CHAIN TRANSITIVITY AND VARIATIONS OF THE
SHADOWING PROPERTY

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ABSTRACT. We show that, under the assumption of chain transitivity, the shadowing property is equivalent to the thick shadowing property. We also show that, if $F$ is a family with the Ramsey property, then an arbitrary sequence of points in a chain transitive space can be $\varepsilon$-shadowed (for any $\varepsilon$) on a set in $F$.

1. Introduction and Preliminaries

Different variations of the shadowing property have appeared and been studied in the recent literature of topological dynamics. Thick shadowing (see [1]), $d$-shadowing (see [1]), and ergodic shadowing (see [7]) are some of the most important examples, and will be defined below. Many more examples can be found ([12] is a particularly rich source). The first main theorem of our paper is to show that several different shadowing properties, most notably shadowing and thick shadowing, are equivalent under the assumption of chain transitivity. Our second main theorem shows that an arbitrary sequence in a chain transitive system can still be shadowed on a "large" set (large in the sense of Ramsey theory).

Suppose that $(X, f)$ is a dynamical system; i.e., $X$ is a compact metric space with a fixed metric $d$, and $f : X \to X$ is a continuous surjection. A sequence $\langle x_i : i \in \mathbb{N} \rangle$ is called a $\delta$-pseudo-orbit if $d(f(x_i), x_{i+1}) < \delta$ for every $i \in \mathbb{N}$. More generally, we say that $\langle x_i : i \in \mathbb{N} \rangle$ is a $\delta$-pseudo-orbit on $A$, for some $A \subseteq \mathbb{N}$, if

$$A \subseteq \{ i \in \mathbb{N} : d(f(x_i), x_{i+1}) < \delta \}.$$ 

Thus, for example, a $\delta$-pseudo-orbit is a $\delta$-pseudo-orbit on $\mathbb{N}$. Pseudo-orbits can be thought of as orbits computed with some error at every step (such as the rounding error inherent to computer models). Sometimes we are interested only in finite segments of some pseudo-orbit: we say that a finite sequence $\langle x_i : i \leq N \rangle$ is a $\delta$-chain if $d(f(x_i), x_{i+1}) < \delta$
for every \( i \leq N \). A system is \textbf{chain transitive} if, for any \( \delta > 0 \), any two points can be connected by a \( \delta \)-chain.

Informally, the shadowing property states that every pseudo-orbit is approximated by an actual orbit. More precisely, \( X \) has \textbf{shadowing} if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \( \xi = \langle x_i : i \in \mathbb{N} \rangle \) is a \( \delta \)-pseudo-orbit then there is a point \( x \in X \) such that \( d(f^i(x), x_i) < \varepsilon \) for all \( i \in \mathbb{N} \). In this case we say that \( x \varepsilon \)-shadows \( \xi \). More generally, we say that \( x \varepsilon \)-shadows \( \xi \) on \( A \) whenever \( A \subseteq \{ i \in \mathbb{N} : d(f^i(x), x_i) < \varepsilon \} \).

The Shadowing Lemma (due to Bowen; see [6]) says, roughly, that shadowing is a common phenomenon in chaotic dynamical systems. In this paper we will be dealing with several different kinds of shadowing, so we will introduce some nomenclature in order to organize them coherently. A \textbf{Furstenberg family}, or simply a \textbf{family} is a set \( F \) of subsets of \( \mathbb{N} \) such that if \( A \in F \) and \( A \subseteq B \) then \( B \in F \). Let \( F \) and \( G \) be two families of subsets of \( \mathbb{N} \). We say that a system \( X \) has \((F, G)\)-shadowing if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that, whenever \( \xi = \langle x_i : i \in \mathbb{N} \rangle \) is a \( \delta \)-pseudo-orbit on a set in \( F \), there is a point \( x \in X \) that \( \varepsilon \)-shadows \( \xi \) on a set in \( G \). Note that, since \( G \) is upwards hereditary, this is equivalent to \( \{ i \in \mathbb{N} : d(f^i(x), x_i) < \varepsilon \} \in G \). Many of the commonly used variants of shadowing have this form, but not all (e.g., “average shadowing”; see [5]).

