leq q \leq s$ and d_{pq} for $1 \leq p \leq t$ and $1 \leq q \leq s$, the complete intersection $V_{(d_{pq})}$ can be made string. When applied systematically, this generates many nice examples of string manifolds, whose Witten genus we study here. See also [Gorbounov and Malikov 2004; Gorbounov and Ochanine 2006] for a study of elliptic genera of complete intersections and the Landau–Ginzburg/Calabi–Yau correspondence.

Set

$$(1-2) \quad D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1s} \\ d_{21} & d_{22} & \cdots & d_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ d_{t1} & d_{t2} & \cdots & d_{ts} \end{bmatrix}.$$

and let m_q be the number of nonzero elements in the q -th column of D . The main result of this paper is this:

Theorem 1.1. *If $m_q + 2 \leq n_q$ for $1 \leq q \leq s$ and $V_{(d_{pq})}$ is string, then the Witten genus $\varphi_W(V_{(d_{pq})})$ vanishes.*

Our result generalizes the known vanishing of the Witten genus of string nonsingular complete intersections of hypersurfaces with degrees d_1, \dots, d_t in a single complex projective space, a result due to Landweber and Stong which is described by Hirzebruch, Berger, and Jung [1992, Section 6.3] by applying the properties of

the sigma function. Stolz [1996] used this as evidence of his celebrated conjecture. In the special case we treat, our result is broader than theirs, since we don't require that the d_1, \dots, d_t are all positive.

Explicitly expanding $\Theta_q(\widetilde{TM} \otimes \mathbb{C})$, we get

$$\widehat{A}(M) \text{ch}(\Theta_q(\widetilde{TM} \otimes \mathbb{C})) = \widehat{A}(M) + \widehat{A}(M)(\text{ch}(TM \otimes \mathbb{C}) - 4k)q^2 + \dots$$

Therefore it's not hard to obtain this corollary from Theorem 1.1:

Corollary 1.2. *If $m_q + 2 \leq n_q$ for $1 \leq q \leq s$ and $V_{(d_{pq})}$ is string, then*

$$\langle \widehat{A}(V_{(d_{pq})}), [V_{(d_{pq})}] \rangle = 0 \quad \text{and} \quad \langle \widehat{A}(V_{(d_{pq})}) \text{ch}(TV_{(d_{pq})} \otimes \mathbb{C}), [V_{(d_{pq})}] \rangle = 0.$$

Let M be a 12-dimensional oriented closed smooth manifold. The signature of M can be expressed by the \widehat{A} -genus and the twisted \widehat{A} -genus as

$$(1-3) \quad \{\widehat{L}(M)\}^{(12)} = \{8\widehat{A}(M) \text{ch}(T_{\mathbb{C}}M) - 32\widehat{A}(M)\}^{(12)};$$

see [Alvarez-Gaumé and Witten 1984; Liu 1995a]. See [Zhang 2001] for a definition of \widehat{L} by curvature in Chern–Weil theory. This formula and its higher generalization have an application to the Rokhlin–Ochanine divisibility of the signature; see [Ochanine 1981; Liu 1995a]. Combining Corollary 1.2 and (1-3), we obtain:

Corollary 1.3. *If $m_q + 2 \leq n_q$ for $1 \leq q \leq s$ and $V_{(d_{pq})}$ is 12-dimensional and string, then the signature of $V_{(d_{pq})}$ vanishes.*

Let M be a 16-dimensional oriented closed smooth manifold. From [Chen and Han 2006] we have the formula

$$(1-4) \quad \{\widehat{L}(M) \text{ch}(T_{\mathbb{C}}M)\}^{(16)} = -2048 \{\widehat{A}(M) \text{ch}(T_{\mathbb{C}}M) - 48\widehat{A}(M)\}^{(16)}.$$

Corollary 1.4. *If $m_q + 2 \leq n_q$, $1 \leq q \leq s$, $V_{(d_{pq})}$ is 16-dimensional and string, then the twisted signature $\text{Sig}(V_{(d_{pq})}, T)$ vanishes.*

This is proved by combining Corollary 1.2 and (1-4) for the twisted signature

$$\text{Sig}(M, T) \triangleq \langle \widehat{L}(TM) \text{ch}(T_{\mathbb{C}}M), [M] \rangle.$$

2. Some preliminaries

This section reviews some tools and the knowledge that we will apply in Section 3 to prove Theorem 1.1. We start with results on residues in complex geometry. See [Griffiths and Harris 1994, Chapter 5] for details.

