SUPERSYMMETRIC QFT, SUPER LOOP SPACES AND BISMUT-CHERN CHARACTER

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ABSTRACT. In this paper, we give a quantum interpretation of the Bismut-Chern character form (the loop space lifting of the Chern character form) as well as the Chern character form associated to a complex vector bundle with connection over a smooth manifold in the framework of supersymmetric quantum field theories developed by Stolz and Teichner [22]. We show that the Bismut-Chern character form comes up via a loop-deloop process when one goes from 1|1D theory over a manifold down to a 0|1D theory over its free loop space. Based on our quantum interpretation of the Bismut-Chern character form and Chern character form, we construct Chern character type maps for SUSY QFTs.

CONTENTS

1. INTRODUCTION

Roughly speaking, a d-dimensional quantum field theory in the sense of Atiyah and Segal (see [3], [18]) gives a way to associate a $(d-1)$-dimensional manifold a Hilbert space and to a bordism between such manifolds a trace class operator between the corresponding Hilbert spaces, such that gluing of bordisms corresponds to composing operators. A field theory based on a manifold M is as before except that the $(d-1)$-manifolds and the bordisms between them come equipped with maps to M. Segal [19] suggests that the conformal field theories (CFT) over a manifold M might be able to provide cocycles for the elliptic cohomology of M.

Stolz and Teichner [20] have developed Segal's idea by adding world-sheet locality and supersymmetry (SUSY) into the picture. They conjecture that the space of certain 2D SUSY QFTs gives the spectrum TMF of topological modular forms [12] and for a smooth manifold M, the space of all 2D SUSY QFTs over M moduled out proper concordance relations gives $\mathrm{Ell}^n(M)$, the elliptic cohomology of M. See also [5], [13] and [9] for other contributions on the aim to understand elliptic cohomology in geometric ways.

The K-theory can be considered as a "case study" for elliptic cohomology. Stolz and Teichner [20] have shown that 1-dimensional SUSY QFTs indeed give the spectrum for K(or KO)-theory. To geometrically understand their 1-dimensional theory, one has to study SUSY 1D field theories over parametrizing manifolds. Along this direction, Florin Dumitrescu [7] has developed the theory of super parallel transport. More precisely, given a vector bundle E with a connection ∇^E over a manifold M, let S be a super manifold and $c : S \times \mathbf{R}^{1|1} \to M$ be a S-family super path in M, he is able to construct a bundle map $SP(c,t) : c_{0,0}^* E \to c_{t,\theta}^* E$, which has similar nice properties as the ordinary parallel transport. In chapter 3, we will briefly introduce (and a little bit modify) his construction to obtain a functor from the category of smooth complex vector bundles over M with connections to the category of SUSY 1D QFTs over M. Florin's construction provides examples of 1|1D SUSY QFTs.

Hohnhold, Kreck, Stolz and Teichner have also studied 0|1D SUSY QFTs over M and related them to the theory of differential forms and de Rham cohomology [14]. More precisely, they show that the set of certain 0D SUSY QFT's is canonically mapped to the set of closed differential forms by a bijection. Concordance between the field theories corresponds to the differential forms being cohomologous to each other.

Stolz and Teichner conjecture that there should be a beautiful quantum interpretation of the Chern character in terms of a map from 1|1D QFTs over M to 0|1D QFTs over the free loop space LM, given by crossing with the standard circle.

We confirm their conjecture in this paper. Starting from a vector bundle E with connection ∇^E over M, applying Dumitrescu's construction of super parallel transport [7], one obtains a SUSY 1|1D QFT over M. Choosing a special S-family super path with S the super loop space and c the super evaluation map, we obtain a special super parallel transport. We then use this super parallel transport to construct a differential form on LM via some loop-deloop process. It turns out that this differential form is just the Bismut-Chern character form [4] over LM. Note that Bismut obtained it by extending the ideas of Witten and Atiyah [2] of interpreting the index of the Dirac operator on the spin complex of a spin manifold as a paring of $1 \in \Omega(LM)$ with certain equivariant current μ_D on the loop space to the situation of twisted spin complex. The restriction of the Bismut-Chern character form to M, the S^1 fixed point set on LM, is the ordinary Chern character form associated to (E, ∇^E). See also [10] for a representation of the Bismut-Chern character form in the cyclic bar complex.

Our construction of the Bismut-Chern character form shows that it actually represents a loop-deloop process in the framework of supersymmetric quantum field theories (see Theorem 4.6 and Section 4.4 in this paper for details). It provides us a new way to understand the Bismut-Chern character form over LM as well as the Chern character form over M. The character form can be considered as a phenomena when one goes from 1|1D field theories to 0|1D field theories. We hope

to understand this kind of phenomena when the dimensions of the field theories are higher. What we did actually gives the Bismut-Chern character form as well as the Chern character form a quantum interpretation (Theorem 4.6 and Theorem 4.7). Based on this quantum interpretation and the 0-dimensional theory in [14], we are able to construct Chern character type maps for SUSY QFTs.

The paper is organized as follows. In section 2, we supply necessary preliminary knowledge on super geometry. In section 3, we briefly introduce SUSY QFTs developed by Stolz and Teichner as well as Dumitrescu's super parallel transport and its slight modification. We at last construct the Bismut-Chern character as well the Chern character and then construct Chern character type maps in the framework of SUSY QFTs in section 4.

2. Preliminaries on Super Geometry

In this section, we briefly survey the theory of super manifolds. We refer interested readers to the standard references [8] and [24] for details.

A *supermanifold* M of dimension $(m|n)$ is a pair $(|M|, \mathcal{O}_M)$, consisting of
1) an m dimensional smooth manifold $|M|$, the so called underlying manifold, or the reduced manifold;
2) a sheaf of $\mathbf{Z}_2$ graded commutative algebras on $|M|$, the "structure sheaf" $\mathcal{O}_M$, such that for any open set $U \subset |M|$,

$$\mathcal{O}_M(U) \cong C^\infty(U) \otimes \Lambda[\theta_1, \theta_2, \cdots, \theta_n].$$

We refer to global sections of the structure sheaf of M as elements in $C^\infty(M) := \Gamma(\mathcal{O}_M)$.

A *morphism between two super manifolds* $M = (|M|, \mathcal{O}_M), N = (|N|, \mathcal{O}_N)$ is a pair $(f, f^\natural)$ such that

$$f : |M| \to |N|$$

is a smooth map between the underlying manifolds and

$$f^\natural : \mathcal{O}_N \to f_* \mathcal{O}_M$$

is a morphism of sheaves.

Let **SM** denote the category of supermanifolds and **SM**(M, N) denote the morphisms between M and N. A basic fact in supergeometry is that maps between two supermanifolds are uniquely determined by the map induced on global sections (see [16], page 208). So we quite often write a map between sheaves as just the map induced on their global sections.

Let E be a vector bundle over an ordinary manifold M. One can canonically construct a super manifold $\Pi E = (M, \mathcal{O}_{\Pi E})$ where $\mathcal{O}_{\Pi E}$ is the sheaf of sections of ΛE^*. This construction provides an important source of supermanifolds. This construction also defines a functor:

$$S : \mathbf{VB} \to \mathbf{SM} : E \mapsto \Pi E.$$

This functor actually induces a bijection on the isomorphism classes of objects however it does not give an equivalence of categories because there are more more morphisms in **SM** then in **VB** (cf. [7]).

Since sheaves are generally hard to work with, one often thinks of super manifolds in terms of their "S-points". To be precise, instead of M itself, one considers the morphisms sets **SM**(S, M), where S varies over all super manifolds S. One can

think of an S-point as a family of points of M parametrized by S. It's not hard to see that M determines a contravariant functor:

$$\mathbf{SM} \to \mathbf{Sets} : S \mapsto \mathbf{SM}(S, M).$$

It's called the *functor of points* of M. A map $f : M \to N$ of super manifolds determines a natural transformation $\mathbf{SM}(\cdot, M) \to \mathbf{SM}(\cdot, N)$. The Yoneda's lemma (cf. [17]) tells us that

$$\mathbf{SM} \to \mathcal{F}(\mathbf{SM}^{op}, \mathbf{Sets}), \ M \mapsto \mathbf{SM}(\cdot, M)$$

is an embedding of categories. This means that to give a map $M \to N$ amounts to give a natural transformation of functors $\mathbf{SM}(-, M) \to \mathbf{SM}(-, N)$. From this point of view, one can think of supermanifold M as a representable functor $\mathbf{SM} \to \mathbf{Sets}$. Such a functor determines M uniquely up to isomorphism. The *product $M \times N$* can be interpreted as the supermanifold representing the functor $\mathbf{SM}(\cdot, M) \times \mathbf{SM}(\cdot, N)$.

An arbitrary contravariant functor $\mathbf{SM} \to \mathbf{Sets}$ will be called a *generalized supermanifold*. Therefore Yoneda's lemma says $\mathbf{SM}$ embeds faithfully into the category $\mathbf{GSM}$ of generalized supermanifolds.

Given two supermanifolds M, N, consider the generalized supermanifold

$$\underline{\mathbf{SM}}(M, N) : \mathbf{SM} \to \mathbf{Sets} : S \to \mathbf{SM}(S \times M, N).$$

If $\underline{\mathbf{SM}}(M, N)$ is an ordinary supermanifold, then by definition we have the following adjunction formula

$$(2.1) \qquad\qquad \mathbf{SM}(S, \underline{\mathbf{SM}}(M, N)) \cong \mathbf{SM}(S \times M, N).$$

Let M and N be two supermanifolds, define the following "evaluation" map

$$(2.2) \qquad\qquad ev : \underline{\mathbf{SM}}(M, N) \times M \to N$$

via it's S-points. That means that for any supermanifold S, define

$$(2.3) \qquad\qquad ev_S : \mathbf{SM}(S \times M, N) \times \mathbf{SM}(S, M) \to \mathbf{SM}(S, N),$$

$$(2.4) \qquad\qquad ev_S(\varphi, m) = \varphi \circ (1 \times m) \circ \Delta \in \mathbf{SM}(S, N).$$

Remark 2.1. *Although supermanifolds have rigorous mathematical foundations now, in practice, physicists often freely and correctly treat supermanifolds as ordinary manifolds with " even and odd variables" without always coming back to the rigorous mathematical definitions.*

On a supermanifold, one can define *tangent sheaf* and *tangent vectors* on it. The tangent sheaf $\mathcal{TM}$ is defined as the sheaf of graded derivations of $\mathcal{O}_M$, i.e. for $U \subseteq |M|$

$$\mathcal{TM}(U) = \{X : \mathcal{O}(U) \to \mathcal{O}(U) \text{ linear } : X(fg) = X(f)g + (-1)^{p(X)p(f)} f X(g)\}.$$

Here $p(X) = 0$ or 1 according to whether X is even, respectively odd vector field on U, and similarly $p(f) = 0$ or 1, for f even, respectively odd, function on M. $\mathcal{TM}$ is then a locally free $\mathcal{O}_M$-module of rank (p, q) the dimension of the supermanifold M. Sections of $\mathcal{TM}$ are the vector fields on M. For X and Y vector fields on M, define their *Lie bracket* $[X, Y]$ by

$$[X, Y](f) = X(Y(f)) - (-1)^{p(X)p(Y)} Y(X(f))$$

for $f \in C^\infty(M) = \mathcal{O}_M(|M|)$.