We now introduce several different families of subsets of \( \mathbb{N} \) that will be used to define our shadowing properties. A set \( A \subseteq \mathbb{N} \) is \textbf{thick} if it contains arbitrarily long intervals. This is equivalent to the property that \( A \) meet every syndetic set, where a syndetic set is one that has bounded gaps (i.e., its complement does not contain arbitrarily long intervals); this is often taken to be the definition of thickness. The family of thick subsets of \( \mathbb{N} \) will be denoted by \( \mathcal{T} \). For any \( A \subseteq \mathbb{N} \), the \textbf{upper density} and \textbf{lower density} of \( A \) are, respectively,

\[
\bar{d}(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n},
\]

\[
d(A) = \liminf_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.
\]

The family of all \( A \subseteq \mathbb{N} \) with positive lower density will be denoted by \( \mathcal{F}_{\bar{d}} \). If \( d(A) = \bar{d}(A) \) then their common value is denoted by \( d(A) \). The family of all \( A \subseteq \mathbb{N} \) with \( d(A) = 1 \) (equivalently, with \( \bar{d}(A) = 1 \)) will be denoted by \( \mathcal{D} \). Finally, we will denote the family \( \{\mathbb{N}\} \) by \( \mathcal{N} \) and the family of all subsets of \( \mathbb{N} \) by \( \mathcal{P}(\mathbb{N}) \).

The following diagram summarizes the relationship between these five families. None of the inclusions are particularly difficult to prove,
nor is it too hard to see that all of the inclusions are strict; these facts are left as an exercise for the reader.

\[ \mathcal{P}(\mathbb{N}) \supseteq \mathcal{T} \supseteq \mathcal{D} \supseteq \mathbb{N} \]

As mentioned already, several familiar types of shadowing can be expressed with this terminology. For example, the shadowing property is \((\mathbb{N}, \mathbb{N})\)-shadowing; thick shadowing (see [1]) is \((\mathcal{D}, \mathcal{T})\)-shadowing; ergodic shadowing (see [7]) is \((\mathcal{D}, \mathcal{D})\)-shadowing; and \(d\)-shadowing (see [1]) is \((\mathcal{D}, \mathcal{F}_d)\)-shadowing. When we refer to one of these types of shadowing in the statement of a theorem, we will use both our name for it and the original name.

The following lemma is obvious, but is worth mentioning nonetheless.

Lemma 1.1. Suppose \(\mathcal{F}_0 \supseteq \mathcal{F}_1\) and \(\mathcal{G}_1 \supseteq \mathcal{G}_0\). Then every space with \((\mathcal{F}_0, \mathcal{G}_0)\)-shadowing also has \((\mathcal{F}_1, \mathcal{G}_1)\)-shadowing.

It is also worth pointing out that the \((\mathcal{F}, \mathcal{G})\)-shadowing property, for any families \(\mathcal{F}\) and \(\mathcal{G}\), is a topological property. Specifically, if \((X, f)\) has this property when \(X\) is taken to have the metric \(d\), then \((X, f)\) also has this property when \(X\) is considered to have any topologically equivalent metric \(d'\). This justifies our writing \((X, f)\) for a dynamical system, as opposed to the more cumbersome \(((X, d), f)\).

Proposition 1.2. Let \((X, f)\) and \((Y, g)\) be dynamical systems, let \(h : X \rightarrow Y\) be a homeomorphism such that \(h \circ f = g \circ h\), and let \(\mathcal{F}\) and \(\mathcal{G}\) be families. Then \(X\) has \((\mathcal{F}, \mathcal{G})\)-shadowing if and only if \(Y\) has \((\mathcal{F}, \mathcal{G})\)-shadowing.