Let U be the ball $\{z \in \mathbb{C}^s : \|z\| < \varepsilon\}$, and let $f_1, \dots, f_s \in \mathcal{O}(\overline{U})$ be functions holomorphic in a neighborhood of the closure \overline{U} of U . We assume that each $f_i(z)$ has an isolated common zero at the origin. Set

$$D_i = (f_i) = \text{divisor of } f_i \quad \text{and} \quad D = D_1 + \dots + D_s.$$

Let

$$\omega = \frac{g(z) dz_1 \wedge \cdots \wedge dz_s}{f_1(z) \cdots f_s(z)}$$

be a meromorphic s -form with polar divisor D . The residue of ω at the origin is

$$\text{Res}_{\{0\}}\omega = \left(\frac{1}{2\pi\sqrt{-1}}\right)^s \int_{\Gamma} \omega,$$

where Γ is the real s -cycle defined by $\Gamma = \{z : |f_i(z)| = \varepsilon \text{ for } 1 \leq i \leq s\}$ and oriented by $d(\arg f_1) \wedge \cdots \wedge d(\arg f_s) \geq 0$.

Let M be a compact complex manifold of dimension s . Suppose that D_i for $i = 1, \dots, s$ are effective divisors, the intersection of which is a finite set of points. Let $D = D_1 + \cdots + D_s$. Let ω be a meromorphic s -form on M with polar divisor D . For each point $P \in D_1 \cap \cdots \cap D_s$, we may restrict ω to a neighborhood of U_P of P and define the residue $\text{Res}_P \omega$ as above. Then ([Griffiths and Harris 1994, Chaper 5]), one has:

Lemma 2.1 (Residue theorem).

$$\sum_{P \in D_1 \cap \cdots \cap D_s} \text{Res}_P \omega = 0.$$

We also need some facts about the Jacobi theta functions. Although we are going to use only one of them, we list for completeness all their definitions and transformation laws.

The four Jacobi theta functions are defined below; see for example [Chandrasekharan 1985].

$$\begin{aligned} \theta(v, \tau) &= 2q^{1/4} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^{2j})(1 - e^{2\pi\sqrt{-1}v} q^{2j})(1 - e^{-2\pi\sqrt{-1}v} q^{2j})], \\ \theta_1(v, \tau) &= 2q^{1/4} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^{2j})(1 + e^{2\pi\sqrt{-1}v} q^{2j})(1 + e^{-2\pi\sqrt{-1}v} q^{2j})], \\ \theta_2(v, \tau) &= \prod_{j=1}^{\infty} [(1 - q^{2j})(1 - e^{2\pi\sqrt{-1}v} q^{2j-1})(1 - e^{-2\pi\sqrt{-1}v} q^{2j-1})], \\ \theta_3(v, \tau) &= \prod_{j=1}^{\infty} [(1 - q^{2j})(1 + e^{2\pi\sqrt{-1}v} q^{2j-1})(1 + e^{-2\pi\sqrt{-1}v} q^{2j-1})], \end{aligned}$$

where $q = e^{\pi i \tau}$, τ is in the upper half plane \mathbb{H} , and $v \in \mathbb{C}$. They are all holomorphic functions for $(v, \tau) \in \mathbb{C} \times \mathbb{H}$. Let $\theta'(0, \tau) = (\partial/\partial v)\theta(v, \tau)|_{v=0}$. The Jacobi theta

functions satisfy the following relations [Chandrasekharan 1985]:

$$\begin{aligned} \theta(v + 1, \tau) &= -\theta(v, \tau), & \theta(v + \tau, \tau) &= -\frac{1}{q}e^{-2\pi iv}\theta(v, \tau), \\ \theta_1(v + 1, \tau) &= -\theta_1(v, \tau), & \theta_1(v + \tau, \tau) &= \frac{1}{q}e^{-2\pi iv}\theta_1(v, \tau), \\ \theta_2(v + 1, \tau) &= \theta_2(v, \tau), & \theta_2(v + \tau, \tau) &= -\frac{1}{q}e^{-2\pi iv}\theta_2(v, \tau), \\ \theta_3(v + 1, \tau) &= \theta_3(v, \tau), & \theta_3(v + \tau, \tau) &= \frac{1}{q}e^{-2\pi iv}\theta_3(v, \tau). \end{aligned}$$

Therefore it's not hard to deduce how the theta functions vary along the lattice $\Gamma = \{m + n\tau \mid m, n \in \mathbb{Z}\}$. We have

$$(2-5) \quad \theta(v + m, \tau) = (-1)^m \theta(v, \tau)$$

and

$$\begin{aligned} (2-6) \quad \theta(v + n\tau, \tau) &= -\frac{1}{q}e^{-2\pi i(v+(n-1)\tau)}\theta(v + (n-1)\tau, \tau) \\ &= -\frac{1}{q}e^{-2\pi i(v+(n-1)\tau)}\left(-\frac{1}{q}\right)e^{-2\pi i(v+(n-2)\tau)}\theta(v + (n-2)\tau, \tau) \\ &= (-1)^n \frac{1}{q^n}e^{-2\pi i[(v+(n-1)\tau)+(v+(n-2)\tau)+\dots+v]}\theta(v, \tau) \\ &= (-1)^n \frac{1}{q^n}e^{-2\pi inv - \pi in(n-1)\tau}\theta(v, \tau) \\ &= (-1)^n e^{-2\pi inv - \pi in^2\tau}\theta(v, \tau). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \theta_1(v + m, \tau) &= (-1)^m \theta_1(v, \tau), & \theta_1(v + n\tau, \tau) &= e^{-2\pi inv - \pi in^2\tau}\theta_1(v, \tau); \\ \theta_2(v + m, \tau) &= \theta_2(v, \tau), & \theta_2(v + n\tau, \tau) &= (-1)^n e^{-2\pi inv - \pi in^2\tau}\theta_2(v, \tau); \\ \theta_3(v + m, \tau) &= \theta_3(v, \tau), & \theta_3(v + n\tau, \tau) &= e^{-2\pi inv - \pi in^2\tau}\theta_3(v, \tau). \end{aligned}$$

3. Proof of Theorem 1.1

Let $i : V_{(d_{pq})} \rightarrow \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \dots \times \mathbb{C}P^{n_s}$ be the inclusion. It's not hard to see that

$$i^*T_{\mathbb{R}}(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \dots \times \mathbb{C}P^{n_s}) \cong TV_{(d_{pq})} \bigoplus i^*\left(\bigoplus_{p=1}^t \left(\bigotimes_{q=1}^s P_q^* \mathbb{C}(d_{pq})\right)\right),$$

where we forget the complex structure of the line bundles $\bigotimes_{q=1}^s P_q^*\mathbb{O}(d_{pq})$ for $1 \leq p \leq t$. Therefore for the total Stiefel–Whitney class, we have

$$i^*w(T_{\mathbb{R}}(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s})) = w(TV_{(d_{pq})}) \prod_{p=1}^t i^*w\left(\bigotimes_{q=1}^s P_q^*\mathbb{O}(d_{pq})\right),$$

or more precisely

$$(3-7) \quad i^*\left(\prod_{q=1}^s (1 + x_q)^{n_q+1}\right) \equiv w(TV_{(d_{pq})}) \prod_{p=1}^t i^*\left(1 + \sum_{q=1}^s d_{pq}x_q\right) \pmod{2}.$$