For example, on $\mathbf{R}^{1|1}$, there are two canonical vector fields $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$ and $Q = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t}$. It's not hard to verify that

$$(2.5) \qquad D^2 = \frac{1}{2}[D,D] = \frac{\partial}{\partial t}, \ Q^2 = \frac{1}{2}[Q,Q] = -\frac{\partial}{\partial t}, \ [D,Q] = 0.$$

Dually one can also define *cotangent sheaf* and *exterior cotangent sheaf* on a supermanifold. Define the cotangent sheaf $\Omega^1(M)$ to be the dual of the tangent sheaf $\mathcal{T}\mathcal{M}$. Sections of $\Omega^1(M)$ are called differential one forms. Let $\langle \ , \ \rangle : \mathcal{T}\mathcal{M} \times \Omega^1(M) \to \mathcal{O}_M$ denote the duality pairing between vector fields and 1-forms. Define the *exterior derivative* $d : \mathcal{O}_M \to \Omega^1(M)$ by

$$\langle X, df \rangle = X(f), \ \text{for } X \in \mathcal{T}\mathcal{M}, f \in \mathcal{O}_M.$$

Let $\Omega^*(M) = \Lambda^*\Omega^1(M)$ be the exterior cotangent sheaf on M, whose sections are called differential forms on M. d extends uniquely to a degree one derivation $d : \Omega^*(M) \to \Omega^*(M)$ by requiring that

$$d^2 = 0,$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta, \ \text{for } \alpha \in \Omega^p(M).$$

The readers can consult [8] for more details.

One has the following proposition (cf. [7], [14] for a proof)

Proposition 2.1. *Let M be an ordinary manifold. Then we can identify*

$$(2.6) \qquad \underline{\mathbf{SM}}(\mathbf{R}^{0|1}, M) \cong \Pi TM.$$

Proof. We want to show that we have isomorphisms

$$\Psi_S : \mathbf{SM}(S \times \mathbf{R}^{0|1}, M) \to \mathbf{SM}(S, \Pi TM),$$

natural in S, where S is an arbitrary supermanifold. The left hand side is the set of grading preserving maps of $\mathbf{Z}_2$-algebras

$$\varphi : C^\infty(M) \to C^\infty(S \times \mathbf{R}^{0|1}) = C^\infty(S)[\theta].$$

If we write $\varphi(f) = \varphi_1(f) + \theta\varphi_2(f)$, for $f \in C^\infty(M)$, then the fact that $\varphi(fg) = \varphi(f)\varphi(g)$ is equivalent to the following conditions:

$$\varphi_1(fg) = \varphi_1(f)\varphi_1(g), \quad \varphi_2(fg) = \varphi_2(f)\varphi_1(g) + (-1)^{p(f)}\varphi_1(f)\varphi_2(g).$$

The first condition is equivalent to $\varphi_1 = a^\sharp$, for some $a : S \to M$. The second condition tells us that φ_2 is an odd tangent vector at $a \in M(S)$, i.e. $\varphi_2 = X_a \in TM_a$. Therefore the left hand side is

$$\mathbf{SM}(S \times \mathbf{R}^{0|1}, M) = \{\text{pairs } (a, X_a) | a \in M(S), X_a \in TM_a, X_a \text{ odd}\}.$$

The right hand side $\mathbf{SM}(S, \Pi TM)$ is the set of $\mathbf{Z}_2$-graded algebra maps $\Omega^*(M) \to C^\infty(S)$. Such maps are determined by their restriction to 0-forms and 1-forms. Define then $\Psi_S(a, X_a)$ to be the map $S \to \Pi TM$ determined by defining it on functions $f \in C^\infty(M)$ by $a^\sharp(f) \in C^\infty(S)$ and on forms ω by $\langle X_a, \omega \rangle \in C^\infty(S)$. One can easily check that Ψ_S is well-defined, bijective and natural in S. $\qquad \square$

3. SUSY QFTs

In this section, we give a very rough introduction to formulations of SUSY QFT's over manifolds developed by Stolz and Teichner ([14], [15], [21], [22]) as well as Dumitrescu's construction of super parallel transport [7] and our slight modifications of his construction. Such formulations of supersymmetric quantum field theories are cocycles for generalized cohomology theories via SUSY QFTs. The purpose of this section is to put our construction of Bismut-Chern character in the next section into this framework of SUSY QFT's.

Quantum field theories in the sense of Atiyah-Segal is a functor from a suitable bordism category to a category of locally convex topological vector spaces satisfying certain axioms. Following Stolz and Teichner, we will briefly first introduce the rendition of Atiyah-Segal's quantum field theories and further enrich it by adding smoothness and supersymmetry. For complete details of all those categories and functors, see [22].

3.1. **Preliminary Definition of QFTs.** Let's first introduce relevant categories.

Definition 3.1. (The Riemannian spin bordism category $\mathbf{RB}^d$) *The objects and morphisms in the category RB^d are as follows:*
objects *are quadruples $(U,\ Y,\ U^-,\ U^+)$, where*

> *U is a Riemaannnian spin manifold of dimension d (typically not closed);*
>
> *Y is a closed codimension 1 smooth submanifold of U;*
>
> *$U^{\pm}$ are disjoint open subsets of $U \setminus Y$ whose union is $U \setminus Y$. Y is r required to contain in the closure of both $U^{\pm}$ (this ensures that $U^{\pm}$ are collars of Y).*

We will often suppress $(U,\ Y,\ U^-,\ U^+)$ in the notation and just write Y instead of $(U,\ Y,\ U^-,\ U^+)$.
morphisms *from Y_1 to Y_2 are equivalence classes of Riemannian spin bordisms from Y_1 to Y_2; here Riemannian spin bordism is a triple $(\Sigma, \iota_1, \iota_2)$, where*

> *Σ is a Riemannian spin manifold of dimension d (not necessarily closed), and*
>
> *$\iota_1 : V_1 \hookrightarrow \Sigma$ and $\iota_2 : V_2 \hookrightarrow \Sigma$ are isometric spin embeddings, where $V_k \subset U_k$ for k=1,2 is some open neighborhood of $Y_k \subset U_k$.*

We define $V_k^{\pm} := U_k^{\pm} \cap V_k$ and require that

> *• the sets $\iota_1(V_1^+ \cup Y_1)$ and $\iota_1(V_2^- \cup Y_2)$ are disjoint and*
>
> *• $\Sigma \setminus (\iota_1(V_1^+) \cup \iota_2(V_2^-))$ is compact.*

Note that $\Sigma \setminus (\iota_1(V_1^+) \cup \iota_2(V_2^-))$ is a compact manifold with boundary $\iota_1(Y_1) \coprod \iota_2(Y_2)$; i.e., it's a bordism between Y_1 and Y_2 in the usual sense. Now suppose that $(\sum, \iota_1, \iota_2)$ and $(\Sigma', \iota'_1, \iota'_2)$ are two Riemaniann spin bordisms from Y_1 to Y_2 with $V_1 \subset V'_1, V_2 \subset V'_2$ and that there is a spin isometry F that makes the following diagram commutative

$$
\begin{array}{ccccc}
V_2 & \xrightarrow{\ \iota_2\ } & \Sigma & \xleftarrow{\ \iota_1\ } & V_1 \\
{\scriptstyle i}\downarrow & & {\scriptstyle F}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle i} \\
V'_2 & \xrightarrow{\ \iota'_2\ } & \Sigma' & \xleftarrow{\ \iota'_1\ } & V'_1
\end{array}
$$

Then we declare that $(\Sigma, \iota_1, \iota_2)$ as equivalent to $(\Sigma', \iota_1', \iota_2')$. A morphism from Y_1 to Y_2 is an equivalence class of Riemannian spin bordisms with this equivalence relation.

composition *of Riemannian spin bordisms is given by gluing; more precisely, let $(\Sigma', \iota_1', \iota_2')$ be a Riemannian spin bordism from Y_1 to Y_2 and $(\Sigma, \iota_2, \iota_3)$ a Riemannian spin bordism from Y_2 to Y_3. Without loss of generality, we can assume that the domains of the isometries ι_2' and ι_2 agree; suppose that $V_2 \subset U_2$ is this common domain. Then identifying $\iota_2'(V_2) \subset \Sigma'$ with $\iota_2(V_2) \subset \Sigma$ via the isometry $\iota_2 \circ (\iota_2')^{-1}$ gives the Riemaniann spin manifold $\Sigma'' := \Sigma \cup_{V_2} \Sigma'$.*

There are some additional structures on the bordism category RB^d. They are

> **symmetric monoidal structure**: disjoint union gives RB^d the structure of a symmetric monoidal category; the unit object is given by the empty $(d-1)$-manifold.

> **the anti-involution** $^\vee$ On objects the anti-involution $^\vee$ is defined by interchanging U^+ and U^- (which can be thought of as flipping the orientation of the normal bundle to Y in U). If $(\Sigma, \iota_1, \iota_2)$ is a Riemannian bordism from Y_1 to Y_2, then $(\Sigma, \iota_1, \iota_2)^\vee = (\Sigma, \iota_2, \iota_1)$ is a Riemannian bordism from $Y_2^\vee$ to $Y_1^\vee$.

> **the involution** $^-$: Replacing the spin structure on the bicollars U as well as the bordism Σ by their opposite defines an involution $^-: \mathrm{RB}^d \to \mathrm{RB}^d$.

The other categories involved in QFT are TV and $\mathrm{TV}^\pm$.

