RAMSEY’S THEOREM FOR PAIRS IN REVERSE MATHEMATICS

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ABSTRACT. We consider the combinatorial principle $RT^2_k$ derived from Ramsey’s Theorem for pairs, and discuss its proof-theoretic strength within the framework of reverse mathematics. Some of the techniques introduced to study this principle are presented.

Let $[\mathbb{N}]^n$ denote the set of $n$-tuples of natural numbers. Ramsey’s Theorem states that any coloring of $[\mathbb{N}]^n$ in $k$ colors (where $n, k \geq 1$) has an infinite set $G$ all of whose $n$-tuples have the same color (Ramsey [37]). Such a $G$ is said to be homogeneous for the coloring. This statement—otherwise called a principle—is abbreviated as $RT^1_k$. It is a natural generalization of the classical pigeonhole principle which is simply $RT^1_1$. $RT^2_k$ is arguably the most extensively studied principle in reverse mathematics concerning combinatorics. Of particular interest is the complexity of a solution of an instance of $RT^2_k$, and the proof-theoretic strength of the principle over subsystems of second-order arithmetic. Our objective here is to provide an overview of this subject, discuss a number of the major mathematical problems in the area, and describe the methodologies developed to resolve them.

We will begin with fixing the notations to be used in the article, and recalling the basic notions, including those that concern subsystem of second-order arithmetic, that are central to our discussion. This will be followed by an exposition of three general problems: (i) The separation of $RT^2_2$ from other subsystems of second-order arithmetic, (ii) the inductive strength of $RT^2_3$, and (iii) conservation strength of $RT^2_2$. As the reader will observe, these three problems are closely related to one another, either by the method of proof or the implication of a theorem. Hence the categorization of topics into three groups is somewhat arbitrary.

1. PRELIMINARIES AND SUBSYSTEMS OF SECOND-ORDER ARITHMETIC

1.1. Notations. The language we use is the language of second-order arithmetic, which includes both number variables $x, y, z, \ldots$ and set variables $X, Y, Z, \ldots$. It also includes a binary relation $\in$ interpreted as “element of”, constant symbols 0 and 1, and function symbols $+, \times$. A structure $\mathcal{M} = (M, S)$ in the language of second-order arithmetic consists of a first-order universe $M$ and a second-order component $S$ contained in the power set of $M$. We use $\omega$ and $\mathbb{N}$ interchangeably to denote the set of (standard) natural numbers. If $\mathcal{M} = (\omega, S)$, then we call it an $\omega$-model. Given $\mathcal{M} = (M, S)$ and $X \subseteq M$, let $\mathcal{M}[X] = (M, (Z : \forall W \in S \wedge Z \subseteq_T X \oplus W))$, where $\oplus$ denotes the join operation and $\subseteq_T$ denotes Turing reducibility.

We reserve $i, j, k, m, n$ for numbers in $\mathbb{N}$. Given $A \subseteq M$ unbounded, let $[A]^n$ denote the set of all $n$-tuples $(c_0, c_1, \ldots, c_{n-1})$ selected from $A$ where $c_0 < c_1 < \cdots < c_{n-1}$. In this paper, we are interested in colorings of $[A]^n$ in $k$ colors, especially for $n = k = 2$.

Given $\mathcal{M}$, we say that $K \subseteq M$ is $\mathcal{M}$-finite if $K$ is coded in $\mathcal{M}$, i.e. there is a $c \in M$ such that for all $x, x \in K$ if and only if $x$ divides $c$. Greek letters $\beta, \nu, \rho, \sigma, \tau, \ldots$ are reserved for $\mathcal{M}$-finite binary strings, and each is identified with its characteristic function, hence an $\mathcal{M}$-finite set (for example $\{x : \sigma(x) = 1\}$) in the obvious way. We write $\sigma \preceq \tau$ if $\sigma$ is an initial segment of $\tau$, and use $< \preceq$ for proper extension. The letter $\alpha$ is used to denote ordinals.

1.2. $\mathbb{Z}_2$ and its subsystems. The system $\mathbb{Z}_2$ of second-order arithmetic consists of the Peano axioms (with free number and set variables in the induction scheme), together with the full comprehension scheme, i.e.

$$\exists x \forall y (y \in x \leftrightarrow \varphi(y))$$

holds for each formula $\varphi$ with possibly free number and set variables. Among the subsystems of $\mathbb{Z}_2$ is $\text{RCA}_0$, the base system, which consists of the following axioms:

- $P^c$, the Peano axioms minus the induction scheme;

2010 Mathematics Subject Classification. 03D32 03F35 03F60 03F30.
Research partially supported by NUS grants C-146-000-042-001 and WBS : R389-000-040-101.
• The $\Delta^0_1$-comprehension scheme:
  \[ \exists X \forall x (x \in X \leftrightarrow \varphi(x)), \]
  where $\varphi$ is $\Sigma^0_1$ and $\varphi(x) \leftrightarrow \neg \psi(x)$ for some $\Sigma^0_0$-formula, and

• The mathematical induction scheme for $\Sigma^0_1$-formulas with free number and set variables.

The induction scheme for $\Sigma^0_n$-formulas ($n \geq 1$) is denoted $I\Sigma^0_n$. Following the proof in Paris and Kirby [34] for first-order language, one can show that $I\Sigma^0_n$ is equivalent (over $P^-$) to the least $\Sigma^0_n$-principle, which says that every nonempty $\Sigma^0_n$-definable set has a least member, and to the statement that every bounded $\Sigma^0_n$-definable set in a model $\mathcal{M}$ of $\text{RCA}_0$ is $\mathcal{M}$-finite. We will use these facts implicitly throughout the paper.

A scheme intermediate between $I\Sigma^0_n$ and $I\Sigma^0_{n+1}$ is $B\Sigma^0_{n+1}$, known as the $\Sigma^0_{n+1}$-bounding scheme, which states that every $\Sigma^0_{n+1}$-definable function maps a (provably) finite set to a (provably) finite set. Semantically, it means that in a model $M$ of $\text{RCA}_0 + B\Sigma^0_n$, every $\Sigma^0_n$-definable function maps an $M$-finite set to an $M$-finite set. Again by Paris and Kirby [34], the following holds over $P^-$:

\[ \cdots \to \Gamma^0_{n+1} \to B\Sigma^0_{n+1} \to \Gamma^0_n \to B\Sigma^0_n \to \cdots. \]

By Slaman [41], $B\Sigma^0_n$ is an induction scheme, since it is equivalent (over $P^-$) to induction for formulas which are provably $\Sigma^0_n$ and $\Pi^0_n$.

$\text{RCA}_0$ is the base system for the “big five systems” studied in Simpson [40]. Two of them are particularly relevant to our main theme:

• $\text{WKL}_0$: $\text{RCA}_0$ together with the principle stating that every infinite binary tree has an infinite path.

• $\text{ACA}_0$: $\text{RCA}_0$ together with the comprehension scheme

  \[ \exists X \forall x (x \in X \leftrightarrow \varphi(x)), \]

  for each $\Sigma^0_1$-formula $\varphi$ with number as well as set parameters.

In $\text{RCA}_0$ one can develop a robust theory of computation (see Chong, Li and Yang [5] for a survey). From the recursion-theoretic point of view, models of $\text{RCA}_0$ are structures $\mathcal{M} = (M, S)$ such that $S$ is closed under the join operation $|$ and Turing reducibility. Furthermore, if $\mathcal{M} \models \text{ACA}_0$ then $S$ is additionally closed under Turing jump.

It is straightforward to verify that every model of $\text{ACA}_0$ is a model of $\text{WKL}_0$, and that every model of $\text{WKL}_0$ is a model of $\text{RCA}_0$. The reverse is however false. Thus $(\omega, \text{REC}) \not\models \text{RCA}_0$ but not $\text{WK}_0$, where $\text{REC} = \text{the collection of recursive sets and}$, as shown by Hirst [22], an application of the Low Basis Theorem of Jockusch and Soare [24] gives a model $(\omega, \text{LOW})$ of $\text{WK}_0$ but not $\text{ACA}_0$, where $\text{LOW} = \text{the collection of low sets.}$ Hence the systems $\text{RCA}_0, \text{WK}_0$ and $\text{ACA}_0$ have improving proof-theoretic strength. The discussion of Ramsey’s theorem in this article is set in the background of these three systems.

1.3. $\text{RCA}_0 + \text{RT}^n_k$. The combinatorial principle $\text{RT}^n_k$ is stated as follows:

Let $k, n \in \mathbb{N}$ and $\mathcal{M} = (M, S) \models \text{RCA}_0$. Then $\mathcal{M} \models \text{RT}^n_k$ if for any $C: [M]^n \to k$ in $S$, there is an infinite $G \in S$ such that $C \upharpoonright [G]^n$ is a constant, i.e. $G$ is homogeneous for $C$.

A simple induction shows that for $n \geq 1$ and $k \geq 2$, $\text{RT}^n_k \to \text{RT}^n_{k+1}$. Thus we confine ourselves to $k = 2$.

The investigation of Ramsey’s Theorem from the logical point of view was inspired by two results that appeared 15 years apart. The first pointed to the case $n = 2$ as the focus for future study:

**Theorem 1.1.** (Jockusch [23])

(i) $\text{RCA}_0 + \text{RT}^2_k$ implies $\text{ACA}_0$ for $n \geq 3$.

(ii) There is a recursive two-coloring of $[\mathbb{N}]^2$ with no $\Sigma^0_2$-definable homogeneous set.

**Sketch of proof of (i).** One exhibiting a recursive coloring of $[\mathbb{N}]^3$ in 2 colors such that any set homogeneous for the coloring computes $\varphi'$. By relativization, this implies that in any model $\mathcal{M} = (M, S)$ of $\text{RCA}_0 + \text{RT}^2_k$, if $X \in S$ then its Turing jump $X'$ is in $S$ as well. \(\square\)

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1. Note that over $\text{RCA}_0$, the fact that $\text{RT}^n_k \to \text{RT}^n_{k+1}$ for $n \geq 1$ and $k \geq 2$ does not yield $\text{RT}^n_2 \to \forall s \geq 2 \text{RT}^n_2$. Indeed if we write $\text{RT}^2_2$ for $\forall s \geq 2 \text{RT}^2_2$, then $\text{RCA}_0 + \text{RT}^2_2 \not\models \text{RT}^2_2$. This was proved by Hirst [22]. It also follows from the combination of Theorem 3.6 below and Theorem 11.4 of Cholak, Jockusch and Slaman [3], which implies that $\text{RCA}_0 + \text{RT}^2_2 \not\models B\Sigma^0_3$. See also the concluding section for remarks on the principle $\text{TT}^1$ which is another illustration of the sharp distinction between a “fixed finite coloring” and “all finite colorings”.

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It should be mentioned that the original context in which Jockush proved Theorem 1.1 was recursion-theoretic (as indicated by the title of the paper). When cast in the setting of reverse mathematics, the theorem characterizes the inductive strength of $RT_2^2$.