By (3-7), we can easily see that

$$w_1(TV_{(d_{pq})}) = 0 \quad \text{and} \quad w_2(TV_{(d_{pq})}) \equiv \sum_{q=1}^s (n_q + 1 - \sum_{p=1}^t d_{pq})i^*x_q \pmod{2}.$$

As for the total rational Pontryagin class, we have

$$i^*p(T_{\mathbb{R}}(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s})) = p(TV_{(d_{pq})}) \prod_{p=1}^t i^*p\left(\bigotimes_{q=1}^s P_q^*\mathbb{O}(d_{pq})\right),$$

or

$$p(V_{(d_{pq})}) = \prod_{q=1}^s (1 + (i^*x_q)^2)^{n_q+1} \prod_{p=1}^t \left(1 + \left(\sum_{q=1}^s d_{pq}i^*x_q\right)^2\right)^{-1}.$$

Hence we have

$$\begin{aligned} p_1(V_{(d_{pq})}) &= \sum_{q=1}^s (n_q + 1)(i^*x_q)^2 - \sum_{p=1}^t \left(\sum_{q=1}^s d_{pq}i^*x_q\right)^2 \\ &= \sum_{q=1}^s (n_q + 1 - \sum_{p=1}^t d_{pq}^2)(i^*x_q)^2 - \sum_{\substack{1 \leq u, v \leq s \\ u \neq v}} \left(\sum_{p=1}^t d_{pu}d_{pv}i^*x_u i^*x_v\right). \end{aligned}$$

Let $i_! : H^*(V_{(d_{pq})}, \mathbb{Q}) \rightarrow H^{*+2t}(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s}, \mathbb{Q})$ be the push forward. Thus if $p_1(V_{(d_{pq})}) = 0$, then

$$i_!p_1(V_{(d_{pq})}) = i_!i^*\left(\sum_{q=1}^s (n_q + 1 - \sum_{p=1}^t d_{pq}^2)x_q^2 - \sum_{\substack{1 \leq u, v \leq s \\ u \neq v}} \left(\sum_{p=1}^t d_{pu}d_{pv}x_u x_v\right)\right) = 0,$$

that is,

$$\left(\prod_{p=1}^t \left(\sum_{q=1}^s d_{pq} x_q \right) \right) \left(\sum_{q=1}^s (n_q + 1 - \sum_{p=1}^t d_{pq}^2) x_q^2 - \sum_{\substack{1 \leq u, v \leq s \\ u \neq v}} \left(\sum_{p=1}^t d_{pu} d_{pv} x_u x_v \right) \right) = 0$$

in $H^{2t+4}(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \dots \times \mathbb{C}P^{n_s}, \mathbb{Q})$. If $m_q + 2 \leq n_q$ for $1 \leq q \leq s$, then the left hand side of the above equality should not only be a zero element in the cohomology ring but should also be a zero polynomial. Note that the polynomial ring is an integral domain. Therefore at least one of its factors should be zero. But $\prod_{p=1}^t (\sum_{q=1}^s d_{pq} x_q)$ is nonzero. This means

$$\sum_{q=1}^s (n_q + 1 - \sum_{p=1}^t d_{pq}^2) x_q^2 - \sum_{\substack{1 \leq u, v \leq s \\ u \neq v}} \left(\sum_{p=1}^t d_{pu} d_{pv} x_u x_v \right) = 0,$$

and consequently these identities hold:

$$(3-8) \quad \begin{aligned} n_q + 1 - \sum_{p=1}^t d_{pq}^2 &= 0 \quad \text{for } 1 \leq q \leq s, \\ \sum_{p=1}^t d_{pu} d_{pv} &= 0 \quad \text{for } 1 \leq u, v \leq s \text{ and } u \neq v. \end{aligned}$$

Note that $n_q + 1 - \sum_{p=1}^t d_{pq}^2 \equiv n_q + 1 - \sum_{p=1}^t d_{pq} \pmod{2}$ for $1 \leq q \leq s$. Hence the first of (3-8) implies that $w_2(TV_{(d_{pq})}) = 0$.