Definition 3.2. (the category TV)
objects *are $\mathbf{Z}/2$-graded locally convex vector spaces;*
morphisms *are graded preserving continuous linear maps.*

Definition 3.3. (the category $\mathrm{TV}^\pm$)
objects *are triples $V = (V^+, V^-, \mu_V)$, where $V^\pm$ are locally convex vector spaces, and $\mu_V : V^- \otimes V^+ \to \mathbf{C}$ is a continuous linear map. Here and in the following, $\otimes$ is the projective tensor product.*
morphisms *from $V = (V^+, V^-, \mu_V)$ to $W = (W^+, W^-, \mu_W)$ are pairs $T = (T^+ : V^+ \to W^+, T^- : W^- \to V^-)$ of continuous linear maps, which are dual to each other in the sense that*

$$\mu_V(T^- w^- \otimes v^+) = \mu_W(w^- \otimes T^+ v^+), \forall v^+ \in V^+, w^- \in W^-.$$

composition *Let $S = (S^+, S^-) : V_1 \to V_2$ and $T = (T^+, T^-) : V_2 \to V_3$ be morphisms in TV. Then their composition $T \circ S : V_1 \to V_3$ is given by*

$$T \circ S := (T^+ \circ S^+, S^- \circ T^-).$$

There are also some additional structures on the category $\mathrm{TV}^\pm$. They are

> **symmetric monoidal structure**: The tensor product of two objects $V = (V^+, V^-, \mu_V)$ and $W = (W^+, W^-, \mu_W)$ is defined as

$$V \otimes W := (V^+ \otimes W^+, V^- \otimes W^-, \mu_{V \otimes W}),$$

> where $V^\pm \otimes W^\pm$ is the projective tensor product and $\mu_{V \otimes W}$ is given by the composition of the usual graded symmetry isomorphism

$$(V^- \otimes W^-) \otimes (V^+ \otimes W^+) \cong V^- \otimes V^+ \otimes W^- \otimes W^+$$

and the linear map

$$V^- \otimes V^+ \otimes W^- \otimes W^+ \xrightarrow{\mu_V \otimes \mu_W} \mathbf{C} \times \mathbf{C} = \mathbf{C}.$$

On morphisms, we define $(T^+, T^-) \otimes (S^+, S^-) := (T^+ \otimes S^+, T^- \otimes S^-)$.

the anti-involution $^\vee$ On objects On objects, it is given by $(V^+, V^-, \mu_V)^\vee :=$
$(V^-, V^+, \mu_V^\vee)$, where $\mu_V^\vee$ is the composition

$$V^+ \otimes V^- \xrightarrow{\cong} V^- \otimes V^+ \xrightarrow{\mu_V} \mathbf{C}$$

of the graded symmetry isomorphism and μ_V. On morphisms the anti-
involution is given by $(T^+, T^-)^\vee := (T^-, T^+)$. We note that $V^\vee \otimes W^\vee =$
$(V \otimes W)^\vee$.

the involution $^-$: If $V = (V^+, V^-, \mu_V)$ is an object of TV$^\pm$, then $\overline{V}$ is
given by complex conjugate vector spaces $\overline{V}^+, \overline{V}^-$ and the paring

$$\overline{V}^- \otimes \overline{V}^+ = \overline{V^- \otimes V^+} \xrightarrow{\overline{\mu}_V} \overline{\mathbf{C}} \cong \mathbf{C} .$$

On morphisms, it is given by $\overline{(T^+, T^-)} = (\overline{T}^+, \overline{T}^-)$, where (as for μ_V) $\overline{T}^\pm$
is the same map as $T^\pm$, but regarded as a complex linear map between the
complex conjugate vector spaces.

There is also a functor TV$\to$TV$^\pm$; on objects, it sends a locally convex vector
space to (V, V', μ), where $\mu : V' \otimes V \to \mathbf{C}$ is the natural paring, V' is the continuous
dual of V. On morphisms, it sends a linear map $T : V \to W$ to the pair (T, T'),
where $T' : W' \to V'$ is the continuous dual to T. This is not a *monoidal* functor due
to the incompatibility of $\otimes$ and $'$ mentioned earlier. However one obtains monoidal
functor if restricting to *finite dimensional* vector spaces. Using this functor, we
can interpret finite dimensional vector spaces and linear maps as objects resp.
morphisms in TV$^\pm$.

With the above preparations, we can give the preliminary definition of QFT.

Definition 3.4. *A quantum field theory of dimensional d is a symmetric monoidal
functor*

$$F^d : \mathrm{RB}^d \to \mathrm{TV}^\pm$$

which is compatible with the involution $^-$ and the anti-involution $^\vee$.

3.2. **QFTs as Smooth Functors.** Now let's enrich the above definition of QFT
by adding **smoothness** into the picture.

Definition 3.5. (Smooth categories and functors) *A smooth category $\mathfrak{C}$ is a
functor*

$$\mathfrak{C} : \mathrm{MAN}^{op} \to \mathrm{CAT}$$

*from the category of smooth manifolds to the category of categories. If S is a
manifold, we will write $\mathfrak{C}_S$ for the category $\mathfrak{C}(S)$; if $f : S' \to S$ is a smooth map,
we will write $f^* : \mathfrak{C}_S \to \mathfrak{C}_{S'}$ for the corresponding functor $\mathfrak{C}(f)$.*

*If $\mathfrak{C}$ and $\mathfrak{D}$ are smooth categories, a smooth functor $\mathfrak{F}$ from $\mathfrak{C}$ to $\mathfrak{D}$ is a natural
transformation. For a manifold S, we will write $\mathfrak{F}_S : \mathfrak{C}_S \to \mathfrak{D}_S$ for the correspond-
ing functor.*

In the following, we proceed to define the smooth version of the above categories and functors involved in the above definition of QFT. Let's first introduce the notion of quasi bundles (the purpose of this concept is to guarantee $(fh)^* = g^*f^*$ for pull backs).

Definition 3.6. *A quasi bundle over a smooth manifold S is a pair (h, V), where $h : S \to T$ is a smooth map and $V \to T$ is a smooth , locally trivial bundle over T. If (h', V') is another quasi bundle over S, a map from (h, V) to (h', V') is a smooth bundle map $F : h^*V \to (h')^*V'$.*

A smooth map $f : S' \to S$ induces a contravariant functor

$$f^* : \mathrm{QBUN}(S) \to \mathrm{QBUN}(S')$$

from the category of quasi bundles over S to those over S'. It is defined by

$$f^*(h, V) = (f \circ h, V)$$

on objects; on morphisms it is given by the usual pullback via f. In particular, the functor $(fg)^*$ is equal to $g^* \circ f^*$. Note that the category of quasi bundles over X is equivalent to the category of bundles over X, sending (h, V) to h^*V provides the equivalence.

The S-family version of the RB^d is the following.

Definition 3.7. (The category RB_S^d) *Let S be a smooth manifold. We want to define the category RB_S^d of S-families of Riemannian bordisms of dimension d in such a way that RB_{pt}^d agree with the category RB^d of the above definition 3.1. In that definition, objects and morphisms were defined in terms of the category $Riem^d$ whose objects are Riemannian spin manifolds of dimension d and morphisms are isometric spin embeddings. Here we replace $Riem^d$ by $Riem_S^d$, whose objects (resp. morphisms) are S-families of objects (resp. morphisms) of $Riem^d$. More precisely,*

- *an object of $Riem_S^d$ is a smooth quasi bundle $U \to S$ with d-dimensional fibers, euqipped with a fiberwise Riemannian metric and spin structure (i.e. a spin structure on the vertical tangent bundle).*

- *A morphism from U to U' is a smooth quasi bundle map $f : U \to U'$ preserving the fiberwise Riemannian metric and spin structure.*

A smooth map $f : S' \to S$ induces a pullback functor $f^* : \mathrm{RB}_S^d \to \mathrm{RB}_{S'}^d$, such that $(fg)^* = g^*f^*$. The fiberwise disjoint union gives RB* the structure of a symmetric monoidal category; the involution $^-$, the anti-involution $^\vee$ and the adjoint transformation generalize from RB^d to RB$_S^d$.

Definition 3.8. (The smooth category $\mathfrak{RB}^d$) *The smooth category $\mathfrak{RB}^d$ is the functor*

$$\mathfrak{RB}^d : \mathrm{MAN}^{op} \to \mathrm{CAT}$$

which sends a smooth manifold S to the category RB_S and a smooth map $f : S \to S'$ to the pullback functor $f^* : RB_S^d \to RB_{S'}^d$.

The S-family version of the TV$^\pm$ is the following.

Definition 3.9. (The category $\mathrm{TV}_S^\pm$) *Let S be a smooth manifold. We want to define the category $TV_S^\pm$ of S-families of (pair) of locally convex vector space in such a way that $TV_{pt}^\pm$ agree with the category $TV^\pm$ of definition 3.3. Just replace locally convex vector spaces by quasi bundles of locally convex vector spaces over S and*

continuous linear maps by smooth bundle maps. In particular, objects are triples
$V = (V^+, V^-, \mu_V)$, *where* $V^\pm$ *are quasi bundles of locally convex vector spaces over*
S, *and* $\mu_V : V^- \otimes V^+ \to \mathbf{C}_S$ *is a smooth bundle map. Here the tensor product of*
quasi bundles over S *is given by*

$$(h : S \to T, V) \otimes (h' : S \to T', V') := (h \times h' : S \to T \times T', p_1^* V \otimes p_2^* V'),$$

where $p_1^* V \otimes p_2^* V' \to T \otimes T'$ *is a fiberwise (projective) tensor product and* p_1 *(res.*
p_2*) is the projection onto the first (res. second) factor. Moreover,* $\mathbf{C}_S$ *is the quasi*
bundle $(p : S \to pt, pt \times \mathbf{C})$*; replacing* $\mathbf{C}$ *by any locally convex vector space* V *gives*
a quasi bundle V_S *over* S. *We note that* C_S *is the unit for the tensor product of*
quasi bundles over S.

A smooth map $f : S' \to S$ induces a functor

$$f^* : \mathrm{TV}_S^\pm \to \mathrm{TV}_{S'}^\pm$$

via pullback of quasi bundle. If $g : S'' \to S'$ is a smooth map, then the func-
tor $(gf)^*$ is equal to $g^* f^*$. The additional structures for the category $\mathrm{TV}^\pm$ (the
symmetric monoidal structure, the (anti-)involution and the adjunction transfor-
mation) generate in a straightforward way to the category $\mathrm{TV}_S^\pm$. These structures
are compatible with pullback functor f^*.