Theorem 1.1 (ii) provides evidence that solutions of instances of $RT_2^2$ could be beyond the computational power of $\varphi'$. This is complemented by the next result which says that even for coloring of pairs, the inductive strength of Ramsey's Theorem is more than $I\Sigma^0_1$:

**Theorem 1.2.** (Hirst [22]) $\text{RCA}_0 + RT_2^2$ implies $B\Sigma^0_2$.

**Sketch of proof.** Suppose for the sake of contradiction that $M$ is a model of $\text{RCA}_0 + RT_2^2$ in which $B\Sigma^0_2$ fails. Let $f : [0, a] \rightarrow M$ be $\Sigma^0_2$-definable in $M$ witnessing this failure. Thus $f$ is total on $[0, a]$ with unbounded range. Using $f$ one can recursively decompose $M$ into $a$-many pairwise disjoint $M$-finite sets $\{A_s : s \in [0, a]\}$. Now define a two-coloring $C : |M|^2 \rightarrow 2$ by setting $C(x, y) = 0$ if $x$ and $y$ belong to the same $A_s$, and let $C(x, y) = 1$ otherwise. It is straightforward to verify that there is no unbounded set homogeneous for $C$ that preserves $I\Sigma^0_1$. Thus $\text{RCA}_0 + RT_2^2$ implies $B\Sigma^0_2$.

Theorems 1.1 and 1.2 set the stage for the study of Ramsey's Theorem for pairs in reverse mathematics. They led to a number of fundamental questions about $RT_2^2$: The first group of questions concerned the strength of $RT_2^2$ *vis-a-vis* $\text{ACA}_0$ (which by Theorem 1.1 (i) is equivalent to $\text{RCA}_0 + RT_2^3$) and $\text{WKL}_0$, the two weakest subsystems among the “big five”. The second group of questions is related to the first and second-order proof-theoretic strength of $RT_2^2$, e.g. how much induction or conservation does $\text{RCA}_0 + RT_2^2$ have? We pause for an elaboration of this question.

Let $M = (M, S)$ be a model of $\text{ACA}_0$. Then $\varphi^{(n)} \in S$ for all $n \in \omega$. Now $I\Sigma^0_1$ relative to $\varphi^{(n)}$ means $M \models I\Sigma^0_1$, and hence $M \models$ Peano arithmetic. This holds in particular for $M \models \text{RCA}_0 + RT_2^2$, for example. Thus whether $\text{RCA}_0 + RT_2^2 \vdash \varphi'$ exists is an interesting question. A negative answer would immediately imply that $\text{RCA}_0 + RT_2^2 \not\vdash RT_2^3$.

The negative answer was given in Seetapun and Slaman [38].

Next, as noted by Hirst [22], $\text{WKL}_0 \not\vdash RT_2^2$. To see this, let $M = (\omega, \text{LOW})$ be the model of $\text{WKL}_0$ introduced earlier. Theorem 1.1 (ii) says that there is a recursive two-coloring of pairs with no $\Sigma^0_2$-definable homogeneous solution in $\text{LOW}$. Hence $M \not\models RT_2^2$, i.e. $\text{WKL}_0 \not\vdash RT_2^2$.

If conversely $\text{RCA}_0 + RT_2^2 \not\vdash \text{WKL}_0$, then one proves as a corollary that there exists a combinatorial principle (indeed many, as it turns out. See Hirschfeldt and Shore [21]) outside the realm of the big five systems. This was resolved in Liu [29].

Let $\Gamma$ be the collection of sentences in the language of second-order arithmetic of a given complexity. A system $T'$ is *$\Gamma$-conservative over* a subsystem $T$ if every $\varphi \in \Gamma$ provable in $T'$ is already provable in $T$. A result due to Harrington states that $\text{WKL}_0$ is $\Pi^1_1$-conservative over $\text{ACA}_0$ (see [3]). On the other hand, $\text{ACA}_0$ is not even $\Pi^1_2$-conservative over $\text{RCA}_0$. Here again, separating $RT_2^3$ from $RT_2^2$ can be achieved by differentiating their first-order strength, for example by showing $\text{RCA}_0 + RT_2^2$ to be $\Pi^1_2$-conservative over $\text{RCA}_0$, which was achieved by Patey and Yokoyama [36]. We will discuss these in §§2–4.

The analysis of $RT_2^3$ versus $RT_2^2$ can be approached from two different angles. The first is by comparing what each principle entails, as described above, and the second is by directly constructing an $M$ that is a model of one but not the other. The latter approach was also the first proof of $\text{RCA}_0 + RT_2^2 \not\vdash RT_2^3$ by Seetapun in Seetapun and Slaman [38]. Since every model of $\text{RCA}_0 + RT_2^3$ contains a set that computes the halting set, to separate $RT_2^3$ from $RT_2^2$ it is sufficient to produce a model of the former with no second-order member that computes $\varphi'$. This was exactly what [38] accomplished. This method of “cone avoidance” has been successfully exploited to investigate other problems such as $\text{WKL}_0$ in Liu [29]. Before diving into the details of model construction for $\text{RCA}_0 + RT_2^2$, we recall the decomposition of $RT_2^3$ into two simpler components introduced in [3]:

Let $M = (M, S)$. $\text{SRT}_2^3$ is the principle of *stable Ramsey's Theorem for pairs*, which states that if $C \subseteq S$ is a 2-coloring of pairs such that for each $x$, the color $C(x, y)$ is the same for all but finitely many $y$’s, then there is a set in $S$ that is homogeneous for $C$. The cohesiveness principle $\text{COH}$ states that if $\{A_s : s \in M\}$ is an array coded as a set in $S$, then there is a $Z \subseteq S$ that is *cohesive* for the array, i.e. for each $s$, all but finitely many members of $Z$ are contained in $A_s$ or its complement. These are among the most important principles known to be weaker than $RT_2^3$ that have been extensively studied in recent years (see Hirschfeldt and Shore [21]). The next proposition will be a basic fact we use in the discussion that follows.
Proposition 1.3. (Cholak, Jockusch and Slaman [3]) Over RCA₀ + BΣ²₀, RT²₂ is equivalent to SRT²₂ + COH.

2. Separating RT²₂ from RT³₂ and WKL₀

2.1. A model of RT²₂ that avoids ϕ′. Given a model ℳ = (M, S) of RCA₀, we call ℳ’ = (M’, S’) an M-extension of ℳ if M’ = M and S’ ⊇ S. We sketch a proof of the theorem that there is an ω-model of RT²₂ whose second-order part does not contain the halting set. The proof proceeds in two parts. Let ℳ₀ = (ω, S₀) be a structure such that S₀ is the collection of recursive subsets of ω. Clearly ℳ₀ |= RCA₀.

Lemma 2.1. There is an ω-extension ℳ₁ = (ω, S₁) of ℳ₀ such that ℳ₁ |= RCA₀ + COH and ϕ′ ∈ S₁.

Proof. Suppose A = {Aₙ : n ∈ ω} is an array coded as a set in S₀. Construct Z that is cohesive for A as follows. Let {Φₑ : e ∈ ω} be an effective list of all partial Σ₀ⁿ-functions.

Stage 0. Suppose A₀ is finite with largest element a₀. See if there exist two finite sets σ and τ, both with least elements greater than a₀, such that for some n ∈ ω, Φ₀ⁿ(Φₙ(τ)) ≠ Φₙ(Φ₀ⁿ(σ)) (we call this a split of Φ₀ⁿ). If the answer is “yes”, denote by σ₀ the string η ∈ {σ, τ} that satisfies Φ₀ⁿ(σ₀) ≠ Φₙ(σ₀) and let X₀ = {y : y > max σ₀}. Otherwise, let σ₀ = ϕ and X₀ = {y : y > a₀}. Note that in this case, for all infinite X with min X > a₀, if Φₙ is total then the range is recursive (by the non-splitting condition) and hence Φₙ ≠ ϕ′.

If A₀ is infinite, let a₀ be the least member of A₀, and search for a pair of strings that split Φ₀ⁿ using numbers within A₀. If a split (σ, τ) exists, let σ₀ ∈ {σ, τ} be such that Φ₀ⁿ(σ₀) ≠ Φₙ(σ₀) for some n. Let X₀ = {y : y ∈ A₀ ∧ y > max σ₀}. If no splitting exists, let σ₀ = ϕ and X₀ = A₀. Thus in all cases, either Φ₀ⁿ(σ₀) ≠ Φₙ(σ₀) for some n, or σ₀ = ϕ and for all infinite X < X₀, if Φₙ is total, then the range is recursive. This ensures that σ₀ ∪ X does not compute ϕ′ via Φ₀ⁿ.

Stage s + 1. Assume that σ₀ ≤ σ₁ ≤ · · · ≤ σₛ and X₀ ⊇ X₁ ⊇ · · · ⊇ Xₛ are defined such that

(i) Xₛ is infinite and min Xₛ > max σₛ;
(ii) For each σ ≤ s, either Xₛ ⊆ Aₛ or Xₛ ∩ Aₛ = ϕ;
(iii) For each s ≤ s′, either there is an n such that Φₙ(σₛ′)(n) ≠ Φₙ(σₛ)(n), or for all X < Xₛ, whenever Φₙσₛ∪X is total, then the range is recursive and hence is not equal to ϕ′.

Define σₛ₊₁ ≥ σₛ and Xₛ₊₁ so that σₛ₊₁ \ σₛ (if nonempty) and Xₛ₊₁ are contained in Xₛ, and

(iv) min Xₛ₊₁ > max σₛ₊₁;
(v) Either Xₛ₊₁ ⊆ Aₛ₊₁ or Xₛ₊₁ ∩ Aₛ₊₁ = ϕ;
(vi) If σₛ₊₁ > σₛ, then there is an n such that Φₙσₛ₊₁(n) ≠ ϕ′(n);
(vii) Suppose σₛ₊₁ = σₛ. For all infinite X < Xₛ₊₁, if Φₙσₛ∪X is total then the range is recursive and hence not equal to ϕ′.

Note that for (vii), if there is already an n such that Φₙσₛ₊₁(n) ≠ ϕ′(n), then one can simply choose σₛ₊₁ > σₛ so that σₛ₊₁ \ σₛ ⊆ Xₛ₊₁.

Let Z = ∪ₖ Xₛ. Then it is straightforward to verify that Z is cohesive for the array {Aₙ : n ∈ ω} and does not compute ϕ′. Let Z ∈ S₁. Repeat the above construction for the structure ℳ₀[Z] to obtain a set Zₐ that is cohesive for the next array in the expanded structure. The modification required in the construction is that, as an example, one now searches for a pair (σ, τ) that forms a split for Φ₀ⁿ in the following sense: Φ₀ⁿσ[Z](n) ≠ Φ₀ⁿτ[Z](n) for some n. If a split exists, let σ₀ be chosen such that Φ₀ⁿσ₀[Z](n) ≠ ϕ′(n). Otherwise, let σ₀ = ϕ. Define X₀ as before. Then for all infinite X < X₀, Φ₀ⁿσ₀⊕Zσ[X] ≠ ϕ′.

Iterating this construction ω-many times produces a model ℳ₁ = (ω, S₁) of RCA₀ + COH in which ϕ′ ∈ S₁.

The principle D₂² was introduced in [3]. It states that for any Σ₀ⁿ-function f : M → 2, there is an infinite subset on which f is a constant. D₂² is equivalent to SRT²₂ over the base system RCA₀ + BΣ²₀ (Chong, Lempp and Yang [4]). We will adopt this version of SRT²₂ henceforth. A notion central to the analysis of SRT²₂ is what is referred to as a Scotch disjunction introduced in Chong, Slaman and Yang [11]. It was used in the construction of a model that separates SRT²₂ from RT²₂ (see Theorem 3.2 below). Although the notion was cast in the language of nonstandard arithmetic, it has a natural analog in standard arithmetic. We present an application of this notion.