In summary:

Proposition 3.1. *If $m_q + 2 \leq n_q$ for $1 \leq q \leq s$, then $p_1(V_{(d_{pq})}) = 0$ implies $V_{(d_{pq})}$ is spin. Therefore if $m_q + 2 \leq n_q$ for $1 \leq q \leq s$, then $V_{(d_{pq})}$ is string if and only if any of the following holds:*

- (1) $p_1(V_{(d_{pq})}) = 0$.
- (2) Equation (3-8) is satisfied.
- (3) The matrix D defined in (1-2) satisfies $\|\text{col}_q D\|^2 = n_q + 1$ for $1 \leq q \leq s$, and any two of its columns are orthogonal, that is,

$$D^t D = \text{diag}(n_1 + 1, \dots, n_s + 1).$$

Proof of Theorem 1.1. Denote the fundamental class of $V_{(d_{pq})}$ in $H_{4k}(V_{(d_{pq})}, \mathbb{Z})$ by $[V_{(d_{pq})}]$. Then according to (1-1) and the multiplicative property of the Witten

genus, up to a constant scalar,

$$\begin{aligned} \varphi_W(V_{(d_{pq})}) &= \left(\left(\prod_{q=1}^s \frac{(i^* x_q)^{n_q+1}}{\left(\frac{\theta(i^* x_q, \tau)}{\theta'(0, \tau)} \right)^{n_q+1}} \right) \left(\prod_{p=1}^t \frac{\left(\sum_{q=1}^s d_{pq} i^* x_q \right)^{-1}}{\left(\frac{\theta(\sum_{q=1}^s d_{pq} i^* x_q, \tau)}{\theta'(0, \tau)} \right)^{-1}} \right) \right) [V_{(d_{pq})}] \\ &= \left(\left(\prod_{q=1}^s \frac{x_q^{n_q+1}}{\left(\frac{\theta(x_q, \tau)}{\theta'(0, \tau)} \right)^{n_q+1}} \right) \left(\prod_{p=1}^t \frac{1}{\left(\frac{\theta(\sum_{q=1}^s d_{pq} x_q, \tau)}{\theta'(0, \tau)} \right)^{-1}} \right) \right) [\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \dots \times \mathbb{C}P^{n_s}] \\ &= \text{coefficient of } x_1^{n_1} \dots x_s^{n_s} \text{ in } \frac{\left(\prod_{q=1}^s x_q^{n_q+1} \right) \left(\prod_{p=1}^t \frac{\theta(\sum_{q=1}^s d_{pq} x_q, \tau)}{\theta'(0, \tau)} \right)}{\prod_{q=1}^s \left(\frac{\theta(x_q, \tau)}{\theta'(0, \tau)} \right)^{n_q+1}} \\ &= \text{Res}_0 \left(\frac{\left(\prod_{p=1}^t \frac{\theta(\sum_{q=1}^s d_{pq} x_q, \tau)}{\theta'(0, \tau)} \right) dx_1 \wedge \dots \wedge dx_s}{\prod_{q=1}^s \left(\frac{\theta(x_q, \tau)}{\theta'(0, \tau)} \right)^{n_q+1}} \right). \end{aligned}$$

The second equality above follows from Poincaré duality. Now, set

$$g(x_1, \dots, x_s) = \prod_{p=1}^t \frac{\theta(\sum_{q=1}^s d_{pq} x_q, \tau)}{\theta'(0, \tau)} \quad \text{and} \quad f_q(x_q) = \left(\frac{\theta(x_q, \tau)}{\theta'(0, \tau)} \right)^{n_q+1}$$

for $1 \leq q \leq s$, and define

$$\omega = \frac{g(x_1, \dots, x_s) dx_1 \wedge \dots \wedge dx_s}{f_1(x_1) \dots f_s(x_s)}.$$