Definition 3.10. (The smooth category $\mathfrak{TV}^\pm$**)** *The smooth category* $\mathfrak{TV}^\pm$ *is*
the functor

$$\mathfrak{TV}^\pm : \mathrm{MAN}^{op} \to \mathrm{CAT}$$

which sends a smooth manifold S *to the category* $TV_S^\pm$ *and a smooth map* $f : S' \to S$
to the pullback functor $f^* : \mathrm{TV}_S^\pm \to \mathrm{TV}_{S'}^\pm$.

Definition 3.11. *A quantum field theory of dimension* d *is a natural transforma-*
tion

$$\mathfrak{F}^d : \mathfrak{RB}^d \to \mathfrak{TV}^\pm,$$

such that for any $S \in \mathrm{MAN}$, $\mathfrak{F}_S^d$ *is a symmetric monoidal functor, which is com-*
patible with the involution $^-$ *and the anti-involution* $^\vee$.

There is also a relative version of the above story. Let M be a smooth manifold.
Define $\mathrm{RB}^d(M)$ to be the category with objects $(U, Y, U^+, U^-, \alpha : U \to M)$, where
$\alpha : U \to M$ is a smooth map; morphisms $(\Sigma, \iota_1, \iota_2, \beta : \Sigma \to M)$, where $\beta : \Sigma \to M$ is
a smooth map. Those relative maps satisfy natural commutativity conditions when
we do bordisms. Similarly, one can define family versions $\mathrm{RB}_S^d(M)$ and $\mathfrak{RB}^d(M)$.
We define a **quantum field theory of dimension** d **over** M to be a natural
transformation

$$\mathfrak{F}^d(M) : \mathfrak{RB}^d(M) \to \mathfrak{TV}^\pm,$$

such that for any $S \in \mathrm{MAN}$, $\mathfrak{F}_S^d(M)$ is a symmetric monoidal functor, which is
compatible with the involution $^-$ and the anti-involution $^\vee$.

3.3. **SUSY QFTs and Examples.** Our next step is to enrich our definitions of
QFT described above by adding supersymmetry into the picture.

3.3.1. *Definitions of SUSY QFTs.* In the following sections, we will always work with **complex super manifolds** or **cs-manifolds** (cf. [8]). A **cs-manifold of dimension** $n|m$ is a topological space M_{red} together with a sheaf $\mathcal{O}_M$ of graded commutative algebras over complex numbers, which is locally isomorphic to $\mathbf{R}_{cs}^{n|m} := (\mathbf{R}^{n|m}, \mathcal{O}^{n|m} \otimes \mathbf{C})$. If $M = (M_{red}, \mathcal{O})$ is a cs-manifold, we denote by $\overline{M} := (M_{red}, \overline{\mathcal{O}})$ the complex conjugate cs-manifold, where $\overline{\mathcal{O}}(U) = \overline{\mathcal{O}(U)}$. A super manifold of dimension $n|m$ leads to a cs-manifold by complexifying its structure sheaf.

To give the definition of SUSY QFT, we will also have to define the so called **super Riemannian structures** on cs-manifolds dimension $d|1$. However we will only describe super Riemannian structures on cs-manifolds of dimension $1|1$ for the purpose for this paper. For general definitions and their physics motivations, the readers are referred to [22].

Definition 3.12. *Let M be a cs-manifold of dimension $1|1$. A super Riemannian structure on M is given by a collection of pair (U_i, D_i) indexed by some set I, where*
1) the U_i's are open subsets of M_{red} whose union is all of M_{red}.
2) the D_i's are sections of the tangent sheaf TM restricted to U satisfying

- *the reduction of the even vector field D_i^2 gives a nowhere vanishing (complex) vector field $(D_i^2)_{red}$ on U_i.*
- *the complex conjugate of $(D_i^2)_{red}$ is $-(D_i^2)_{red}$.*

3) the restrictions of D_i and D_j to $U_i \cap U_j$ are equal up to a possible sign.
Tow such collections define the same structure if their union is again such a structure.

Let S be a cs-manifold. Let's define the category $\mathrm{RB}_S^{d|1}$, which is a super analogy to Definition 3.7 of RB_S^d.

Definition 3.13. (The category $\mathrm{RB}_S^{d|1}$) *The objects and morphisms are the followings:*
objects *of $\mathrm{RB}_S^{d|1}$ are smooth quasi bundles of cs-manifolds $U \to S$ with fibers of dimension $d|1$ which are equipped with a fiberwise super Riemannian struture.*
morphisms *from U to U' are embeddings $U \hookrightarrow U'$ of cs-manifolds that are bundle maps (i.e. commutes with the projection S) and preserve the fiberwise Riemannian structure.*

Let's present some examples of objects and morphisms.
the super point $\mathrm{spt}_S \in \mathrm{RB}_S^{1|1}$. The quadruple

$$\mathrm{spt} := (U, Y, U^+, U^-) = (S \times \mathbf{R}_{cs}^{1|1}, S \times \mathbf{R}_{cs}^{0|1}, S \times \mathbf{R}_{cs,-}^{1|1}, S \times \mathbf{R}_{cs,+}^{1|1})$$

is an object $\mathrm{RB}_S^{1|1}$; here $\mathbf{R}_{cs,\pm}^{1|1} \subset \mathbf{R}_{cs}^{1|1}$ is the super submanifold whose reduced manifold is $\mathbf{R}_{\pm}^1 \subset \mathbf{R}^1$.
the super interval $I_l^{1|1} \in \mathrm{RB}_S^{1|1}(\mathrm{spt}, \mathrm{spt})$. For $l \in \mathbf{R}_{cs,+}^{1|1}(S)$ the pair of bundle maps

$$U = S \times \mathbf{R}_{cs}^{1|1} \xrightarrow{\ \mathrm{id}\ } \Sigma = S \times \mathbf{R}_{cs}^{1|1} \xleftarrow{\ l\ } U = S \times \mathbf{R}_{cs}^{1|1} \ ,$$

is a super Riemannian bordism from spt_S to spt_S. We will use the notation $I_l^{1|1}$ for this morphism. The l in above diagram is actually $(1 \times \mu) \circ (1 \times l \times 1) \circ (\Delta \times 1)$,

where μ is the product for a standard group structure on $\mathbf{R}_{cs}^{1|1}$:

$$\mu : \mathbf{R}_{cs}^{1|1} \times \mathbf{R}_{cs}^{1|1} \to \mathbf{R}_{cs}^{1|1}, \ ((t_1, \theta_1), (t_2, \theta_2)) \mapsto (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2).$$

Definition 3.14. (The super smooth category $\mathfrak{SRB}^{d|1}$) *The smooth category $\mathfrak{SRB}^{d|1}$ is the functor*

$$\mathfrak{SRB}^{d|1} : \mathrm{cs} - \mathrm{SMAN}^{op} \to \mathrm{CAT}$$

which sends a cs-manifold S to the category $RB_S^{d|1}$ and $f : S \to S'$ a smooth map between cs-manifolds to the pullback functor $f^* : RB_S^{d|1} \to RB_{S'}^{d|1}$.

A *vector bundle* over a cs-manifold S is a sheaf of modules over the structure sheaf $\mathcal{O}_S$ which is locally isomorphic to the (projective, graded) tensor product $\mathcal{O}_S \otimes V$, where V is a $\mathbf{Z}/2$-graded locally convex vector space. These modules are equipped with a locally convex topology and the local isomorphism is bi-continuous. Here $C^\infty(S)$ comes with its usual Frechet topology. A *vector bundle map* is a continuous map between these sheaves.

One can very similarly define the category $\mathrm{TV}_S^\pm$ as in definition 3.9 by using the notion of quasi vector bundles over S and consequently define the super smooth category $\mathfrak{STV}^\pm$.

Definition 3.15. (The super smooth category $\mathfrak{STV}^\pm$) *The super smooth category $\mathfrak{STV}^\pm$ is the functor*

$$\mathfrak{STV}^\pm : \mathrm{cs} - \mathrm{SMAN}^{op} \to \mathrm{CAT}$$

which sends a cs-manifold S to $TV_S^\pm$ and a smooth map $f' : S' \to S$ to the pull-back functor $f^* : TV_S^\pm \to TV_{S'}^\pm$.

Definition 3.16. *A super symmetric quantum field theory of dimension $d|1$ is a natural transformation*

$$\mathfrak{SF}^{d|1} : \mathfrak{SRB}^{d|1} \to \mathfrak{STV}^\pm$$

such that for any $S \in \mathrm{cs} - \mathrm{SMAN}$, $\mathfrak{SF}_S^{d|1}$ is a symmetric monoidal functor, which is compatible with the involution $^-$ and the anti-involution $^\vee$.

Let M be a smooth manifold. We can also similarly as what we did after definition 3.11 define relevant categories relative to M and define a **super symmetric quantum field theory of dimension $d|1$ over M** to be a natural transformation

$$\mathfrak{SF}^{d|1}(M) : \mathfrak{SRB}^{d|1}(M) \to \mathfrak{STV}^\pm,$$

such that for any $S \in \mathrm{cs} - \mathrm{SMAN}$, $\mathfrak{SF}_S^{d|1}(M)$ is a symmetric monoidal functor, which is compatible with the involution $^-$ and the anti-involution $^\vee$.

In the following two subsections, we talk about some examples of SUSY QFTs arising from classical geometric objects.

3.3.2. *Super Parallel Transport.* Let E be a complex vector bundle over M and ∇^E be a connection over E. Dumitrescu has introduced the super parallel transport as follows. His construction with some modifications can be viewed as an example of the 1|1D QFT over M defined above.

Let $c : S \times \mathbf{R}^{1|1} \to M$ be a family of *supercurves* parametrized by S in M. Let c^*E and $c^*\nabla^E$ be the pull back of the vector bundle and connection to $S \times \mathbf{R}^{1|1}$ respectively. The vector field $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$ extends trivially to $S \times \mathbf{R}^{1|1}$. Consider

the derivation $(c^*\nabla^E)_D : \Gamma(c^*E) \to \Gamma(c^*E)$. An element ψ of $\Gamma(c^*E)$ is called a *section of E along c* and called super parallel if moreover it satisfies

$$(3.1) \qquad (c^*\nabla^E)_D\psi = 0.$$

In local coordinates, one can think of the above equation as a so called (by Dumitrescu) *half-order differential equation*. He named this because of two reasons: first $D^2 = \frac{\partial}{\partial t}$; secondly, for $2n$ unknowns functions we only need n values as initial data.