2.1.1. Scotch disjunction. Let C : N → 2 be a Σ₀ⁿ-definable (possibly with parameters) coloring. Call a number x red (resp. blue) if C(x) = 0 (resp. C(x) = 1). Let ρ and β denote finite binary strings which we identify with finite sets through their respective characteristic functions (ρ will usually denote a red string while β will usually denote...
a blue string). Given $(\rho, \beta)$, an infinite set $X$ such that $\min X > \max \{\rho, \beta\}$, and given a $\Sigma^0_1$-predicate $\varphi \equiv \exists u \psi(u, x)$ (with parameters) where $x$ is a free variable, define the following collection $\{o_s : s \in \omega\}$ of “blobs”:

- $o_0 \subset X$ is the least finite set $o$ such that
  $$\min o > \max \rho \cup \beta, \text{ and } \exists u (\max u \leq \max o \land \psi(u, \rho \cup o))$$
  holds;

- $o_{s+1} \subset X$ is the least finite set $o$ enumerated such that
  $$\min o > \max o_s \text{ and } \exists u (\max u \leq \max o \land \psi(u, \rho \cup o))$$
  holds.

Simultaneously, define the Seetapun tree $T_s$ associated with $\{o_s' : s' \leq s\}$ as follows: Let $\beta$ be the root of $T_s$. Put $\beta \cup n$ at level 0 of $T_s$ for each $n \in o_0$. Inductively, if $\eta$ is a node of $T_s$ at level $s'$, where $s' < s$, and $n$ is a number in $o_{s'+1}$, then $\eta \cup n$ is a node of $T_s$ at level $s' + 1$. Declare the configuration $(\delta_s, T_s)$, where $\delta_s = \{o_0, \ldots, o_s\}$, to be a Seetapun disjunction if for every maximal path $p$ in $T_s$, there exists a subset $\nu$ such that $\exists u (\max u \leq \max \nu \land \psi(u, \beta \cup \nu))$ holds. Note that each of the enumerations of $o_s$, $T_s$ and of a Seetapun disjunction is $\Sigma^0_1$ relative to $X$.

![Figure 1. A Seetapun disjunction](image)

Figure 1 is an illustration of a Seetapun disjunction consisting of finite sets $o_0, o_1, \ldots$, a maximal path $p$ and a finite subset $\nu$ of $p$. The point of computing a Seetapun disjunction is encapsulated in the following claim:

**Claim 2.2.** If $(\delta_s, T_s)$ is a Seetapun disjunction, where $\rho$ is red and $\beta$ is blue, then either there is a red solution for $\varphi$ or a blue solution for $\varphi$.

A red solution is a blob $o_{s'}$, where $s' \leq s$, consisting only of red numbers, while a blue solution is a blue set $\nu$ such that $\exists u (\max u \leq \max \nu \land \psi(u, \beta \cup \nu))$ holds. If there is no red solution on the $\delta_s$ side, then it must be the case that each $o_{s'}$ contains a blue number. Hence there is a maximal path in $T_s$ consisting only of blue members (a “blue path”). On this path there is a blue solution for $\varphi$. In other words, if a Seetapun disjunction is enumerated, then either the red side or the blue side makes progress on $\varphi$.

Now there are two ways in which one may fail to enumerate a Seetapun disjunction. One is that for some $s$, $o_s$ is enumerated but not $o_{s+1}$. This would occur if there is no $o$ such that $\min o > \max o_s$ and $\exists u (\max u \leq \max \nu \land \psi(u, \beta \cup \nu))$ holds. The other is that for some $s'$, $o_{s'}$ is enumerated but not $o_{s'+1}$. This would occur if there is no $\nu$ such that $\exists u (\max u \leq \max \nu \land \psi(u, \beta \cup \nu))$ holds.
max \( o \land \psi(u, \rho \cup o) \) holds. In particular, this applies to red \( o \). The other way is for \( o_s \) and hence \( T_s \) to be defined for each \( s \in \omega \) but there is a maximal path in \( T_s \) with no subset \( v \) for 

\[ \exists u (\max u \leq \max v \lor \psi(u, \beta \cup v)) \] 

holds. Then for any infinite path \( Y \) on \( T = \bigcup_i T_i \), for any finite \( v < Y \) such that \( \min v > \max \beta, \neg \exists u (\max u \leq \max v \lor \psi(u, \beta \cup v)) \) holds. In either case we ensure a “\( \Pi_1 \)-outcome” for \( \varphi \). We give the first application of Seeatapun disjunction to \( \text{RT}_2^2 \):

2.1.2. \( \text{RCA}_0 + \text{RT}_2^2 \psi \text{RT}_2^3 \).

**Theorem 2.3.** (Seeatapun and Slaman [38]) There is a model \( \mathcal{M} = (\omega, S) \) of \( \text{RCA}_0 + \text{RT}_2^2 \) in which \( \varphi' \notin S \).

**Proof.** We take as ground model \( \mathcal{M}_0 = (\omega, S_0) \) where \( S_0 \) is the collection of recursive sets. Let \( C \in S \) be a two-coloring (red and blue) of pairs of numbers. We construct a G homogeneous for \( C \) such that \( G \not\prec \varphi' \). Let \( A_s = \{ y : C(s, y) = \text{red} \} \). By the proof of Theorem 2.1, there is a \( Z \) cohesive for the array \( \{ A_s : s \in \omega \} \) and \( Z \not\prec \varphi' \). Then \( C \upharpoonright |Z|^2 \) is a stable two-coloring of pairs in \( Z \), and any \( G \subseteq Z \) that is homogeneous for \( C \upharpoonright |Z|^2 \) is homogeneous for \( C \). Hence it is sufficient to show that if \( C \leq_T Z \) is a stable two-coloring of pairs and \( Z \not\prec \varphi' \), then there is a \( G \) homogeneous for \( C \) such that \( G \not\prec \varphi' \). We will sketch a proof of this.

Call a number \( x \) “red” if \( \lim_{s \to o} C(x, s) = \text{red} \), and “blue” if \( \lim_{s \to o} C(x, s) = \text{blue} \). Given \( \rho, \beta \) and reduction procedure \( \Phi \), let \( \varphi \in \mathcal{S}_1 \) be:

\[ \exists n \leq \max \{ \exists \delta_0, \delta_1 \delta_0, \delta_1 \subseteq o \land \Phi(\rho \cup \delta_0) = Z(n) \neq \Phi(\rho \cup \delta_1) = Z(n) \} \]

We say in this case that \( (\delta_0, \delta_1) \) is a \( Z \)-splitting \( \Phi \) over \( \rho' \). A Seeatapun disjunction is declared for \( (\delta_0, \delta_1) \) if for each maximal path \( p \in T_0 \), there exists a \( v \) such that \( p = \rho \cup v \) holds, i.e. there exist \( v_0, v_1 \) such that \( v_0, v_1 \subseteq v \) and \( \Phi(\rho \cup v_0) = Z(n) \neq \Phi(\rho \cup v_1) = Z(n) \) for some \( n \leq \max v \). We say in this case that \( (v_0, v_1) \) \( Z \)-splits \( \Phi \) over \( \beta' \).

Let \( \{ \Phi_0, \Phi_1, \ldots \} \) be an effective list of reduction procedures. The construction of the homogeneous set \( G \) proceeds in stages. Let \( X \) in the definition of Seeatapun disjunction be the cohesive set \( Z \). Hence the blobs \( o_s \) are finite sets enumerated in \( Z \). We will repeatedly apply the following due to Jockusch and Soare [24]:

(*) Let \( Z, A \in \omega \) such that \( Z \not\prec_T A \). If \( T \) is an infinite, \( Z \)-recursively bounded, \( Z \)-recursive tree, then \( T \) has an infinite path \( W \) such that \( W \not\prec Z \prec_T A \).

At stage 0, let \( \rho = \beta = \phi \). Enumerate \( (\delta_0, \delta_1) \) for \( Z \)-splitting of \( \Phi = \Phi_0 \) as described above. If a Seeatapun disjunction \( (\delta_0, \delta_1) \) is enumerated, choose a red \( \delta_1 \subseteq o \) for some \( s' \leq s \) and \( i = 2 \), or a blue \( v_1 \subseteq v \) for some \( i < 2 \), as the case may be (according to the Claim), so that \( \Phi(\rho \cup v) = Z(n) \neq \Phi(\rho \cup \delta) = Z(n) \) for some \( n \leq \max \delta \). Hence the \( \delta \)’s are defined for each \( s \). Then \( T = \bigcup_i T_i \) is an infinite \( Z \)-recursively bounded \( Z \)-recursive tree. By (*) select an infinite path \( Y_0 \in T \) such that \( Y_0 \not\prec Z \not\prec \varphi' \). Let \( \beta_0 = \{ n_0 \} \) where \( n_0 > t_0 \) is red, and let \( \beta_0 = \phi \) (note that if \( n_0 \) does not exist, then all numbers in \( Z \) above \( t_0 \) are blue and we will immediately have a blue homogeneous set \( Z \not\prec \varphi' \) contained in \( Z \)).

Note 1. No infinite \( V < Z \) such that \( \min V > t_0 \) computes \( \varphi' \) relative to \( Z \) via \( \Phi_0 \). Suppose otherwise and \( W \) is a counterexample. Since \( Z \)-splitting of \( \Phi_0 \) exists for any \( o \in Z \) with \( \min o > t_0 \), it means that for any finite set \( o \subseteq Z \) and \( n \), if \( \Phi_0^{\sigma}Z(n) \) for some \( \sigma \in Z \), then \( \Phi_0^{\sigma}Z(n) = \Phi_0^{\sigma}Z(n) = \varphi'(n) \). This gives an algorithm to compute \( \varphi' \) from \( Z \), which is a contradiction.

Case 2. \( o_s \) is defined for each \( s \). Then \( T = \bigcup_i T_i \) is an infinite \( Z \)-recursively bounded \( Z \)-recursive tree. By (*) select an infinite path \( Y_0 \) in \( T \) such that \( Y_0 \not\prec Z \not\prec \varphi' \). Let \( \beta_0 = \{ n_0 \} \) where \( n_0 \) is a blue number in \( Y_0 \) and let \( \rho_0 = \phi \). Again if \( n_0 \) does not exist then we will have a red homogeneous set \( W \subseteq Y_0 \) such that \( W \not\prec Z \not\prec \varphi' \).

Note 2. No infinite subset \( V \) of \( Y_0 \) satisfies \( \Phi_0^{\sigma}Z = \varphi' \). Suppose otherwise and \( W \) is a counterexample. Then there is no \( Z \)-splitting of \( \Phi_0 \) in \( Y_0 \), for any \( v < Y_0 \) and any \( n \), if \( \Phi_0^{\sigma}Z(n) \) for some \( \varphi' \) from \( Y_0 \not\prec Z \), which is a contradiction.