Then up to a constant scalar,

$$(3-9) \quad \varphi_W(V_{(d_{pq})}) = \text{Res}_{(0,0,\dots,0)} \omega.$$

By (2-5),

$$\begin{aligned} g(x_1 + 1, x_2, \dots, x_s) &= \prod_{p=1}^t \frac{\theta(\sum_{q=1}^s d_{pq} x_q + d_{p1}, \tau)}{\theta'(0, \tau)} \\ &= (-1)^{d_{11} + \dots + d_{t1}} \prod_{p=1}^t \frac{\theta(\sum_{q=1}^s d_{pq} x_q, \tau)}{\theta'(0, \tau)} \end{aligned}$$

and

$$f_1(x_1 + 1) = \left(\frac{\theta(x_1 + 1, \tau)}{\theta'(0, \tau)} \right)^{n_1+1} = (-1)^{n_1+1} \left(\frac{\theta(x_1, \tau)}{\theta'(0, \tau)} \right)^{n_1+1}.$$

Thus

$$\frac{g(x_1 + 1, \dots, x_s)}{f_1(x_1 + 1) \dots f_s(x_s)} = (-1)^{(d_{11} + \dots + d_{t1}) - (n_1+1)} \frac{g(x_1, \dots, x_s)}{f_1(x_1) \dots f_s(x_s)}.$$

By (3-8), $(d_{11} + \dots + d_{t1}) - (n_1 + 1) \equiv (d_{11}^2 + \dots + d_{t1}^2) - (n_1 + 1) = 0 \pmod{2}$. Thus one obtains that

$$\frac{g(x_1 + 1, \dots, x_s)}{f_1(x_1 + 1) \cdots f_s(x_s)} = \frac{g(x_1, \dots, x_s)}{f_1(x_1) \cdots f_s(x_s)}.$$

Similarly, we have

$$(3-10) \quad \frac{g(x_1, \dots, x_q + 1, \dots, x_s)}{f_1(x_1) \cdots f_q(x_q + 1) \cdots f_s(x_s)} = \frac{g(x_1, \dots, x_s)}{f_1(x_1) \cdots f_s(x_s)} \quad \text{for } 1 \leq q \leq s.$$

On the other hand, by (2-5) and (2-6),

$$\begin{aligned} g(x_1 + \tau, x_2, \dots, x_s) &= \prod_{p=1}^t \frac{\theta(\sum_{q=1}^s d_{pq}x_q + d_{p1}\tau, \tau)}{\theta'(0, \tau)} \\ &= \prod_{p=1}^t (-1)^{d_{p1}} e^{-2\pi i d_{p1}(\sum_{q=1}^s d_{pq}x_q) - \pi i d_{p1}^2 \tau} \cdot \frac{\theta(\sum_{q=1}^s d_{pq}x_q, \tau)}{\theta'(0, \tau)} \\ &= (-1)^{d_{11} + \dots + d_{t1}} e^{-2\pi i \sum_{p=1}^t d_{p1}(\sum_{q=1}^s d_{pq}x_q) - \pi i \tau(\sum_{p=1}^t d_{p1}^2)} \cdot \frac{\theta(\sum_{q=1}^s d_{pq}x_q, \tau)}{\theta'(0, \tau)} \\ &= (-1)^{d_{11} + \dots + d_{t1}} e^{-2\pi i \sum_{p=1}^t d_{p1}(\sum_{q=1}^s d_{pq}x_q) - \pi i \tau(\sum_{p=1}^t d_{p1}^2)} g(x_1, x_2, \dots, x_s) \end{aligned}$$

and

$$\begin{aligned} f_1(x_1 + \tau) &= \left(\frac{\theta(x_1 + \tau, \tau)}{\theta'(0, \tau)} \right)^{n_1 + 1} \\ &= \left(-e^{-2\pi i x_1 - \pi i \tau} \frac{\theta(x_1, \tau)}{\theta'(0, \tau)} \right)^{n_1 + 1} \\ &= (-1)^{n_1 + 1} e^{-2\pi i(n_1 + 1)x_1 - \pi i \tau(n_1 + 1)} \left(\frac{\theta(x_1, \tau)}{\theta'(0, \tau)} \right)^{n_1 + 1} \\ &= (-1)^{n_1 + 1} e^{-2\pi i(n_1 + 1)x_1 - \pi i \tau(n_1 + 1)} f_1(x_1). \end{aligned}$$