Theorem 3.1. *(Dumitrescu, [7, Prop. 4.1]) Let S be a supermanifold and $c : S \times \mathbf{R}^{1|1} \to M$ be a family of supercurves parametrized by S in M. Let $\psi_0 \in \Gamma(c_{0,0}^*E)$ be a section of E along $c_{0,0} : S \to S \times \mathbf{R}^{1|1} \to M$, with the first map the standard inclusion $i_{0,0} : S \to S \times \mathbf{R}^{1|1}$. Then there exists a unique super parallel section ψ of E along c such that $\psi(0,0) = \psi_0$.*

This theorem is proved in [7] by writing the equation (3.3) in local coordinates and reducing it to a system of first ordinary differential equations.

Let S be a supermanifold and $(t,\theta) \in \mathbf{R}_+^{1|1}(S)$ be an S-point of $\mathbf{R}_+^{1|1}$. Consider the super triplet

$$S \xrightarrow{\ i_{(0,0)}\ } S \times \mathbf{R}_+^{1|1} \xleftarrow{\ i_{(t,\theta)}\ } S$$

with $i_{(0,0)}(s) = (s,0,0)$ and $i_{(t,\theta)}(s) = (s,t(s),\theta(s))$. Denote this (family of) super intervals by $I_{(t,\theta)}$. Let x and y be S-points of M. A super path in M parametrized by $I_{(t,\theta)}$ and with endpoints x and y is a super curve $c : S \times \mathbf{R}^{1|1} \to M$ with $c \circ i_{(0,0)} = c(0,0) = x$ and $c \circ i_{(t,\theta)} = c(t,\theta) = y$.

Theorem 3.2 tells us that to any superpath $c : S \times I_{t,\theta} \to M$ in M, one can associate a bundle map

$$x^*E \xrightarrow{\ SP(c)\ } y*E$$

with both mapping down to S.

which is called *super parallel tranport*. It can be showed that the super parallel transport satisfies the following properties (see details in [7]):

(1) The correspondence $c \to SP(c)$ is smooth and natural in S. Smoothness means: if c is a family of smooth superpaths parametrized by a supermanifold S, then the map $SP(c) : c_{0,0}^*E \to c_{t,\theta}^*E$ is a smooth bundle map over S.

(2) Compatibility under glueing: If $c : I_{t,\theta} \to M$ and $c' : I_{t',\theta'} \to M$ are two superpaths in M such that $c' = c \circ R_{t,\theta}$ on some neighborhood $S \times U$ of $S \times (0,0) \hookrightarrow S \times \mathbf{R}^{1|1}$, with U an open subsupermanifold in $\mathbf{R}^{1|1}$ containing $(0,0)$, we have

$$SP(c' \cdot c) = SP(c') \circ SP(c),$$

where $c' \cdot c : I_{t'+t+\theta'\theta,\theta'+\theta} \to M$ is obtained from c and c' by gluing them along their common endpoint. Here $R_{t,\theta} : S \times \mathbf{R}^{1|1} \to S \times \mathbf{R}^{1|1}$ is the right translation by (t,θ) in the $\mathbf{R}^{1|1}$ direction.

(3) For any superpath $c : I_{t,\theta} \to M$, the bundle map $SP(c) : c_{0,0}^*E \to c_{t,\theta}^*E$ is an isomorphism.

(4) Invariance under geometric reparametrization: Given $c : I_{t,\theta} \to M$ a superpath

in M and $\varphi : I_{s,\eta} \to I_{t,\theta}$ a family of diffeomorphisms of superintervals that preserve the vertical distribution, we have $SP(c \circ \varphi) = SP(c)$.

We want to point out that some modified version of Florin's super parallel transport canonically associates a 1|1D QFT to (E, ∇^E). Let's explain the modifications. Since in SUSY QFT, we work with cs-manifolds, the S in Florin's setting should be changed to cs-manifold. Also we will have to use $\mathbf{R}_{cs}^{1|1}$ instead of $\mathbf{R}^{1|1}$. Note that $\mathbf{R}_{cs}^{1|1}$ has a standard super Riemannian structure, i.e. the odd vector field $D_{cs} = \frac{1}{2\pi} \frac{\partial}{\partial \theta} - i\theta \frac{\partial}{\partial t}$ ($\frac{1}{2\pi}$ is a normalization constant we need in this paper; $\frac{\partial}{\partial \theta} - i\theta \frac{\partial}{\partial t}$ is the one used in [22] and the physics motivation of applying it can be found there). Therefore we have to use D_{cs} instead of D is Florin's setting. It's not hard to see that with these modifications, Florin's construction goes through and the resulted new super parallel transport still satisfies those properties. We will use the same notation SP in the following. Another thing we want to point out is that to only consider super paths in Florin's setting is enough for all super bordisms because super parallel is only a local construction and the isometric group preserving the super Riemannian structure $\mathrm{Isom}(\mathbf{R}_{cs}^{1|1}, D_{cs}) \cong \mathbf{R}_{cs}^{1|1}$.

Therefore (E, ∇^E) canonically gives us a 1|1D QFT $\mathfrak{S}\mathfrak{F}^{1|1}(M)^{(E, \nabla^E)}$. Let's present some examples to show $\mathfrak{S}\mathfrak{F}^{1|1}(M)^{(E, \nabla^E)}$ explicitly. Let $i : S \to U = S \times \mathbf{R}_{cs}^{1|1}$ be the standard inclusion $i(s) = (s, 0, 0)$. Then

$$\mathfrak{S}\mathfrak{F}_S^{1|1}(M)^{(E, \nabla)}(\alpha : U \to M) = i^* \alpha^* E,$$

where $\alpha : U \to M$ is a super point over M and $i^* \alpha^* E$ is the pull back bundle over S. For a super interval $(I_{cs,l}^{1|1}, \beta)$ over M,

$$U = S \times \mathbf{R}_{cs}^{1|1} \xrightarrow{\mathrm{id}} \Sigma = S \times \mathbf{R}_{cs}^{1|1} \xleftarrow{l} U = S \times \mathbf{R}_{cs}^{1|1} ,$$
$$\downarrow{\beta}$$
$$M$$

$\mathfrak{S}\mathfrak{F}^{1|1}(M)^{(E, \nabla)}$ sends $(I_{cs,l}^{1|1}, \beta)$ to

$$i^* \beta^* E \xrightarrow{SP((I_{cs,l}^{1|1}, \beta))} i^* l^* \beta^* E .$$
$$S$$

3.3.3. 0|1*D Theories.* Let M be a smooth manifold. Hohnhold, Kreck, Stolz and Teichner have also studied 0|1**D QFT's over** M which in spirit are very similar to the above SUSY QFT's of dimension $d|1$, $d = 1, 2$. Let S be a cs-manifold. Let's first define super categories $\mathrm{RB}_S^{0|1}(M)$ and TV_S^0.

• $\mathrm{RB}_S^{0|1}(M)$: For an cs-manifold S, the objects of $\mathrm{RB}_S^{0|1}(M)$ are S-families of "super points in M, i.e. an object of $\mathrm{RB}_S^{0|1}(M)$ is a pair $(S \times \mathbf{R}_{cs}^{0|1}, f)$, where $f : S \times \mathbf{R}_{cs}^{0|1} \to M$ is a morphism between supermanifolds. A morphism from $(S \times \mathbf{R}_{cs}^{0|1}, f)$ to $(S \times \mathbf{R}_{cs}^{0|1}, f')$ is a diffeomorphism G making the following diagram

commutative:

• TV^0_S: For any cs-manifold S, the objects of the category TV^0_S are $C^\infty(S)$; the morphisms are identity morphisms.

Similarly as in the positive dimension cases, we can define a super smooth category $\mathfrak{SRB}^{0|1}(M) : \mathrm{cs} - \mathrm{SMAN} \to \mathrm{CAT}$ such that $\mathfrak{SRB}^{0|1}(M)(S) = RB_S^{0|1}(M)$ and a super smooth category $\mathfrak{STV}^0 : \mathrm{cs} - \mathrm{SMAN} \to \mathrm{CAT}$ such that $\mathfrak{STV}^0(S) = \mathrm{TV}^0_S$.

A **supersymmetric quantum field theory of dimension** $0|1$ **over** M is a symmetric monoidal functor:

$$\mathfrak{SF}^{0|1}(M) : \mathfrak{SRB}^{0|1}(M) \to \mathfrak{STV}^0.$$

The following theorem relates $0|1$D QFT's over M to differential forms on M.

Theorem 3.2. *(Hohnhold, Kreck, Stolz and Teichner, [14]) There is a bijection*

$$\{\mathfrak{SF}^{0|1}(M) : \mathfrak{SRB}^{0|1}(M) \to \mathfrak{STV}^0\} \xrightarrow{\ e\ } \{\omega \in \Omega^*(M)|\, d\omega = 0\}\,,$$

where e is given by

$e(\mathfrak{SF}^{0|1}(M))$

$=\mathfrak{SF}^{0|1}(M)(\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M))(\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M) \times \mathbf{R}_{cs}^{0|1}, ev) \in C^\infty(\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M)) \cong \Omega^*(M).$

Here $ev : \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M) \times \mathbf{R}_{cs}^{0|1} \to M$ is the evaluation map.

Remark 3.1. *There is also an equivariant version of the above $0|1$D theory.*

4. SUSY QFTs AND CHERN CHARACTER

In this section we construct the Bismut-Chern character form via our modified super parallel transport and super loop spaces. Our construction gives the Bismut-Chern character a quantum interpretation. It shows that the Bismut-Chern character can be viewed as a map from the 1D SUSY QFT induced by a vector bundle with connection over M to a 0D SUSY QFT over the loop space of M. With the quantum interpretation of the Bismut-Chern character, we are able to construct Chern character type maps in the world of SUSY QFTs.

4.1. **The Chern Character and Bismut-Chern Character.** Let E be a vector bundle over a smooth manifold M. Let ∇^E be a connection on E and $R^E = (\nabla^E)^2$ be the curvature. The Chern character form associated to (E, ∇^E) is defined as

$$\mathrm{Ch}(E, \nabla^E) := \mathrm{Tr}\left(e^{\frac{\sqrt{-1}}{2\pi}R^E}\right),$$

which is an even closed differential form on M (by the Chern-Weil theory). The Chern character form induces the Chern character homomorphism:

$$\mathrm{Ch} : K(M) \to H_{dR}^{even}(M, \mathbf{C}).$$

The importance of this homomorphism lie in the following result due to Atiyah and Hirzebruch, which says that if one ignores the torsion elements in $K(M)$, the induced homomorphism:

$$\mathrm{Ch} : K(M) \otimes \mathbf{C} \to H_{dR}^{even}(M, \mathbf{C})$$

is actually an isomorphism. The Chern character plays an important role in geometry and topology.