This proposition follows from a routine application of uniform continuity. Note that the result for \((\mathbb{N}, \mathbb{N})\)-shadowing (i.e., normal shadowing) is given in [3], Remark 1.2.4.

2. Shadowing with Chain Transitivity

The following result is fully articulated and proved in [11] (see Theorem 5), but much of it is implicit in [2] (see pp. 175-176). Recall that \((X, f)\) factors over \((Y, g)\) if there is a continuous surjection \(h : X \rightarrow Y\) such that \(h \circ f = g \circ h\).

Lemma 2.1 ([2, 11]). Let \((X, f)\) be a chain transitive dynamical system. Then either
(1) There is a $k \geq 1$ such that $f$ cyclically permutes $k$ clopen components of $X$, and $f^k$ restricted to each component is chain mixing.

(2) $f$ factors onto an adding machine map.

If a chain transitive space $X$ admits a decomposition as described in (1), we will say that $X$ is **piecewise chain mixing**. In the course of the proof of Lemma 2.1, Richeson and Wiseman develop a few ideas that will be useful for proving our main theorem. First, the following result is Lemma 6 in [11] and is also given in [2].

**Lemma 2.2.** Let $f : X \to X$ be a chain transitive map, and let $\delta > 0$. There exists $k_\delta \in \mathbb{N}$ such that for any $x \in X$, $k_\delta$ is the greatest common divisor of the lengths of all $\delta$-chains from $x$ to $x$.

This lemma leads Richeson and Wiseman in [11] to define a $\delta$-dependent relation on $X$: $x \sim_\delta y$ if and only if there is a $\delta$-chain from $x$ to $y$ whose length is a multiple of $k_\delta$. They prove that, for any $\delta > 0$, there are precisely $k_\delta$ equivalence classes of $\sim_\delta$, these equivalence classes are clopen, and they are permuted cyclically by $f$.

If a chain transitive space $X$ is piecewise chain mixing, then there is some $\delta$ such that $\sim_\delta$ gives exactly the decomposition described in Lemma 2.1(1). If not, then the $\sim_\delta$ equivalence classes still provide a decomposition of $X$ into pieces that are “approximately” chain mixing. This, at least, is the intuitive content of the following technical lemma, which we will need for the proof of our main result.

**Lemma 2.3.** Let $X$ be chain transitive. For each $\delta > 0$ there is some $M \in \mathbb{N}$ such that, for any $m \geq M$ and any $x, y \in X$ with $x \sim_\delta y$, there is a $\delta$-chain in $(X, f)$ of length $mk_\delta$ from $x$ to $y$.

It should be noted that, while Richeson and Wiseman do not state or prove this lemma, its proof is an adaptation of the second paragraph of the proof of their Theorem 5.

**Proof of Lemma 2.3.** As there are only finitely many equivalence classes of $\sim_\delta$, it suffices to show that for each equivalence class $X_i$ of $\sim_\delta$, there exists $M_i \in \mathbb{N}$ satisfying the statement of the lemma for any pair $x, y \in X_i$. Then for $M$ equal to the maximum of the $M_i$, the desired result holds for any pair $x, y \in X$ with $x \sim_\delta y$.