Therefore

$$(3-11) \quad \frac{g(x_1 + \tau, \dots, x_s)}{f_1(x_1 + \tau) \cdots f_s(x_s)} = (-1)^{d_{11} + \dots + d_{t1} - n_1 - 1} \frac{g(x_1, \dots, x_s)}{f_1(x_1) \cdots f_s(x_s)} \\ \times \exp\left(-2\pi i \sum_{p=1}^t d_{p1} \left(\sum_{q=1}^s d_{pq}x_q\right) - \pi i \tau \left(\sum_{p=1}^t d_{p1}^2\right) + 2\pi i(n_1 + 1)x_1 + \pi i \tau(n_1 + 1)\right).$$

However, $d_{11} + \dots + d_{t1} - n_1 - 1 \equiv d_{11}^2 + \dots + d_{t1}^2 - n_1 - 1 \pmod{2}$, and the argument of exp in (3-11) is equal to

$$\pi i \tau \left((n_1 + 1) - \sum_{p=1}^t d_{p1}^2 \right) + 2\pi i \left((n_1 + 1) - \sum_{p=1}^t d_{p1}^2 \right) x_1 - 2\pi i \sum_{q=2}^s \left(\sum_{p=1}^t d_{p1} d_{pq} x_q \right).$$

Therefore by (3-8), we have $d_{11} + \dots + d_{t1} - n_1 - 1 \equiv 0 \pmod{2}$, and

$$-2\pi i \sum_{p=1}^t d_{p1} \left(\sum_{q=1}^s d_{pq} x_q \right) - \pi i \tau \left(\sum_{p=1}^t d_{p1}^2 \right) + 2\pi i (n_1 + 1) x_1 + \pi i \tau (n_1 + 1) = 0.$$

Consequently, by (3-11), we obtain that

$$\frac{g(x_1 + \tau, \dots, x_s)}{f_1(x_1 + \tau) \cdots f_s(x_s)} = \frac{g(x_1, \dots, x_s)}{f_1(x_1) \cdots f_s(x_s)}.$$

Similarly, one also obtains that

$$(3-12) \quad \frac{g(x_1, \dots, x_q + \tau, \dots, x_s)}{f_1(x_1) \cdots f_q(x_q + \tau) \cdots f_s(x_s)} = \frac{g(x_1, \dots, x_s)}{f_1(x_1) \cdots f_s(x_s)} \quad \text{for } 1 \leq q \leq s.$$

Therefore from (3-10) and (3-12), we see that ω can be viewed as a meromorphic s -form defined on the product $(\mathbb{C}/\Gamma)^s$, an s -tori which is a compact complex manifold.

$\theta(v, \tau)$ has the lattice points $m + n\tau$, $m, n \in \mathbb{Z}$ as its simple zero points [Chandrasekharan 1985]. We therefore see that ω has pole divisors

$$\{0\} \times (\mathbb{C}/\Gamma)^{s-1}, (\mathbb{C}/\Gamma) \times \{0\} \times (\mathbb{C}/\Gamma)^{s-2}, \dots, (\mathbb{C}/\Gamma)^{s-1} \times \{0\}.$$

So $(0, 0, \dots, 0)$ is the unique intersection point of these polar divisors.

Therefore by the residue theorem on compact complex manifolds, we directly deduce $\text{Res}_{(0,0,\dots,0)} \omega = 0$. By (3-9), we get $\varphi_W(V_{(d_{pq})}) = \text{Res}_{(0,0,\dots,0)} \omega = 0$. \square

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