Let LM be the free loop space of M. It is the set of C^∞ mappings $t \in S^1 \to x_t \in M$. If $x \in M$, the tangent space $T_x LM$ is identified with the space of smooth periodic vector fields X over x so that $X_t \in T_{x_t} M$. LX is modeled as a Frechét manifold.

The Chern character form has a loop space lifting, the Bismut-Chern character form [4]. Let's roughly explain the motivation of this lifting (cf. [10]). Let S be a Clifford module on M with Dirac operator D. Witten observed that it should be possible to associate an equivariantly closed current μ_D on the free loop space of M such that $\mathrm{Ind}D = \langle \mu_D, 1 \rangle$, where $1 \in \Omega^0(LM)$. The source of this current is the formalism of path-integrals in supersymmetric quantum mechanics. Bismut [4] showed how to generalize this formula by associating to a vector bundle E over M, equipped with a connection ∇^E, an equivariantly closed differential form $\mathrm{Bch}(E, \nabla^E)$ on LM such that $\mathrm{Ind}(D \otimes E) = \langle \mu_D, \mathrm{Bch}(E, \nabla^E) \rangle$, and moreover $i^* \mathrm{Bch}(E, \nabla^E) = \mathrm{Ch}(E, \nabla^E)$, where $i : M \to LM$ is the inclusion of the point loops. Let's call $\mathrm{Bch}(E, \nabla^E)$ the *Bismut-Chern character form* associated to (E, ∇^E). Getzler, Jones and Petrack [10] give a formula for the Bismut-Chern character form from the point of view of their model of equivariant differential forms on loop space. In their model, they reformulate equivariant differential forms and equivariant currents on LM as cyclic chains and cochains over differential graded algebra $\Omega(M)$ of differential forms on M.

The Bismut-Chern character form $\mathrm{BCh}(E, \nabla^E)$ is defined as follows ([4], cf.[10]):

Definition 4.1. *Let* $\varphi(t) \in \Omega(LM) \otimes \mathrm{Hom}(E_{\gamma(0)}, E_{\gamma(t)})$ *be the solution of the ordinary differential equation*

$$(4.1) \qquad\qquad \nabla_{\frac{\partial}{\partial t}} \varphi(t) = \varphi(t) \frac{iF(t)}{2\pi},$$

with initial condition $\varphi(0) = 1$, *where* $F(t)$ *is the pull back of the curvature of* ∇^E *at* $\gamma(t)$ *by the parallel transport along the loop* γ. *Then*

$$(4.2) \qquad\qquad \mathrm{BCh}(E, \nabla^E) := \mathrm{Tr}(\varphi(1)) \in \Omega(LM).$$

Bismut showed that $\mathrm{BCh}(E, \nabla^E)$ defined in this way is an equivariant closed form over LM and the restriction to M, the fixed point set of LM, is the ordinary Chern character form $\mathrm{Ch}(E, \nabla^E)$ over M. In the following we will provide a new understanding of $\mathrm{BCh}(E, \nabla^E)$ via modified super parallel transport.

4.2. **Super Loop Spaces.** Let $S_{cs}^{1|1} = S^1 \times \mathbf{R}_{cs}^{0|1}$ be the standard super circle. Let $\underline{\mathbf{SM}}(S_{cs}^{1|1}, M)$ be the *super loop space* of M (which is a generalized super manifold currently). Let $\Pi T LM_{cs}$ be the supermanifold defined as in Section 2.1

such that functions on it are sections $\Lambda^*(T^*LM) \otimes \mathbf{C}$. Since we are now working with infinite dimensional case, let's explain it a little bit. We equip the fibre $T_\gamma LM$ with the Frechét topology and the fibre $T_\gamma^* LM$ with the induced weak topology. Note that with this weak topology, $(T_\gamma^* LM)^* = T_\gamma LM$ (cf. [6]). Let $\Lambda^*(T^*LM) := \overline{\oplus_{k=0}^\infty \Lambda^k(T^*LM)}$, the fibrewise completion of $\oplus_{k=0}^\infty \Lambda^k(T^*LM)$. We call ΠTLM_{cs} a complex super Frechét manifold.

For any super manifold S, by definition, one has

$$\begin{aligned}
&\mathbf{SM}(S, \underline{\mathbf{SM}}(S_{cs}^{1|1}, M)) \\
=&\mathbf{SM}(S \times S_{cs}^{1|1}, M) \\
=&\mathbf{SM}(S \times S^1 \times \mathbf{R}_{cs}^{0|1}, M) \\
=&\mathbf{SM}(S \times \mathbf{R}_{cs}^{0|1}, LM) \\
=&\{\text{pairs } (a, X_a) | a \in LM(S), X_a \in TLM_a \otimes \mathbf{C}, X_a \text{ odd}\} \\
=&\mathbf{SM}(S, \Pi TLM_{cs}),
\end{aligned}$$

(4.3)

where we can see the last equality holds from the proof of Proposition 2.1.1 by requiring all the maps between algebras to be maps between Frechét algebras. Therefore we have

Proposition 4.1.

(4.4) $$\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \cong \Pi TLM_{cs}.$$

This essentially means that ΠTLM_{cs} is a model for the generalized super manifold $\mathbf{SM}(- \times S_{cs}^{1|1}, M)$.

Definition 4.2. *Let ω be a degree p differential form over M. Given any $t \in S^1$, ω defines a differential form ω_t over LM:*

(4.5) $$\omega_t|_x(X^1, X^2, \cdots, X^p) := \omega(X_t^1, \cdots, X_t^p),$$

where $X^1, \cdots, X^p$ are tangent vectors of LM at x. If ω is degree 1, given $t \in S^1$, we can also canonically define a smooth function $\omega(t)$ on LM by

(4.6) $$\omega(t)(x) := \omega(\dot{x}(t)).$$

As t runs over all of S^1, $\omega(t)$ then defines a smooth function on $LM \times S^1$, which we denote by $\widetilde{\omega}$.

As t runs over all of S^1, one can view ω_t defined above as a C^∞-function on $\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S^1 \cong \Pi TLM_{cs} \times S^1$. Denote it by $\widehat{\omega}$. Note that $\widehat{\omega}$ can be viewed as a function on $\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1}$. Similarly, $\widetilde{\omega}$ can also be viewed as a function on $\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1}$.

Let $slev : \underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1} \to M$ and $lev : LM \times S^1 \to M$ be the evaluation maps defined after (2.3)-(2.5) above. Here $slev$ and lev represent super loop evaluation and loop evaluation respectively.

The following theorem characterizes the map $slev$ on function spaces.

Theorem 4.1. *The super loop evaluation map $slev : \underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1} \to M$ is characterized on functions by*

$$C^\infty(M) \to C^\infty(\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1}) = C^\infty(\Pi TLM \times S_{cs}^{1|1}),$$

(4.7) $$f \mapsto \widehat{f} + \theta(\widehat{df}).$$

Proof. Let S be any supermanifold, $\varphi \in \mathbf{SM}(S \times S_{cs}^{1|1}, M), \phi \in \mathbf{SM}(S, S_{cs}^{1|1})$. Let

$$r : \mathbf{SM}(S \times S_{cs}^{1|1}, M) \to \mathbf{SM}(S, \underline{\mathbf{SM}}(S_{cs}^{1|1}, M))$$

be the representation map.

Note that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbf{SM}(S \times S_{cs}^{1|1}, M) \times \mathbf{SM}(S, S_{cs}^{1|1}) & \xrightarrow{\;slev_S\;} & \mathbf{SM}(S, M) \\
{\scriptstyle r \times id} \downarrow & & \uparrow {\scriptstyle slev(S)} \\
\mathbf{SM}(S, \underline{\mathbf{SM}}(S_{cs}^{1|1}, M)) \times \mathbf{SM}(S, S_{cs}^{1|1}) & \xrightarrow[m]{} & \mathbf{SM}(S, \underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1})
\end{array}
$$

From (2.4), (2.5), we see that

$$S \xrightarrow{\;\Delta\;} S \times S \xrightarrow{\;1 \times \phi\;} S \times S_{cs}^{1|1} \xrightarrow{\;\varphi\;} M \xrightarrow{\;f\;} \mathbf{C}$$

characterizes $slev_S$, or in other words

(4.8) $$slev_S^*(f) = f \circ \varphi \circ (1 \times \phi) \circ \Delta.$$

To prove our theorem, we only have to verify that

$$slev_S^*(f)(s) = (\widehat{f} + \theta \widehat{df})(r(\varphi)(s), \phi(s)),$$

i.e.

$$f(\varphi(s, \phi(s))) = (\widehat{f} + \theta \widehat{df})(r(\varphi)(s), \phi(s)).$$

We can actually prove that

(4.9) $$f(\varphi(s, t, \theta)) = (\widehat{f} + \theta \widehat{df})(r(\varphi)(s), t, \theta) = \widehat{f}(r(\varphi)(s), t) + \theta \widehat{df}(r(\varphi)(s), t).$$

This is essentially only an explicit explanation of the the canonical isomorphism

$$\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M) \cong \Pi T M_{cs}$$

in our situation.

Let

(4.10) $$f(\varphi(s, t, \theta)) = \varphi_1(f)(s, t) + \theta \varphi_2(f)(s, t),$$

where $\varphi_1(f)(s, t)$ and $\varphi_2(f)(s, t)$ are two functions on $S \times S^1$. Let $g \in C^\infty(M)$ be another function. We should have that

$$\varphi_1(fg)(s, t) + \theta \varphi_2(fg)(s, t) = (\varphi_1(f)(s, t) + \theta \varphi_2(f)(s, t))(\varphi_1(g)(s, t) + \theta \varphi_2(g)(s, t)).$$

Therefore

$$\varphi_1(fg) = \varphi_1(f)\varphi_1(g),$$
$$\varphi_2(fg) = \varphi_1(f)\varphi_2(g) + \varphi_1(g)\varphi_2(f).$$

We can see from above that φ gives two maps $\varphi_1, \varphi_2 : C^\infty(M) \to C^\infty(S \times S^1)$ such that φ_1 is a homomorphism of rings and φ_2 is $\varphi_1 \circ X_\varphi$, where X_φ is a tangent vector field over M.