Next, we describe the construction at stage 1. This will give an idea of the steps to take at an arbitrary stage. There are several scenarios to consider, depending on the outcome at stage 0:

- Suppose a Seeatapun disjunction was enumerated at stage 0. If \( \rho_0 = \phi \), let \( \Phi = \Phi_0 \). If \( \rho_0 \neq \phi \), let \( \Phi = \Phi_1 \). Suppose no Seeatapun disjunction was enumerated but Case 1 holds. Let \( \Phi = \Phi_1 \). Now do the following: Define \( \varphi \) as in Stage 0, upon setting \( (\rho, \beta) = (\rho_0, \beta_0) \). Enumerate “blobs” \( o_s \subseteq Z \) with min \( o_s \geq \min \{ \rho_0, \beta_0, o_{s-1} \} \) and define the trees \( T_i \) as before, with \( \beta_0 \) as root. A Seeatapun disjunction for \( (\delta_0, \delta_1) \) is declared if the following modified condition is satisfied: Every maximal path in \( T_i \) has a subset \( v \) that \( Z \)-splits \( \Phi \) over \( \beta_0 \), where \( \Phi = \Phi_0 \) if \( \beta_0 = \phi \) and \( \Phi = \Phi_1 \) otherwise. There are now three possible outcomes:
(i) If a Seetapun disjunction is enumerated during this process and there is a red $\alpha'$, where $s' \leq s$, such that $\varphi'(p_0 \cup \alpha')$ holds, let $\bar{\alpha}_i < \alpha'$, for some $i < 2$, be chosen such that $\Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i}(n) \neq \varphi'(n)$ for some $n$. Let $\rho_1 = p_0 \cup \bar{\alpha}_1$, $\beta_1 = \rho_0$. If there is a blue $\nu$ such that $\varphi'(p_0 \cup \nu)$ holds, let $\nu_1 < \nu$, for some $i < 2$, be chosen such that $\Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i}(n) \neq \varphi'(n)$ for some $n$, and let $\beta_1 = p_0 \cup \nu_1$, $\rho_1 = \rho_0$.

(ii) If no disjunction is enumerated, but there is a (least) $s$ such that $\alpha_0$ is not defined, denote this by $s_1$ and let $t_1$ be an upper bound of numbers in $\alpha_{s-1}$, and let $\rho_1 = p_0 \cup [n_1]$ where $n_1 \in Z$ is a red number larger than $t_1$. Let $\beta_1 = \rho_0$.

Note that in either (i) or (ii), if $\rho_1 > \rho_0$ then either $\Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i}$ has already diagonalized against $\varphi'$ for $\Phi$, or for any infinite $W \subset Z$ such that $\min W > \max \rho_1$, $\Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i \cup W}$ is either partial or has range recursive in $Z$, hence nor equal to $\varphi'$. The same holds if $\rho_0, \rho_1$ are replaced by $\beta_0, \beta_1$ respectively.

(iii) If no disjunction is enumerated, and $\alpha_0$ is defined for all $s$, then $T = \bigcup_i T_i$ is an infinite $Z$-recursively bounded, $Z$-recursive tree. By (v), let $Y_1$ be an infinite path in $T$ such that $Y_1 \cup \eta \neq \varphi'$. Let $\beta_1 = \rho_0 \cup [n_1]$, where $n_1$ is a blue number in $Y_1$ larger than $\max \beta_0$. Let $\rho_1 = \rho_0$. Then for all infinite $W \subset Y_1$ such that $\min W > \max \rho_1$, we have $\Phi^0_W \neq \varphi'$ to be either partial or with range recursive in $Z$, hence not equal to $\varphi'$.

- If Case 2 above holds at Stage 0, enumerate $s_0 \subset Y_0$ after making the following modification: In $\varphi$, let $(p, \beta) = (p_0, \beta_0)$, $\Phi = \Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i}$ and $\rho \subset Y_0$. Thus if $s_{s-1}$ is defined, search for an $\alpha \subset Y_0$ with $\min \alpha > \max s_{s-1}, p_0, \beta_0$ such that there exist $\bar{\alpha}_i, \alpha_i \subset \alpha$ which $Y_0 \bullet Z$-splits $\Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i} \setminus \rho \setminus \Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i} \setminus \rho$.

(v) If Case 2 above holds at Stage 0, then $s_0 \subset Y_0$ after making the following modification: In $\varphi$, let $(p, \beta) = (p_0, \beta_0)$, $\Phi = \Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i}$ and $\rho \subset Y_0$. Thus if $s_{s-1}$ is defined, search for an $\alpha \subset Y_0$ with $\min \alpha > \max s_{s-1}, p_0, \beta_0$ such that there exist $\bar{\alpha}_i, \alpha_i \subset \alpha$ which $Y_0 \bullet Z$-splits $\Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i} \setminus \rho \setminus \Phi^0_{\bar{\alpha}_i \cup \bar{\alpha}_i} \setminus \rho$.

The construction at stage $k > 0$ is carried out similarly. Thus $\rho_k, \beta_k$ are defined for each $k$ and either $\rho_{k+1} > \rho_k$ or $\beta_{k+1} > \beta_k$. If $\rho_{k+1} > \rho_k$ for infinitely many $k$, let $G = \bigcup_k \rho_k$. Otherwise, let $G = \bigcup_k \beta_k$. Then $G$ is a red or blue homogeneous set for $C$ and $G \not\equiv Z \neq \varphi'$.

Next, take $\mathcal{M}_0[Z, G]$ and apply the argument in Theorem 2.1 to the next array $A$ in the structure, to obtain an infinite set cohesive for $A$ and whose join with $G \equiv Z$, denoted $Z^*$, does not compute $\varphi'$. Take the structure $\mathcal{M}_0[Z^*]$ and apply the construction above to the next instance of stable two-coloring in the structure. This produces an infinite homogeneous set (for the coloring) whose join with $Z^*$ does not compute $\varphi'$.

We complete the proof of the theorem by iterating this construction, arriving at a countable $\omega$-extension $\mathcal{M} = (\omega, S)$ of $\mathcal{M}_0$ such that $\mathcal{M} \models \text{RCA}_0 + \text{RT}_2^2$ and $\varphi' \not\in S$.

\[\square\]

2.2. Weak König’s Lemma $\text{WKL}_0$. As seen earlier, $\text{WKL}_0$ is stronger than $\text{RCA}_0$. Where or how $\text{RT}_2^2$ fits in the lower end of the “big five picture” is an interesting question in the reverse mathematics of Ramsey’s Theorem. The question whether $\text{RCA}_0 + \text{RT}_2^2 \rightarrow \text{WKL}_0$ was a major open problem. The solution appeared in 2012:

**Theorem 2.4.** (Liu [29]) $\text{RCA}_0 + \text{RT}_2^2 \not\rightarrow \text{WKL}_0$.  

To construct a model of $\text{RCA}_0 + \text{RT}_2^2 + \neg \text{WKL}_0$, first observe that there is a recursive set which codes an infinite tree of binary strings in which every infinite path computes a completion of Peano arithmetic. This is the candidate to use for diagonalization, i.e. construct a model $\mathcal{M} = (\omega, \emptyset)$ of $\text{RCA}_0 + \text{RT}_2^2$ such that no path in the tree belongs
A natural way to approach this is to use the cone avoidance strategy in [38]. It was clear, however, that the technique required to achieve this would have to be different and more intricate than that introduced to prove Theorem 2.3. For a start, instead of avoiding a non-recursive Turing degree, one now has to avoid uncountably many degrees in a construction to be completed in \( \omega \) steps. Resolving this apparent conflict is a key challenge. Apart from [29], the reader may also refer to Hirschfeldt [20] for an excellent exposition of the proof.

In a subsequent paper [30], Liu showed that, indeed, \( \text{RT}^2 \) does not even imply a weak form of \( \text{WKL}_0 \) called the weak Weak König’s Lemma Principle \( \text{WWKL} \): If \( T \) is an infinite tree of binary strings such that

\[
\lim_{s \in T \land |s| = \omega} \frac{1}{2^{|s|}} \neq 0,
\]

then there is an infinite path in \( T \). This measure-theoretic principle was introduced in Yu and Simpson [43] and is weaker than \( \text{WKL}_0 \) over \( \text{RCA}_0 \). It is yet another evidence of the limited proof-theoretic strength of \( \text{RT}^2 \).\(^2\)

3. Separating \( \text{SRT}^2 \) from \( \text{RT}^2 \)

On the face of its definition, one would naturally expect \( \text{SRT}^2 \) to be weaker than \( \text{RT}^2 \) over the base system \( \text{RCA}_0 \). However, confirming this intuition mathematically turned out to require a certain amount of effort. From the model-theoretic point of view, one can establish \( \text{RCA}_0 + \text{SRT}^2 \not\text{RT}^2 \) by exhibiting a model \( \mathcal{M} \) for it. In particular, \( \mathcal{M} \) need not be an \( \omega \)-model. This was the philosophical position and approach taken in [11]. There are two facts about two-coloring of pairs that shed light on the problem at hand:

(I) The result of Jockusch [23] stated earlier on the non-existence of a \( \Sigma^0_2 \)-definable homogeneous solution for a particular recursive two-coloring of pairs;

(II) In contrast, every stable two-coloring of pairs has a \( \Delta^0_2 \)-solution.

However, Downey, Hirschfeldt, Lempp and Solomon [17] gave an example of one with no low solution. A natural first approach, in view of (I), is to seek a model of \( \text{RCA}_0 + \text{SRT}^2 \not\text{RT}^2 \) consisting only of \( \Delta^0_2 \)-sets. This approach has an obvious drawback: Suppose \( \mathcal{M} = (\omega, S) \) is such a model. Then if \( G \leq_T \varphi' \) is a member of \( S \), a two-coloring \( C \) of pairs which is stable relative to \( G \) may not have a solution in \( S \), i.e. one that is recursive in \( \varphi' \), since the property of being a \( \Delta^0_2 \)-set is not transitive under the relation “\( \Delta^0_2 \) in”.

A modified approach is for \( S \) to consist only of low sets. This approach would seem to have been ruled out by (II). However, a closer examination reveals that the proof in [17] applies an infinite injury priority argument that requires \( I \Sigma^0_2 \). It exploits the fact that under \( I \Sigma^0_2 \) there exist nonlow incomplete \( \Delta^0_2 \)-sets, a property which actually fails in some models without \( \Sigma^0_2 \)-induction. One can trace the history of this mathematical observation back to the development of \( \alpha \)-recursion theory in the 1970’s: Firstly, Shore [39] showed that if \( \alpha \) is an admissible ordinal such as \( \kappa^\omega \) or \( \kappa_\omega^\omega \), constructible cardinals where \( \Sigma^2 \)-replacement fails, then every incomplete \( \alpha \)-r.e. set is low. The parallel between \( \alpha \)-recursion theory and recursion theory in fragments of arithmetic, especially where it pertains to the jump of an incomplete r.e. set, was investigated in Mytilinaios and Slaman [32] and Chong and Yang [13]. It turns out that many of the ideas and methods in \( \alpha \)-recursion theory are applicable in the arithmetic setting. The overall picture is that the failure of \( I \Sigma^0_2 \) in a model of \( B \Sigma^0_2 \) is the arithmetic counterpart of the failure of \( \Sigma^0 \)-replacement in an admissible ordinal with \( \Sigma^{n-1} \)-replacement. In particular, there exist models of \( B \Sigma^0_2 + \gamma \Sigma^0_2 \) in which every incomplete r.e. set is low. To elaborate on how one might find and use such a model, we pause to recall some notions in nonstandard arithmetic (see [5]).

Given a nonstandard model \( \mathcal{M} \) of \( \text{RCA}_0 \), call \( I \subset M \) a \( \Sigma^0_n \)-cut if it is \( \Sigma^0_n \)-definable, closed downwards, includes 0 and is closed under the successor operation. A set \( X \subseteq I \) is coded by an \( \mathcal{M} \)-finite set \( \hat{X} \) if \( \hat{X} \cap I = X \). We say in this case that \( X \) is coded (by some \( \hat{X} \)) for short. Now every model of \( \text{RCA}_0 + B \Sigma^0_2 + \gamma \Sigma^0_2 \) is already endowed with a basic collection of coded sets that have a bearing on the jump of an r.e. set (see Chong and Mourad [9], Chong and Yang [13]), but a richer collection of coded sets, such as that found in a saturated structure with \( \omega \) as a \( \Sigma^0_2 \)-cut, would ensure that every incomplete r.e.-set is low ([32]). The method of proof is reminiscent of that in [39] and generalizes to showing that every incomplete \( \Delta^0_2 \)-set is low.