On the other hand, by the definition of the representation map:

$$r : \mathbf{SM}(S \times S_{cs}^{1|1}, M) \to \mathbf{SM}(S, \underline{\mathbf{SM}}(S_{cs}^{1|1}, M)),$$

it's not hard to see that

$$\widehat{f}(r(\varphi)(s), t) = f(\varphi(s, t, 0)) = \varphi_1(f)(s, t),$$

and
$$\widehat{df}(r(\varphi)(s), t) = X_\varphi(f)(\varphi(s, t, 0)) = \varphi_1(X_\varphi(f))(s, t) = \varphi_2(f)(s, t).$$

This finishes the proof. □

We have the following diagram of maps (which does not commute):

$$\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1}$$

$$p \downarrow \qquad \searrow^{slev}$$

$$LM \times S^1 \xrightarrow{\ \ lev\ \ } M$$

Let's adopt the Deligne-Morgan sign convention in the following. The paring of vectors and 1-forms is written with the vector on the left with the rule

$$\langle uD, v\omega \rangle = (-1)^{p(D)p(v)} uv \langle D, \omega \rangle.$$

When computing with differential forms on super space, we sue a bigraded point of view. This means objects have a "cohomological degree" and a parity. The permutation of objects of parity p_1, p_2 and cohomological degree d_1, d_2 introduce two signs $(-1)^{d_1 d_2}$ and an additional factor $(-1)^{p_1 p_2}$. See [8] for details.

We find that *slev* also has the following property on differential one forms.

Theorem 4.2. *Let $\omega \in \Omega^1(M)$, then one has*

$$(4.11) \qquad \langle D_{cs}, slev^*(\omega) \rangle = \frac{1}{2\pi}\widehat{\omega} + \frac{1}{2\pi}\theta(\widehat{d\omega}) - i\theta\widetilde{\omega} \in C^\infty(\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1}),$$

where $D_{cs} = \frac{1}{2\pi}\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}$.

Proof. By partition of unity, ω can be written as

$$\omega = \sum_i f_i dg_i,$$

where f_i's, g_i's are smooth functions over M, f_i's are nonnegative, $\sum_i f_i = 1$ and near each point, there exists a neighborhood such that there are only finite many f_i's nonzero in it. Therefore we only have to prove the theorem for differential forms like $f dg$.

Actually we have

(4.12)
$$\langle D_{cs}, slev^*(fdg) \rangle = \left\langle \frac{1}{2\pi}\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}, (\widehat{f} + \theta\widehat{df})d(\widehat{g} + \theta\widehat{dg}) \right\rangle$$

$$= \frac{1}{2\pi}\left\langle \frac{\partial}{\partial\theta}, (\widehat{f} + \theta\widehat{df})d(\widehat{g} + \theta\widehat{dg}) \right\rangle - i\left\langle \theta\frac{\partial}{\partial t}, (\widehat{f} + \theta\widehat{df})d(\widehat{g} + \theta\widehat{dg}) \right\rangle$$

$$= \frac{1}{2\pi}\widehat{fdg} + \frac{1}{2\pi}\theta\widehat{df dg} - i\theta\left\langle \frac{\partial}{\partial t}, \widehat{f d\widehat{g}} \right\rangle$$

$$= \frac{1}{2\pi}\widehat{fdg} + \frac{1}{2\pi}\theta\widehat{d(fdg)} - i\theta\widehat{f}\left(\frac{\partial\widehat{g}}{\partial t}\right)$$

$$= \frac{1}{2\pi}\widehat{fdg} + \frac{1}{2\pi}\theta\widehat{d(fdg)} - i\theta\widetilde{fdg}.$$

This completes the proof. □

4.3. **A Quantum Interpretation of the Bismut-Chern Character and Chern Character.** Applying the super parallel transport equation (3.3), one can identify the ordinary differential equation that the Bosonic part of the super parallel transport of $slev$ should satisfy in local coordinates.

Theorem 4.3. *Locally suppose* $\nabla^E = d + A$, *where* $A \in \Omega^1(M, \mathrm{End}(E))$ *and let* $R = (d + A)^2$ *be the curvature. Then the Bosonic super parallel transport* $SP(slev, t, 0)$ *satisfies the following ordinary differential equation*

$$(4.13) \qquad \frac{d}{dt} SP(slev, t, 0) = -\frac{i}{2\pi} \widehat{R} SP(slev, t, 0) - \widetilde{A} SP(slev, t, 0).$$

Proof. By definition of super parallel transport,

$$(slev^* \nabla^E)_{D_{cs}} SP(slev, t, \theta) = 0.$$

We therefore have

$$(4.14) \qquad \left\langle \frac{1}{2\pi} \frac{\partial}{\partial \theta} - i\theta \frac{\partial}{\partial t}, dSP(slev, t, \theta) + (slev^* A) SP(slev, t, \theta) \right\rangle = 0.$$

Let $SP(slev, t, \theta) = SP(slev, t, 0) + \theta SP'(slev, t)$. Then we have

$$(4.15) \quad \left\langle \frac{1}{2\pi} \frac{\partial}{\partial \theta} - i\theta \frac{\partial}{\partial t}, dSP(slev, t, \theta) \right\rangle = \frac{1}{2\pi} SP'(slev, t) - i\theta \frac{d}{dt} SP(slev, t, 0),$$

and by Theorem 4.2

$$(4.16)$$
$$\left\langle \frac{1}{2\pi} \frac{\partial}{\partial \theta} - i\theta \frac{\partial}{\partial t}, (slev^* A) SP(slev, t, \theta) \right\rangle$$
$$= \left(\frac{1}{2\pi} \widehat{A} + \frac{1}{2\pi} \theta \widehat{dA} - i\theta \widetilde{A} \right) (SP(slev, t, 0) + \theta SP'(slev, t))$$
$$= \frac{1}{2\pi} \widehat{A} SP(slev, t, 0) + \theta \left[-\frac{1}{2\pi} \widehat{A} SP'(slev, t) + \left(\frac{1}{2\pi} \widehat{dA} - i\widetilde{A} \right) SP(slev, t, 0) \right].$$

From (4.14)-(4.16), we see that

$$SP'(slev, t) = -\widehat{A} SP(slev, t, 0),$$

$$\frac{d}{dt} SP(slev, t, 0) = \frac{i}{2\pi} \widehat{A} SP'(slev, t) - \left(\frac{i}{2\pi} \widehat{dA} + \widetilde{A} \right) SP(slev, t, 0).$$

Hence we obtain that

$$\frac{d}{dt} SP(slev, t, 0) = -\frac{i}{2\pi} (\widehat{A^2} + \widehat{dA}) SP(slev, t, 0) - \widetilde{A} SP(slev, t, 0),$$

or

$$(4.17) \qquad \frac{d}{dt} SP(slev, t, 0) = -\frac{i}{2\pi} \widehat{R} SP(slev, t, 0) - \widetilde{A} SP(slev, t, 0).$$

$$\square$$

From Theorem 4.3, we see that super parallel transport along the fermionic direction D_{cs} on the super evaluation curve is different from the ordinary transport along the time direction. However we will see that the super parallel transport along the curve $lev \circ p$ degenerates to the ordinary transport on $LM \times S^1$.

Theorem 4.4. *Using the same notations as in theorem 4.3, we have*

$$\frac{d}{dt}SP(lev \circ p, t, 0) + \widetilde{A}SP(lev \circ p, t, 0) = 0.$$

Proof. It's not hard to see that the map $lev \circ p : \underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1} \to M$ is characterized on functions by

$$C^\infty(M) \to C^\infty(\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1}), \qquad f \mapsto \widehat{f}.$$

Similarly to what we did in Theorem 4.2, we have

$$\begin{aligned}
\langle D_{cs}, (lev \circ p)^*(fdg)\rangle &= \left\langle \frac{1}{2\pi}\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}, \widehat{f}d\widehat{g}\right\rangle \\
&= -i\theta\left\langle \frac{\partial}{\partial t}, \widehat{f}d\widehat{g}\right\rangle \\
&= -i\theta\widehat{f}\left(\frac{\partial\widehat{g}}{\partial t}\right) \\
&= -i\theta\widetilde{fdg}.
\end{aligned}$$

Therefore one has for $\omega \in \Omega^1(M)$,

$$\langle D_{cs}, (lev \circ p)^*(\omega)\rangle = -i\theta\widetilde{\omega} \in C^\infty(\underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1}).$$

By definition of super parallel transport along the curve $lev \circ p$,

$$((lev \circ p)^*\nabla^E)_{D_{cs}}SP(lev \circ p, t, \theta) = 0.$$

We therefore have

$$\left\langle \frac{1}{2\pi}\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}, dSP(lev \circ p, t, \theta) + ((lev \circ p)^*A)SP(lev \circ p, t, \theta)\right\rangle = 0.$$

Let $SP(lev \circ p, t, \theta) = SP(lev \circ p, t, 0) + \theta SP'(lev \circ p, t)$. Then we have

$$\left\langle \frac{1}{2\pi}\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}, dSP(lev \circ p, t, \theta)\right\rangle = \frac{1}{2\pi}SP'(lev \circ p, t) - i\theta\frac{d}{dt}SP(lev \circ p, t, 0),$$

and

$$\begin{aligned}
&\left\langle \frac{1}{2\pi}\frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}, ((lev \circ p)^*A)SP((lev \circ p), t, \theta)\right\rangle \\
&= \left(-i\theta\widetilde{A}\right)(SP(lev \circ p, t, 0) + \theta SP'(lev \circ p, t)) \\
&= \left(-i\theta\widetilde{A}\right)SP(lev \circ p, t, 0).
\end{aligned}$$

Hence we see that

$$SP'(lev \circ p, t) = 0,$$

$$\frac{d}{dt}SP(lev \circ p, t, 0) + \widetilde{A}SP(lev \circ p, t, 0) = 0.$$

$\square$

Combining the above super parallel transports along the two super curves, we have

Theorem 4.5. *Let's use the same notations as in theorem 4.3. The following ordinary differential equations hold:*

(4.18)
$$\frac{d}{dt}[SP^{-1}(lev \circ p, t, 0)SP(slev, t, 0)]$$
$$= -SP^{-1}(lev \circ p, t, 0)\left(\frac{i}{2\pi}\widehat{R}\right)[SP^{-1}(lev \circ p, t, 0)SP(slev, t, 0)],$$