\(^2\)Flood [18] introduced a principle called \( \text{RWKL} \) (Ramsey like Weak König’s Lemma) which was shown to be a consequence of both \( \text{WKL}_0 \) and \( \text{SRT}^2 \) but implies the principle \( \text{DNR} \) (Diagonally non-Recursive principle). See Bienvenue, Patey and Shafer [1] for a further study of \( \text{RWKL} \), where among other things, it was shown that the implication \( \text{RCA}_0 + \text{RWKL} \rightarrow \text{DNR} \) is strict.
The strategy for producing a model of RCA$_0$ that separates SRT$_2^2$ from RT$_2^2$ is therefore (i) select as ground model an appropriate $\mathfrak{M}_0 = (M, S_0) \models$ RCA$_0 + B \Sigma^0_2 + \neg \Sigma^0_2$, with a sufficient number of codes so that every incomplete $\Delta^0_2$-set in $\mathfrak{M}_0$ is low, and (ii) construct an $M$-extension that satisfies SRT$_2^2$ by adding only low $\Delta^0_2$-homogeneous sets. By (i) above, RT$_2^2$ fails in the $M$-extension.

In addition to having a sufficient number of coded sets, the model $\mathfrak{M}_0$ for (i) will also be equipped with an extra feature described below (this is a technical device introduced to make the proof work. See remarks following the proof of Theorem 3.6). Let $b \in M$ and assume that $P \subset \{<\}^2$ is a $\Sigma^0_2$-definable relation (with parameters) such that for each $\mathfrak{M}_0$-finite set $E$, $P^0(E) = \{E' : P(E, E')\}$ holds is $\mathfrak{M}_0$-finite. Inductively for $\nu < b$, define $P_{\nu+1}(E) = \{E'' : \exists E' \in P^\nu(E) (P(E', E'') \text{ holds})\}$.

**Definition 3.1.** Let $\mathfrak{M}_0$ be a model of RCA$_0 + B \Sigma^0_2$. We say that $\mathfrak{M}_0$ satisfies the bounded monotone enumeration principle (BME$_1$) if for all $b \in M$, $\Sigma^0_2$-definable relations $P$ (with parameters), and any $\mathfrak{M}_0$-finite set $E$, there is an $s^*$ such that $\bigcup_{s \in b} P^0_s(E) = \bigcup_{s \in b} P^0_s(E)$ for all $s > s^*$, where $P^0_s(E)$ is the set of $\mathfrak{M}_0$-finite sets enumerated into $P^\nu(E)$ by stage $s$. In other words, $\bigcup_{s \in b} P^\nu(E) = \bigcup_{s \in b} P^\nu_s(E)$ is $\mathfrak{M}_0$-finite.

In general, a model of RCA$_0 + B \Sigma^0_2$ need not satisfy BME$_1$ (for example, the principle fails in the model of $B \Sigma^0_2 + \neg \Sigma^0_2$ defined in [34]). By Kreutzer and Yokoyama [26], BNME$_1$ is equivalent over RCA$_0$ to the principle $P \Sigma^0_1$ (see Hájek and Pudlák [19] for a discussion of this principle), a version of the Pigeonhole principle, as well as totality of the Ackermann function.

**Theorem 3.2.** (Chong, Slaman and Yang [11]) There is a model $\mathfrak{M}$ of RCA$_0 +$ SRT$_2^2 + \neg$RT$_2^2$.

**Proof.** A countable model $\mathfrak{M}_0 = (M, S_0)$ of RCA$_0 + B \Sigma^0_2 + \neg \Sigma^0_2$ with the following properties was constructed in [11]:

1. $\omega$ is a $\Sigma^0_2$-cut;
2. Every arithmetically definable subset of $\omega$ is coded;
3. $S_0$ is the collection of all recursive sets;
4. $\mathfrak{M}_0 \models$ BME$_1$.

We construct an $M$-extension of $\mathfrak{M}_0$, denoted $\mathfrak{M} = (M, S)$, such that $\mathfrak{M} \models$ SRT$_2^2$ and every member of $S$ is low. By (i) above, RT$_2^2$ fails in $\mathfrak{M}$. Let $C \subseteq_T \varnothing'$ be a two-coloring of $M$. Call a number $x \in M$ “red” or “blue” according to $C(x) = 0$ or $C(x) = 1$. We implement a Seetapun disjunction-type construction, but now in the nonstandard domain.

We first describe the overall procedure. Let $X \subseteq M$ be $\mathfrak{M}_0$-finite and low, and let $B$ be an $\mathfrak{M}_0$-finite collection of $\Sigma^0_2$-formulas (with parameters) from a fixed set variable $G$. Given a pair $(\rho, \beta)$ of $\mathfrak{M}_0$-finite sets (identified respectively with the binary strings which are their characteristic functions), enumerate a sequence of “blobs” $\alpha_s \subset X$, $s \in M$, such that $\min \alpha_s > \max_{s < x} \{\rho, \beta, \rho \cup \beta\}$ and $\varphi(\rho \cup \alpha_s)$ holds for some $\varphi \in B$. Simultaneously, define the Seetapun tree $T_s$ associated with $(\alpha_0, \ldots, \alpha_s)$ as follows: The root of $T_s$ is $\beta$. If $\rho$ is a node in $T_s$, where $s' < s$, then for each $n \in \alpha_{s'+1}$, $\sigma \cup n$ is a node in $T_{s'+1}$. Declare $(\alpha_s, T_s)$, where $\alpha_s = (\alpha_0, \ldots, \alpha_s)$, to be a Seetapun disjunction if for each maximal path $\rho$ in $T_s$, there is a subset $\nu$ such that $\varphi(\beta \cup \nu)$ holds for some $\varphi \in B$. In the notation of Definition 3.1, we let $E = (\rho, \beta)$, $b = |B|$ and

$$P^0_s(\rho, \beta) = \{s' \leq s \land \exists \varphi \in B(\varphi(\rho \cup \alpha_s)) \cup \{\{\rho, \beta \cup \nu\} : \nu = \text{least subset of a maximal path in } T_s \text{ s. t. } \exists \varphi \in B(\varphi(\beta \cup \nu))\},$$

and call it the set of exits over $(\rho, \beta)$. Note that if $(\rho', \beta')$ is an exit, then $\rho' > \rho$ implies $\beta' = \beta$ and vice versa. As before, one has a

**Fact.** Suppose $\rho$ is red and $\beta$ is blue. If $(\alpha_s, T_s)$ is a Seetapun disjunction, then either there is a red $\alpha_{s'}$ for some $s' \leq s$, or there is a blue maximal path in $T_s$.

Upon enumeration of a Seetapun disjunction, one repeats the above steps as follows: For each exit $(\rho', \beta') \in P^0(\rho, \beta)$, replace $B$ by $B \setminus \{\varphi\}$, where $\varphi$ is satisfied by $\rho'$ if $\rho' > \rho$ or by $\beta'$ if $\beta' > \beta$ (recall if $\rho' > \rho$ then $\beta' = \beta$ and vice versa), and enumerate blobs over $\rho'$ and define simultaneously a Seetapun tree with root $\beta'$. If and when a Seetapun disjunction is enumerated, members of $P^0(\rho', \beta')$ are defined. Then $P_1(\rho, \beta) = \{P^0(\rho', \beta') : (\rho', \beta') \in P^0(\rho, \beta)\}$. In this way one obtains $P^2(\rho, \beta), P^3(\rho, \beta), \ldots$. Applying BME$_1$ in $\mathfrak{M}_0$, one concludes that there is an $s^*$ such that

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3We adopt the convention that if $\varphi = \exists u \psi(u, x)$, then to say that $\varphi(\rho \cup \alpha_s)$ holds means that there is a $u$ with $\max u \leq \max \alpha_s$ such that $\psi(u, \rho \cup \alpha_s)$ holds. The same applies to $\varphi(\beta \cup \nu)$. 

The union $\bigcup_{\rho \in \mathcal{B}} P^\nu(\rho, \beta) = \bigcup_{\rho \in \mathcal{B}} P^\nu_\nu(\rho, \beta)$. In $P^\nu_\nu(\rho, \beta)$, let $(\rho^*, \beta^*)$ be a maximal pair such that $\rho^*$ is red and $\beta^*$ is blue. There are two cases to consider:

**Case 1.** In the enumeration of blobs over $\rho^*$, there is a largest $s$, denoted $s_0$, such that $\omega_{s_0}$ is defined. Let $B_{\rho^*} \subset B$ be the set of formulas not satisfied by $\rho^*$. Then there is no set $o$ such that $\min o > \max o$, and $\rho^* \cup o$ satisfies a formula in $B_{\rho^*}$. This means that every formula in $B_{\rho^*}$ has a $\Pi^0_1$-outcome for any $G \in X$ extending $\rho^*$. Thus for any $\phi \in B$, $\phi(G)$ if and only if $\phi(\rho^*)$. We say in this case that $\rho^*$ “decides” $B$.

**Case 2.** $\omega_s$ is defined for all $s$.

Then the tree $T = \bigcup T_s$ with root $\beta^*$ is $\mathcal{M}_0[X]$-infinite. Let $B_{\beta^*}$ be the set of formulas in $B$ not satisfied by $\beta^*$. Then $U = \{ \sigma \in T : \forall \phi \in B_{\beta^*}(\neg \phi(\sigma)) \}$ is an $X$-recursively bounded, $X$-recursive $\mathcal{M}_0[X]$-infinite tree. By the Low Basis Theorem [24] which holds in models of $\text{RCA}_0 + \Sigma^0_2$, one may choose an unbounded path $Z$ in $U$ that is low relative to $X$, and hence low as well. Then for $\rho^*$, the blue set $\beta^*$ decides $B$ in the sense that for all $G \in Z$ extending $\beta^*$, for all $\phi \in B$, $\phi(G)$ if and only if $\phi(\beta^*)$.

The above outline sets the stage for the construction of our desired model. Since $\omega$ is a $\Sigma^0_2$ cut in $\mathcal{M}_0$, there is a $\Sigma^0_2$-definable, strictly increasing and cofinal map $g : \omega \to M$.

**Stage 0.** Let $B = B_0 = \{ \phi : \phi \text{ a $\Sigma^0_1$-formula with parameters in } g(0) \}$ and a free set variable $\hat{G}$. Let $\rho = \beta = \emptyset$ and let $X = M$. Execute the steps above to obtain $(\rho^*, \beta^*) = (\rho_0^*, \beta_0^*)$.

The next step is to preserve $\text{BME}_1$, for instances with parameters less than $g(0)$, relative to the generic set $G$ we are constructing. We weave in a construction to achieve this. The construction has two versions, depending on whether $(\rho_0^*, \beta_0^*)$ was obtained via Case 1 or Case 2 above.

Let $\hat{P}(x, x', \hat{G})$ be a “universal instance” of relativized $\text{BME}_1$, with a $\Sigma^0_1$ definition containing parameter $b \in g(0)$ (the existence of a universal instance was shown in [11]). The numbers $x$ and $x'$ will code $\mathcal{M}_0$-finite sets $E, E'$; The $\Sigma^0_1$ property of $\hat{P}$ makes it amenable to a Seetapun disjunction style analysis.

First suppose Case 1 holds. Let $s_0$ be the largest $s$ for which $\omega_s$ is defined over $\rho_0^*$ in Case 1. Let $E = \emptyset$ and let $\omega_s > \max o$. Enumerate blobs $\omega_s$ (by abuse of notation, we continue to use $\omega_s$ in this part of the construction), where $\min o > \max \{ s_0, o_{s-1} \}$, such that for some $E'$, $\hat{P}(\sigma, E', \rho_0^* \cup \omega_s)$ holds after max $\omega_s$ steps of computation (write this as $E' \in \hat{P}_{\rho_0^* \cup \omega_s}$), and simultaneously define the Seetapun tree $T_s$ with root $\beta_0^*$ as before. Declare $(\omega_s, T_s)$ to be a Seetapun disjunction if for each maximal path in $T_s$ there is a subset $v$ such that an $E'$ is enumerated in $\hat{P}_{\rho_0^* \cup \omega_s}$, $\forall \beta \in v$, i.e., $\hat{P}(\phi, E', \beta \cup v)$ holds for some $E'$ after max $v$ steps of computation.

If and when this occurs, call a pair $(\rho', \beta')$, where $(\rho', \beta') = (\rho_0^* \cup \omega_s, \beta_0^*)$ or $(\rho', \beta') = (\rho_0^*, \beta_0^* \cup v)$, an exit (for uniqueness, in the latter case choose the least $v$ for each maximal path in $T_s$). Note that by the definition of an exit, at least one of the exits will be a red $\rho' > \rho_0^*$ or a blue $\beta' > \beta_0^*$. Furthermore, if $\rho' > \rho_0^*$ then $\beta' = \beta_0^*$, and vice versa. If $\rho' > \rho_0^*$ then there exists an $E' \in \hat{P}_{\rho_0^* \cup \omega_s}$. If $\beta' > \beta_0^*$ then there is an $E' \in \hat{P}_{\rho_0^* \cup \omega_s}$. Given an exit $(\rho', \beta')$, let

$$E_0(\rho') = \{ E' : E' \in \hat{P}_{\rho_0^* \cup \omega_s}(x, x'), \text{ where } \rho' > \rho_0^* \land \beta' = \beta_0^* \},$$

$$E_0(\beta') = \{ E' : E' \in \hat{P}_{\rho_0^* \cup \omega_s}(\phi, \beta), \text{ where } \rho' = \rho_0^* \land \beta' > \beta_0^* \}.$$

Each exit $(\rho', \beta')$ and its accompanying sets $E_0(\rho')$, $E_0(\beta')$ are then used to replace

$$(\rho_0^*, \beta_0^*, \emptyset)$$

to generate the next collection of Seetapun disjunctions and their exits. More precisely, we do the following:

First suppose $\rho' > \rho_0^*$, $\beta' = \beta_0^*$. Enumerate blobs $\omega_s$, with $\min o > \max \{ \rho', \beta', o_{s-1} \}$, for which there is an $E' \in E_0(\rho')$ and an $E' \in \hat{P}_{\rho_0^* \cup \omega_s}(E', \rho' \cup \omega_s)$, i.e., $\hat{P}(E', E'_0, \rho' \cup \omega_s)$. In this case we say that $E_0(\rho')$ is in $\rho_0^* \cup \omega_s$. Simultaneously define an associated tree $T_s$ with root $\beta'$. A Seetapun disjunction for $(\omega_s, T_s)$ is declared if every maximal path in $T_s$ contains a subset $v$ for which there is an $E' \in \hat{P}_{\rho_0^* \cup \omega_s}(x, \beta \cup v)$. An exit in the Seetapun disjunction is a pair $(\rho'', \beta'')$ where $\rho'' = \rho' \cup \omega_s$ and $\beta'' = \beta'$, or $\rho'' = \rho'$ and $\beta'' = \beta' \cup v$ for some least $v$ contained in a maximal path of $T_s$. Define $E_1(\rho'') = \{ E'' : E'' \in \hat{P}_{\rho_0^* \cup \omega_s}(x, x'), \text{ if } \rho'' > \rho', \text{ and } E_0(\beta'') = \{ E' : E' \in \hat{P}_{\rho_0^* \cup \omega_s}(x, \beta') \text{ if } \beta'' > \beta' \}$. Now suppose $\rho' = \rho_0^*$ and $\beta' > \beta_0^*$.

Enumerate blobs $\omega_s$ such that $\min o > \max \{ \rho', \beta', o_{s-1} \}$ for which there is an $E' \in \hat{P}_{\rho_0^* \cup \omega_s}(x, x')$. Simultaneously enumerate an associated tree $T_s$ with root $\beta'$ and declare $(\omega_s, T_s)$ to be a Seetapun disjunction if for every maximal path in $T_s$, there is a subset $v$ for which there is an $E' \in E_0(\beta')$ and an $E'' \in \hat{P}_{\rho_0^* \cup \omega_s}(x, \beta' \cup v)$. By definition
we say that $E'' \in \mathcal{P}_\text{max}^1(\emptyset, \beta' \cup \nu)$. Define exits $(\rho'', \beta'')$ as before and $E_0(\rho'')$ for $\rho'' > \rho'$ and $E_1(\beta'')$ for $\beta'' > \beta'$ similarly. In this way, uniformly in $v \leq b = |B_0|$, one defines exits $(\rho, \beta)$, sets $\mathcal{P}^i(\emptyset, \rho)$ and $\mathcal{P}(\emptyset, \beta)$, as well as $E_i(\rho), E_i(\beta)$ (where applicable). By BME$_1$, there is a least $t$, denoted $t^*$, such that $\mathcal{P}^n = \mathcal{P}^n_0$, for $v \leq b$. Choose an exit $(\rho_0, \beta_0)$ that is maximal and enumerated by $t^*$, such that $\rho_0$ is red and $\beta_0$ is blue. Note that $\rho_0 \geq \rho_0$ and $\beta_0 \geq \beta_0$. Assume $(\rho_0, \beta_0) \in \mathcal{P}^n$.

Now for $(\rho_0, \beta_0)$, there are two possible outcomes:

(i) There is a largest $s$ for which $\alpha_s$ is defined over $\rho_0$, i.e. $\mathcal{P}^n_{max}(\emptyset, \rho_0 \cup \alpha_s)$ enumerates an output. Let $s_0$ denote the largest such $s$ and let $t_0 > max \, \alpha_{s_0}$. Proceed to Stage 1 of the construction.

(ii) $\alpha_s$ is defined for each $s$. Let $T = \bigcup T_s$. Then

$$U_0 = \{\sigma : \sigma \in T \wedge \sigma \text{ contains no subset } v \text{ s. t. } \mathcal{P}^{n+1}(\emptyset, \rho_0 \cup v) \text{ enumerates an output}\}$$

is an $\mathfrak{M}_0$-infinite, recursively bounded recursive tree and hence contains an $\mathfrak{M}_0$-infinite low path $Z_0$ by the Low Basis Theorem. Proceed to Stage 1 below.

Note that in either (i) or (ii), any $\mathfrak{M}_0$-infinite $G$ extending $\rho_0$ or $\beta_0$, where min $G \setminus \rho_0 > t_0$ if (i) applies, and $G \subset Z_0$ if (ii) applies, preserves instances of BME$_1$ with parameters in $g(0)$.

Now suppose Case 2 holds. Let $Z$ be an $\mathfrak{M}_0$-infinite low path in the tree $U$ extending $\rho_0$ chosen for this case. We enumerate blobs $\alpha_s$, where min $\alpha_s > max \, (\rho_0, \beta_0, \alpha_{s-1})$, for which an $\alpha'_s$ is enumerated in $\mathcal{P}^n_{max}(\emptyset, \rho_0 \cup \alpha_s)$ as before, but now $\alpha_s \subset Z$. Simultaneously define $T_s$ over $\mathcal{P}^n_{max}(\emptyset, \rho_0 \cup \alpha_s)$ in the same fashion. Declare $(\alpha_s, T_s)$ to be a Seetapun disjunction if the same conditions are met, and use each exit in the disjunction to enumerate the next Seetapun disjunction, and so on. Then BME$_1$ in $\mathfrak{M}_0$ ensures that there is a $t^*$ such that $\mathcal{P}^n = \mathcal{P}^n_{s^*}$, $\alpha_s \subset Z$, for all $v \leq |B_0|$. Then again there are two possible outcomes similar to (i) and (ii) which we now label as (iii) and (iv) respectively. If (iv) holds, then there is a low path $Z_0$ selected (as in (ii)) will be low relative to $Z$, hence low. This completes the construction at stage 0.

**Stage 1.** First suppose (i) or (iii) above holds. Let $\rho = \rho_0 \cup a$ where $a > max \, \alpha_{s_0}$ is red (and belongs to $Z_0$ if (iii) holds). Such a number must exist else one obtains immediately a low blue solution for the coloring.

Let $\beta = \beta_0, B = B_1 = \{\varphi : \varphi \in \Sigma^0_l \}$ with a free set variable and parameters in $g(1)$. Let $X = M$ if (i) holds and $X = Z_0$ if (iii) holds. Now run the machine enumerating blobs $\alpha_s$ in $X$ and Seetapun trees $T_s$ to arrive at an $s^*$ such that $U_{\rho} \cup B_1 \cap P^\rho(\emptyset, \beta) = U_{\rho} \cup B_1 \cap P^\beta(\rho, \beta)$. As in Stage 0, let $(\rho^*, \beta^*)$ be a maximal pair that is an exit, with $\rho^* = \text{red}$ and $\beta^* = \text{blue}$. Set $(\rho^*_1, \beta^*_1) = (\rho^*, \beta^*)$ and note that there are now two scenarios (Case 1 and Case 2) to consider. In each case, proceed to the construction that enumerates outputs of the universal instance $\mathcal{P}$ of BME$_1$ with parameters less than $g(1)$. This step results in the extension of $(\rho^*_1, \beta^*_1)$ to $(\rho_1, \beta_1)$ which preserves BME$_1$ for instances with parameters less than $g(1)$. Now continue to the next stage.

If (ii) or (iv) holds, let $X = Z_0$ where $Z_0 \subset U_0$ is an $\mathfrak{M}_0$-infinite low path and extends $\beta_0$. Let $a > max \, \rho_0$ be a blue number in $Z_0$ (which must exist else $Z_0$ is a red solution for the coloring). Let $B = B_1$ defined above, $(\rho, \beta) = (\rho_0, \beta_0 \cup a)$. Enumerate $\alpha_s, T_s$ as before to arrive at an $s^*$ and $(\rho^*, \beta^*)$ which we accordingly denote as $(\rho^*_1, \beta^*_1)$. The next step is to preserve BME$_1$ for instances with parameters less than $g(1)$. Again, to achieve this, there are Case 1 and Case 2 to consider as before. Each case bifurcates into two possibilities and all result in the extension of $(\rho^*_1, \beta^*_1)$ to $(\rho_1, \beta_1)$ that preserves BME$_1$ for parameters less than $g(1)$.

In general, the construction proceeds by induction on $n \in \omega$. The countability of $\mathfrak{M}_0$ ensures that the construction is completed in $\omega$ steps. Let $(\rho_n, \beta_n)$ be the pair of strings constructed at the end of stage $n$. Then either $\rho_n \prec \rho_{n+1}$ for infinitely many $n$, or $\beta_n \prec \beta_{n+1}$ for infinitely many $n$. Suppose it is the former. Then the set $K = \{n : \rho_n \prec \rho_{n+1}\}$ is a definable subset of $\omega$ and hence coded in $\mathfrak{M}_0$. Using the code, $\varphi'$ can compute $g(n)$ for each $n \in K$ and thereby recover $\rho_n$. Then $G = \bigcup_{n \in K} \rho_n$ is a low and red homogeneous set for the coloring. The argument for the $\beta$-side is similar. Then $\mathfrak{M}_0[G]$ is a model of $\text{RCA}_0 + \text{BME}_1 + B^\Sigma_2$ and solves an instance of a stable two-coloring with a low set. To obtain the model $\mathfrak{M}$, we iterate the construction above by successively adjoining a homogeneous solution preserving BME$_1$ every step of the way. Each solution will also be low relative to those constructed earlier.