(4.19)
$$\frac{d}{dt}[SP^{-1}(slev, t, 0)SP(lev \circ p, t, 0)]$$
$$= [SP^{-1}(slev, t, 0)SP(lev \circ p, t, 0)]SP^{-1}(lev \circ p, t, 0)\left(\frac{i}{2\pi}\widehat{R}\right).$$

Proof. Differentiating the identity $SP^{-1}(lev \circ p, t, 0)SP(lev \circ p, t, 0) = I$, we have that

$$\left(\frac{d}{dt}SP^{-1}(lev \circ p, t, 0)\right)SP(lev \circ p, t, 0) = -SP^{-1}(lev \circ p, t, 0)\left(\frac{d}{dt}SP(lev \circ p, t, 0)\right).$$

However by Theorem 4.4, we see that

$$\frac{d}{dt}SP(lev \circ p, t, 0) = -\widetilde{A}SP(lev \circ p, t, 0).$$

Therefore, one has

(4.20)
$$\frac{d}{dt}SP^{-1}(lev \circ p, t, 0) = SP^{-1}(lev \circ p, t, 0)\widetilde{A}.$$

Hence by (4.13) and (4.20), we obtain that

(4.21)
$$\frac{d}{dt}[SP^{-1}(lev \circ p, t, 0)SP(slev, t, 0)]$$
$$= \left(\frac{d}{dt}SP^{-1}(lev \circ p, t, 0)\right)SP(slev, t, 0) + SP^{-1}(lev \circ p, t, 0)\frac{d}{dt}SP(slev, t, 0)$$
$$= SP^{-1}(lev \circ p, t, 0)\widetilde{A}SP(slev, t, 0)$$
$$\quad + SP^{-1}(lev \circ p, t, 0)\left[-\frac{i}{2\pi}\widehat{R}SP(slev, t, 0) - \widetilde{A}SP(slev, t, 0)\right]$$
$$= -SP^{-1}(lev \circ p, t, 0) \circ \left(\frac{i}{2\pi}\widehat{R}\right) \circ SP(slev, t, 0)$$
$$= -\left[SP^{-1}(lev \circ p, t, 0) \circ \frac{i}{2\pi}\widehat{R} \circ SP(lev \circ p, t, 0)\right][SP^{-1}(lev \circ p, t, 0)SP(slev, t, 0)]$$
$$= -SP^{-1}(lev \circ p, t, 0)\left(\frac{i}{2\pi}\widehat{R}\right)[SP^{-1}(lev \circ p, t, 0)SP(slev, t, 0)],$$

which proves (4.18).

From (4.18), it's not hard to obtain (4.19) as how we obtained (4.20). $\square$

Comparing Definition 4.1 with Theorem 4.5, we obtain that

Theorem 4.6. *The following identity holds:*
(4.22)
$$\mathrm{BCh}(E, \nabla^E) = \mathrm{Tr}[SP^{-1}(slev, 1, 0) \circ SP(lev \circ p, 1, 0)] \in C^\infty(\underline{\mathbf{SM}}(S^{1|1}, M)) = \Omega^*(LM).$$

Up to some sign, the following identity holds:

$$\mathrm{BCh}(E, \nabla^E) = \mathrm{Tr}[SP^{-1}(lev{\circ}p, 1, 0){\circ}SP(slev, 1, 0)] \in C^\infty(\underline{\mathbf{SM}}(S_{cs}^{1|1}, M)) = \Omega^*(LM).$$

Remark 4.1. *In certain sense, $SP^{-1}(slev, 1, 0) \circ SP(lev \circ p, 1, 0)$ is a loop-deloop process. Therefore, from Theorem 4.6, we see that the Bismut-Chern character is a phenomena related to this loop-deloop process when one moves from $1|1D$ theories down to $0|1D$ theories. This theorem also shows us the supersymmetric aspect of the Bismut-Chern character form.*

Let $i : \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M) \to \underline{\mathbf{SM}}(S_{cs}^{1|1}, M)$ be the inclusion of super constant loops. Let

$$\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M) \times S_{cs}^{1|1} \xrightarrow{i \times 1} \underline{\mathbf{SM}}(S_{cs}^{1|1}, M) \times S_{cs}^{1|1} \xrightarrow{slev} M$$

be the restriction of the super evaluation curve on the constant loops. Then it's not hard to see that

Theorem 4.7. *The following identity holds:*

$$(4.23) \quad \mathrm{Ch}(E, \nabla^E) = \mathrm{Tr}[SP^{-1}(slev \circ (i \times 1), 1, 0)] \in C^\infty(\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M)) = \Omega^*(M).$$

Up to some sign, the following identity holds:

$$\mathrm{Ch}(E, \nabla^E) = \mathrm{Tr}[SP(slev \circ (i \times 1), 1, 0)] \in C^\infty(\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, M)) = \Omega^*(M).$$

4.4. **Chern Character in SUSY QFTs.** In view of Theorem 4.6, Remark 4.1 and Theorem 4.7, we are motivated to define Chern character type maps for SUSY QFTs in the following.

Given any $1|1D$ QFT $\mathfrak{SF}^{1|1}(M)$ over M, let's construct a $0|1D$ QFT $\mathfrak{SF}^{0|1}(LM)$ over LM as follows.

Let $\widetilde{r} : LM \to LM$ be the reverse map, which sends $\gamma : S^1 \to M$ to

$$S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} M \ ,$$

where $r(t) = 1 - t$. Then $\widetilde{r}$ induces a map, we still denote it by $\widetilde{r}$, on the super loop space $\widetilde{r} : \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \to \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM)$. Let's pick out two particular bordisms:

$$\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \xrightarrow{\mathrm{id}} \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \xleftarrow{1} \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \ ,$$

$$\downarrow{\widetilde{r} \times \mathrm{id}}$$

$$\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1}$$

$$\downarrow{slev}$$

$$M$$

and

$$\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \xrightarrow{\ \mathrm{id}\ } \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \xleftarrow{\ 1\ } \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \ ,$$

$$\downarrow p$$

$$LM \times S^1$$

$$\downarrow lev$$

$$M$$

where $1 : \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \to \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1}$ is the constant map given by $1(s, t, \theta) = (s, t+1, \theta)$. Let's denote these two bordisms by $b(1, \widetilde{r}, slev)$ and $b(1, p, lev)$ respectively.

Let S be any cs-manifold. An object $f \in \mathbf{SM}(S \times \mathbf{R}_{cs}^{0|1}, LM)$ determines a map $\widetilde{f} \in \mathbf{SM}(S, \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM)) = \mathbf{SM}(S, \underline{\mathbf{SM}}(S_{cs}^{1|1}, M))$. With this $\widetilde{f}$, one has two new bordisms (of S-families):

$$
\begin{array}{ccccc}
S \times \mathbf{R}_{cs}^{1|1} & \xrightarrow{\ \mathrm{id}\ } & S \times \mathbf{R}_{cs}^{1|1} & \xleftarrow{\ 1\ } & S \times \mathbf{R}_{cs}^{1|1} \\
\downarrow {\scriptstyle \widetilde{f} \times \mathrm{id}} & & \downarrow {\scriptstyle \widetilde{f} \times \mathrm{id}} & & \downarrow {\scriptstyle \widetilde{f} \times \mathrm{id}} \\
\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} & \xrightarrow{\ \mathrm{id}\ } & \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} & \xleftarrow{\ 1\ } & \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \\
& & \downarrow {\scriptstyle \widetilde{r} \times \mathrm{id}} & & \\
& & \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} & & \\
& & \downarrow {\scriptstyle slev} & & \\
& & M & &
\end{array}
$$

and

$$
\begin{array}{ccccc}
S \times \mathbf{R}_{cs}^{1|1} & \xrightarrow{\ \mathrm{id}\ } & S \times \mathbf{R}_{cs}^{1|1} & \xleftarrow{\ 1\ } & S \times \mathbf{R}_{cs}^{1|1} \\
\downarrow {\scriptstyle \widetilde{f} \times \mathrm{id}} & & \downarrow {\scriptstyle \widetilde{f} \times \mathrm{id}} & & \downarrow {\scriptstyle \widetilde{f} \times \mathrm{id}} \\
\underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} & \xrightarrow{\ \mathrm{id}\ } & \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} & \xleftarrow{\ 1\ } & \underline{\mathbf{SM}}(\mathbf{R}_{cs}^{0|1}, LM) \times \mathbf{R}_{cs}^{1|1} \\
& & \downarrow {\scriptstyle p} & & \\
& & LM \times S^1 & & \\
& & \downarrow {\scriptstyle lev} & & \\
& & M & &
\end{array}
$$

Denote them by $b(\widetilde{f}, 1, \widetilde{r}, slev)$ and $b(\widetilde{f}, 1, p, lev)$ respectively.

Define

$$(4.24) \qquad \begin{aligned} &\mathfrak{S}\mathfrak{F}_S^{0|1}(LM)((S \times \mathbf{R}_{cs}^{0|1}, f)) \\ &:= \mathrm{Str}\left[\mathfrak{S}\mathfrak{F}_S^{1|1}(M)(b(\widetilde{f}, 1, \widetilde{r}, slev)) \circ \mathfrak{S}\mathfrak{F}_S^{1|1}(M)(b(\widetilde{f}, 1, p, lev)) \right] \in C^\infty(S). \end{aligned}$$

It's not hard to check that this indeed gives us a 0|1D QFT over LM. In other words, we have canonically constructed a loop-deloop map:

$$LD: \{1|1D \text{ QFTs over } M\} \longrightarrow \{ 0|1D \text{ QFTs over } LM \},$$

which makes the following diagram commutative:

$$
\begin{array}{ccc}
\{1|1D \text{ QFTs over } M\} & \xrightarrow{\quad LD \quad} & \{0|1D \text{ QFTs over } LM\} \\
{\scriptstyle SP}\big\uparrow & & \big\downarrow{\scriptstyle HKST} \\
\{\text{vector bundles with connections over } M\} & \xrightarrow{\; BCh \;} & \{S^1 - \text{closed forms on } LM\} \\
& {\scriptstyle Ch}\searrow & \big\downarrow{\scriptstyle res} \\
& & \{\text{closed forms on } M\}
\end{array}
$$

Let $i: M \to LM$ be the inclusion of constant loops and i^* be the pull back of field theories on LM to field theories on M. Then the map $i^* \circ LD$ from 1|1D SUSY QFTs to 0|1D SUSY QFTs plays the role of the Chern character in the framework of SUSY QFTs.

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