**Remark 3.3.** It was also shown in [11] that $\text{RCA}_0 + \text{SRT}_2^+ + \text{WKL}_0 \not\equiv \text{RT}_2^+$. The proof of Theorem 3.2 immediately implies

**Corollary 3.4.** $\text{RCA}_0 + \text{SRT}_2^+ \not\models I\Sigma_2^0$. 

Remark 3.5. Monin and Patey [31] have recently posted a proof of Theorem 3.2 using an $\omega$-model. The model does not yield Corollary 3.4. It is not known if there is a model $(\omega, S)$ separating $\text{SRT}_2$ from $\text{RT}_2^2$ in which every member of $S$ is low$_2$.

The limited inductive strength of $\text{SRT}_2^2$ extends to full scale Ramsey’s Theorem for pairs. It is another manifestation of the sharp contrast between $\text{RT}_2^2$ and its junior sibling.

Theorem 3.6. (Chong, Slaman and Yang [12]) $\text{RCA}_0 + \text{RT}_2^2 \nvdash \text{IS}_2^0$.

Proof. We give sketch of the proof. By Chong, Slaman and Yang [10], every model $\mathcal{M}$ of $\text{RCA}_0 + \Sigma^0_2$ has an $M$-extension that satisfies $\text{COH}$ while preserving $\Sigma^0_2$. This implies that $\text{RCA}_0 + \text{COH} + \Sigma^0_2$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + \Sigma^0_2$ (see §4). The proof can be strengthened to include preservation of $\text{BME}_1$ if $\mathcal{M} \models \text{BME}_1$ ([12]). By Proposition 1.3, it means that to establish the theorem, one starts with an appropriate ground model and alternates construction sets so that instances of $\text{COH}$ and $\text{SRT}_2^2$. The resulting structure will be a model of $\text{RT}_2^2$.

Denote by $\mathcal{M}_0 = (M, S_0)$ the structure in Theorem 3.2. Given a two-coloring $C$ of pairs in $\mathcal{M}_0$, one defines from it an array $\langle A_s : s \in M \rangle$ such that $C$ is stable on any set cohesive for the array. By [12] there is a solution $X_0$ for the array such that $\mathcal{M}_0[X_0] \models \text{RCA}_0 + \text{BME}_1 + \Sigma^0_2$. Now $C$ is stable on $X_0$ and we can implement the construction in Theorem 3.2 relative to $X_0$ to obtain a $G$ homogeneous for $C$ which preserves $\text{BME}_1$. We will show in fact $G \leq_T X_0'$ so that $\mathcal{M}_0[X_0, G] \models \text{BME}_1 + \Sigma^0_2$ as well.

For the model $\mathcal{M}_0$, the function $g : \omega \rightarrow M$ is $\Sigma^0_2$ without set parameters. A closer inspection of the construction, which is carried out relative to $X_0$ and follows the steps in Theorem 3.2, reveals that it is recursive in $X_0''$. Now the homogeneous set $G$ is either $\bigcup \rho$ or $\bigcup \beta$, and it can be computed from $X_0''$ because the latter is able to compute $W_\rho = \{ n : \rho_{n-1} < \rho_n \}$ and $W_\beta = \{ n : \beta_{n-1} < \beta_n \}$, where we set $\rho_{-1} = \beta_{-1} = \emptyset$. To see this, note that at each stage, the construction splits into Case 1 and Case 2, and each splits into two further cases (Case (i) and (ii) for Case 1 and Case (iii) and (iv) for Case 2). At stage $n$, the split into Case 1 or 2 depends on the $\Sigma^0_2(X_0)$-question: Is there a largest $s$ where $o_s$ is enumerated over $\rho_n$? Similarly the split into (i) or (ii), and the split into (iii) or (iv), depends on the $\Sigma^0_2(X_0)$-question: Is there a largest $s$ where $o_s$ is enumerated over $\rho_n$? Then $\rho_n > \rho_{n-1}$ if the answer to the first question is affirmative, and $\beta_n > \beta_{n-1}$ otherwise (note that $\beta_n = \beta_{n-1}$ only if Case 1 and Case (i) hold). Also, $\rho_{n+1} > \rho_n$ if the answer to the second question is affirmative (note that in this case one may still have $\beta_n > \beta_{n-1}$ if Case 2 holds at stage $n$). As an illustration, suppose Case 1 holds at stage $n$ for infinitely many $n$, and Case (i) never holds. Consider the set

$K_0 = \{ (n, m) : \exists s(o_s \text{ is enumerated over } \rho_n \text{ and } g(m) \leq s < g(m+1) \text{ at stage } n) \}.$

Then $K \subseteq \omega \times \omega$ and is $\Delta^0_2(X_0)$, hence coded by an $\mathcal{M}_0$-finite set $K_0$ in $\mathcal{M}_0$ according to [9]. Now define

$K_1 = \{ n : n \in \omega \land \lim_{m \to \omega} \langle n, m \rangle \in \hat{E}_0 \text{ exists} \}.$

Now $K_1$ is definable over $\mathcal{M}_0$ and hence coded. Then $X_0'$ can use the code to compute the homogeneous set $G$. It follows that $G$ is low relative to $X_0$. Thus $\mathcal{M}[X_0, G] \models \Sigma^0_2$. 

Remark 3.7. Theorem 3.6 also follows from Theorem 4.5 in the next section. Namely, if $\text{RCA}_0 + \text{RT}_2^2$ proves $\text{IS}_2^0$, then it would imply $\text{BME}_1$ as well, since the latter follows from $\text{IS}_2^0$. However, since $\text{BME}_1$ is a $\Pi^0_3$-sentence, the $\Pi^0_3$-conservation of $\text{RCA}_0 + \text{RT}_2^2$ over $\text{RCA}_0$ would then lead one to conclude $\text{RCA}_0 \vdash \text{BME}_1$, which is false.

Theorem 4.5 also implies that there is a model of $\text{RCA}_0 + \text{RT}_2^2$ in which $\text{BME}_1$ fails. It is an interesting problem to construct such a model.

4. Conservation Property of $\text{RT}_2^2$

The quest to understand the relationship between infinitistic mathematics and the finitistic is, in the view of many logicians, the heart of logic. It is also a major endeavour of philosophy of mathematics. In the context of the subject matter of this paper, one may pose the following question: how much light does infinitistic combinatorial principles such as $\text{RT}_2^n$ shed on finite mathematics? For $n \geq 3$, one knows that there is much more that $\text{RCA}_0 + \text{RT}_k^n$ proves than what $\text{RCA}_0$ does. The corresponding question for $\text{RT}_2^2$ is, however, a challenging one.
We give a sketch of the simpler proof given in [27]. Suppose the theorem is false and over \( \text{RCA}_0 \). Theorem 4.5. (Ko, Lemma 4.4. RT)

Let \( \text{RCA}_0 + \text{RT}^2 \) be a \( \Pi^1_2 \)-conservative over \( \text{RCA}_0 \). Theorem 4.5 is a major step towards resolving this question. To discuss this result, we first recall the notion of largeness introduced by Ketone and Solovay [25] in their analysis of the Paris-Harrington principle via the hierarchy of fast growing functions:

**Definition 4.2.** Let \( X \) be a finite set.

i. \( X \) is \( \omega \)-large if \( |X| > \text{min} X \);

ii. \( X \) is \( \omega^{n+1} \)-large if \( X \setminus \{ \text{min} X \} \) is the union of sets \( X_1, \ldots, X_{\text{min} X} \) such that

- \( \max X_i < \text{min} X_{i+1} \) for each \( i < \text{min} X \);
- Each \( X_i \) is \( \omega^n \)-large.

iii. \( X \) is \( \text{RT}^n_k \)-large if every \( k \)-coloring of pairs in \( X \) has a homogeneous subset that is \( \omega^n \)-large.

Clearly \( \text{RT}^2_k \) implies (iii), i.e. for any model of \( \text{RCA}_0 + \text{RT}^2 \) and each \( k, n \in \omega \), there is an \( X \) that is \( \text{RT}^2_k \)-\( \omega^n \)-large. The heart of the proof of Theorem 4.5 is in deriving a version of the converse; namely, beginning with an \( \text{RCA}_0 + \text{RT}^2 \)-\( \omega^n \)-large set, construct a model of \( \text{RCA}_0 + \text{RT}^2 \).

In the absence of \( \text{RT}^2_k \), we have the following finite version of Ramsey’s theorem for pairs:

**Theorem 4.3.** (Ketone and Solovay [25]) \( \text{I}^0_4 \) implies that if \( k \geq 2 \) and \( X \) is \( \omega^{k+4} \)-large (with \( \text{min} X \geq 3 \)), then \( X \) is \( \text{RT}^n_k \)-\( \omega^n \)-large.

The next lemma is an effective generalization of this result:

**Lemma 4.4.** (Ko, Lemma 4.4. RT)

Denote by \( \forall \mathcal{X} \Pi^n_0 \) the class of sentences \( \varphi \) of the form \( \forall \mathcal{X} \forall \exists y \forall z \varphi_0 \), where \( \varphi_0 \) is bounded. Clearly \( \forall \mathcal{X} \Pi^n_0 \) properly contains the class of \( \Pi^n_0 \)-sentences.

**Theorem 4.5.** (Ko and Solovay [25]) \( \text{RCA}_0 + \text{RT}^2 + \text{WKL}_0 \) is \( \forall \mathcal{X} \Pi^n_0 \)-conservative over \( \text{RCA}_0 \).

**Proof.** We give the sketch of the simpler proof given in [27]. Suppose the theorem is false and \( \varphi \) is a witness to this, i.e. \( \text{RCA}_0 + \text{RT}^2 + \text{WKL}_0 \vdash \varphi \) while \( \text{RCA}_0 \not\vdash \varphi \), where \( \varphi \equiv \forall \mathcal{X} \forall x \exists y \forall z \varphi_0 (X \mid z, x, y, z) \) and \( \varphi_0 = \Delta^0_0 \). Then there is a countable model \( \mathcal{M} = (M, S) \) of \( \text{RCA}_0 \) such that

\[
\mathcal{M} \models \exists \mathcal{X} \exists x \forall y \exists z \varphi_0 (X \mid z, x, y, z).
\]

Let \( X_0 \in S \) and \( a \in M \) be such that \( \mathcal{M} \models \forall y \exists z \varphi_0 (X_0 \mid z, a, y, z) \). We will produce a cut \( I \subseteq \omega \) such that \( I = (I, S_I) \models \text{RCA}_0 + \text{RT}^2 + \text{WKL}_0 + \neg \varphi \), where \( S_I = \{ \mathcal{E} : \mathcal{E} \| \mathcal{E} \| I \wedge I \subseteq \mathcal{M} \} \). This will give us the contradiction required.

First let \( \{ a_s : s \in M \} \) be a primitive recursive strictly increasing sequence such that for all \( s, y < a_s \), there is a \( z < a_{s+1} \) such that \( \mathcal{M} \models \neg \varphi_0 (X_0 \mid z, a, y, z) \). Then \( \mathcal{M} = \{ a_s : s \in M \} \) in \( S \) and \( \mathcal{M} \) is \( \forall \mathcal{X} \Pi^n_0 \)-finite. Let \( \omega \) be the least infinite ordinal formalized in \( \text{RCA}_0 \). Then for each \( n \in \omega \), there is an \( \forall \mathcal{X} \Pi^n_0 \)-finite subset of \( Y \) that is \( \omega^n \)-large. By overspill, there is a nonstandard \( d \) and an \( \forall \mathcal{X} \Pi^n_0 \)-finite set \( Z \subseteq Y \) such that \( Z \) is \( \omega^{3d} \)-large. Let \( \{ E_i : i \in \omega \} \) list all \( \forall \mathcal{X} \Pi^n_0 \)-finite sets which are not \( \omega^n \)-large. Let \( \{ C_i : i \in \omega \} \) list all \( \forall \mathcal{X} \Pi^n_0 \)-two-colorings of pairs in \( Z \). We construct an \( \omega \)-sequence of \( \forall \mathcal{X} \Pi^n_0 \)-finite sets \( Z \) such that \( Z_0 \supseteq Z_1 \supseteq \cdots \) such that

- Each \( Z_i \) is \( \omega^{3d-1} \)-large;
- \( C_i \mid [Z_{2i+1}]^2 \) is constant;
- \( \{ x : \text{min} Z_{2i+2} \leq x \leq \text{max} Z_{2i+2} \} \cap E_i = \emptyset \).

To define \( Z_1 \), apply Lemma 4.4 by letting \( n = 300^{d-1} \) and so \( Z_i \subseteq Z_0 \) is \( \omega^{3d-1} \)-large and homogeneous for \( C_0 \). On \( Z_1 \), define a two-coloring \( C \) of pairs such that \( C(u, v) = \text{red} \) if and only if \( E_0 \cap \{ w : u \leq w \leq v \} \neq \emptyset \). Let \( Z_2 \subseteq Z_1 \) be homogeneous for \( C \) such that \( Z_2 \) is \( \omega^{3d-2} \)-large (by Lemma 4.4).

**Claim 1.** \( C \mid |Z_2|^2 \) is constantly blue.

Suppose otherwise. Then for any \( u < v \) in \( Z_2 \), \( E_0 \cap \{ w : u \leq w \leq v \} \neq \emptyset \). But this would immediately imply that \( E_0 \) is \( \omega \)-large, which is not possible.
In this way one defines inductively $Z_i$ for each $i \in \omega$. Now let $I = \{u : \exists i \in \omega (u \leq \min Z_i)\}$. It is not difficult to verify that $I$ is a cut. Let $S_I = \{E \cap I : E \in \mathfrak{M}\text{-finite}\}$.

**Claim 2.** $I = (I, S_I) \models \text{RCA}_0 + \text{RT}^2_2 + \text{WKL}_0$.

We first show that $I \models I\Sigma^0_1$. For this, it is sufficient to show that every $\Sigma^0_1(I)$-function defined on a bounded set in $I$ is bounded in $I$. Let $f$ be $\Sigma^0_1$ with parameter $E \in S_I$ defined on $K = \{x : x \leq e\}$ for some $e \in I$. Suppose $f(K)$ is unbounded in $I$. Then $f(K) \in S_I$ and so there is an $i$ such that $f(K) = E_i \cap I$. However, the construction ensures that $E_i$ is disjoint from $Z_{2i}$ and is therefore bounded in $I$.

Now let $C$ be a two-coloring of $[I]^2$ that belongs to $S_I$. Then there is an $i$ such that $C_i \cap I = C$. Then by construction $Z_{2i+1}$ is homogeneous for $C_i$ and hence so is $Z_{2i+1} \cap I$, which belongs to $S_I$. Thus $I \models \text{RT}^2_2$.

To show that $I \models \text{WKL}_0$, let $T \in S_I$ be an unbounded binary tree. If for each $\sigma \in T$, the collection $T_\sigma = \{\tau : \tau > \sigma \land \tau \in T\}$ is bounded in $I$, then by $B\Sigma^0_2$ which holds in $I$ (by Fact 1.2), $T$ would be bounded in $I$. Thus there is a $\sigma_0 \in T$ such that $T_{\sigma_0}$ is unbounded. By induction, one obtains an unbounded $\omega$-sequence $\sigma_0 < \sigma_1 < \cdots < \sigma_i < \cdots$ such that $p = \bigcup_{j < i} \sigma_j$ is unbounded in $I$. Since $T \subseteq S_I$, there is an $\mathfrak{M}$-finite set $\hat{T}$ such that $T = \hat{T} \cap I$ and a path $\hat{p} \subseteq \hat{T}$ such that $p = \hat{p} \cap I$, and hence $p \in S_I$. Thus $I \models \text{WKL}_0$.

Finally, since $a_i \in I$ for $s \in I$, we have $I \models \neg \varphi$, which contradicts the assumption that $\text{RCA}_0 + \text{RT}^2_2 + \text{WKL}_0 \models \varphi$.

5. CONCLUDING REMARKS

The study of the reverse mathematics of Ramsey's Theorem has opened up a wide area of research that connects combinatorial theory to logic. New tools and techniques have been developed to investigate problems in this fertile field. This article has considered only those directly related to Ramsey's Theorem itself. Nevertheless, studies have also been made on combinatorial principles known to be weaker than $\text{RT}^2_2$, such as the ascending and descending sequence principle $\text{ADS}$, which states that every infinite linearly ordered set has an infinite ascending or descending subsequence, and the Chain/Anti-chain principle $\text{CAC}$, which states that every infinite partially ordered set contains an infinite chain or anti-chain (see Hirschfeldt and Shore [21] for a discussion of these principles, Lerman, Solomon and Towsner [28] for a proof that $\text{RCA}_0 + \text{ADS} \not\models \text{CAC}$, and Chong, Slaman and Yang [10] for $\Pi^1_1$-conservation of these principles over $\text{RCA}_0 + B\Sigma^0_2$). Separately, Wang [42] showed that both the Free Set Theorem Principle $\text{FS}$ (for every $n \geq 1$ and coloring $C$ of $[\mathbb{N}]^n$ in infinitely many colors, there is an infinite $A \subseteq \mathbb{N}$ such that for any $\hat{x} = (x_0, \ldots, x_{n-1}) \in [A]^n$, $C(\hat{x}) \in A \iff C(\hat{x}) \in \hat{x}$, and the Thin Set Theorem Principle $\text{TS}$ (for every $n \geq 1$ and infinite coloring $C$ of $[\mathbb{N}]^n$, there is an infinite $A \subseteq \mathbb{N}$ such that $C[A]^n \not\subseteq \mathbb{N}$) are strictly weaker than $\text{ACA}_0$. A striking point about these two results is that they hold for arbitrary $n$-tuples—a phenomenon that is quite different from Ramsey's Theorem where $\text{RT}^2_2 \models \text{ACA}_0$ for $n \geq 3$.$^4$

More recently, there is increasing interest in the generalization of Ramsey type combinatorial principles to other structures such as trees. Preserving the topological structure of a tree introduces a new dimension to the complexity of issues involved in constructing a homogeneous solution with the prescribed property. We give two examples below.

Let $2^{<\mathbb{M}}$ denote the full binary tree in $\mathfrak{M} = (M, S)$.

**Example 1.** The $\text{TT}^1$ principle: Any finite coloring of the full binary tree has a homogeneous solution, i.e. a tree isomorphic to the full binary tree all of whose nodes have the same color. $\text{TT}^1$ holds in any structure of $\text{RCA}_0 + \Pi^0_2$, as can be easily verified. However, in the absence of $\Sigma^0_2$-induction, the existence of a homogeneous tree is a nontrivial problem. Corduan, Groszek and Milet [15] showed that $\text{TT}^1$ is not provable in $\text{RCA}_0 + B\Sigma^0_2$. Chong, Li, Wang and Yang [8] have recently shown that in the base system $\text{RCA}_0$, $\text{TT}^1$ is $\Pi^1_1$-conservative over $\text{BME}_1 + B\Sigma^0_2$. As a consequence, $\text{RCA}_0 + \text{TT}^1$ does not prove $\Pi^0_2$. A rather interesting point to note is that if $\Pi^0_2$ fails, then there is a recursive finite coloring of the full binary tree for which there is no $\varphi''$-computable homogeneous tree. Since little is known about the structure of Turing degrees $\not\equiv \emptyset''$, it may explain why solutions constructed in [8] are in general non-definable over the ground model.

**Example 2.** $\text{TT}^2_k$: Denote by $[2^{<\mathbb{M}}]^n$ the collection of $n$-tuples of pairwise compatible nodes in the full binary tree. This principle states that for any coloring $C : [2^{<\mathbb{M}}]^n \rightarrow k$, there is a tree $T \equiv 2^{<\mathbb{M}}$ in $S$ such that $C \upharpoonright [T]^n$ is constant, where $[T]^n$ is the set of $n$-tuples of pairwise compatible nodes in $T$. Patey [35] showed that $\text{TT}^2_k$ is strictly stronger than $\text{RT}^2_2$, while Dzhafarov and Patey [16] showed that $\text{TT}^2_2$ is strictly weaker than $\text{TT}^3_2$, which is

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$^4$Cholak, Gusto, Hirst and Jockusch [2] have shown that the free set theorem for pairs, i.e. when $n = 2$, is a consequence of $\text{RT}^2_2$. 
equivalent to ACA. Thus TT is a Ramsey type principle whose strength lies between ACA and RT. Recently Chong, Li, Liu and Yang [6] showed that despite its greater strength over RT, TT still fails to imply the weak Weak König’s Lemma principle WWKL, and hence WKL as well. In a related work [7], it is shown that TT does not imply /Σ₂, generalizing Theorem 3.6. In the following figure, we give a summary of the proof-theoretic strengths of combinatorial principles discussed in the previous sections, with RT at the center. Numbered square brackets [n], n > 0, refer to the papers cited. [0] refers to the fact that TT \( \not\rightarrow \) RT trivially, since the former has an \( \omega \)-model whose second-order members are the recursive sets. Note that Theorem 4.5, which concerns another aspect of RT, i.e. \( \Pi^0_1 \)-conservation over RCA, is not incorporated in the figure.

![Figure 2. Combinatorial principles related to RT]

We end this article with some questions:

**Question 1.** Is there an \( \omega \)-model of SRT + ¬RT consisting only of \( \text{low}_2 \) sets?

The model constructed in [31] selects, roughly, paths in a \( \varphi'' \)-recursive tree as solutions of instances of SRT, and are therefore not necessarily \( \text{low}_2 \).

**Question 2.** Is RCA + TT or RCA + TT \( \Pi^0_1 \)-conservative over RCA?

To follow the strategy adopted in [36], one needs an analysis of the ordinal bounds of finite versions of these principles as carried out in [25] and [27]. It is not clear how this may be achieved. A closely related problem concerns \( \Pi^1_1 \)-conservation:

**Question 3.** Is RCA + RT or RCA + TT \( \Pi^1_1 \)-conservative over RCA + BΣ²?

**Question 4.** What is the relative proof-theoretic strength between TT and RT or TT over the base system RCA?

While TT has a trivial solution over RCA + IΣ², so that RT \( \not\rightarrow \) TT (strictly) over RCA + IΣ², the situation is unknown without \( \Sigma^0_2 \)-induction.
REFERENCES

[26] Alexander P. Kreuzer, and Keita Yokoyama. On principles between $\Sigma^1_1$- and $\Sigma^1_2$-induction, and monotone enumerations. Journal of Mathematical Logic, 16 (1):