Let $X_0$ be an equivalence class of $\sim_\delta$. Fix $x \in X_0$. Since $k_\delta$ is the greatest common divisor of the lengths of all $\delta$-chains from $x$ to $x$, there exist $\delta$-chains from $x$ to $x$ of lengths $p_ik_\delta$, $0 \leq i < n$, such that $\gcd \{p_i : 0 \leq i < n\} = 1$. By continuity, there is a neighborhood $U_x$ of $x$ such that, for each $x' \in U_x$, each of these $n$ chains can be...
made into a $\delta$-chain from $x'$ to $x'$ simply by replacing the first and last items in the chain, which are both $x$, by $x'$. Concatenating these chains, we can obtain, for any $m_0, \ldots, m_{n-1} \in \mathbb{N}$, a $\delta$-chain from $x'$ to $x'$ of length $\sum_{0 \leq i < n} m_ip_i\delta$. By a well-known result of Issai Schur, we can obtain a $\delta$-chain from $x'$ to $x'$ of length $mk\delta$ for any sufficiently large $m$ (this is related to what is often called the “coin problem”; see [10] for a thorough discussion). That is, for any $x \in X_0$ there is an open $U_x \ni x$ and some $M(U_x) \in \mathbb{N}$ such that, for all $x' \in U_x$ and $m \geq M(U_x)$, there is a $\delta$-chain from $x'$ to $x'$ of length $mk\delta$. It follows from the compactness of $X_0$ that there is some $M_0 \in \mathbb{N}$ such that, for any $x \in X_0$ and $m \geq M_0$, there is a $\delta$-chain from $x$ to $x$ of length $mk\delta$.

By chain transitivity and the compactness of $X_0$, there is some $M_1 \in \mathbb{N}$ such that between any two points in $X_0$ there is a $\delta$-chain of length at most $M_1k\delta$. Moreover, as pointed out in [11], any such chain must have a length that is a multiple of $k\delta$. Let $x, y \in X_0$. By concatenating a $\delta$-chain from $x$ to $x$ with a $\delta$-chain from $x$ to $y$, we can obtain a $\delta$-chain from $x$ to $y$ of length $mk\delta$ for any $m \geq M = M_0 + M_1$.

\[ \square \]

**Theorem 2.4.** The following properties are equivalent for a chain transitive dynamical system:

1. shadowing (i.e., $(\mathbb{N}, \mathbb{N})$-shadowing).
2. thick shadowing (i.e., $(\mathcal{D}, \mathcal{T})$-shadowing).
3. $(\mathcal{T}, \mathcal{T})$-shadowing.
4. $(\mathbb{N}, \mathcal{T})$-shadowing.

**Proof.** By Lemma 1.1, (1)-(3) all imply (4). Also by Lemma 1.1, (3) implies (2). It is enough, therefore, to show that (4) implies (1) and (1) implies (3).

First we show that (4) implies (1). The basic idea for the proof of this implication is found in the proof of Lemma 3.2 in [7]. Let $\varepsilon > 0$ and fix $\delta > 0$ such that any $\delta$-pseudo-orbit in $X$ can be shadowed on a thick set. Let $\langle x_i: i \leq n \rangle$ be any finite $\delta$-chain. Because $X$ is chain transitive, there is a $\delta$-chain $\langle y_i: i \leq m \rangle$ in $X$ such that $y_0 = x_n$ and $y_m = x_0$. The sequence

$$\langle x_0, \ldots, x_n, y_1, \ldots, y_{m-1}, x_0, \ldots, x_n, y_1, \ldots, y_{m-1}, x_0, \ldots, x_n, \ldots \rangle$$

is a $\delta$-pseudo-orbit in $X$. By (4), there is some $y \in X$ that $\varepsilon$-shadows this sequence on a thick set $T$. Because $T$ contains arbitrarily large intervals, there is some $k$ such that

$$[k(m+n), k(m+n) + n] \subseteq T.$$
Let \( x = f^{k(m+n)}(y) \). Because \( y \varepsilon \)-shadows the above sequence on \( T \), if \( 0 \leq i \leq n \) then \( d(f^i(x), x_i) = d(f^{k(m+n)+i}(y), x_{k(m+n)+i}) < \varepsilon \). Thus any finite \( \delta \)-chain in \( X \) can be \( \varepsilon \)-shadowed in \( X \). It follows that \( X \) has the shadowing property (see the proof of Theorem 1.2.1 in [3]).

In order to obtain \((1) \Rightarrow (3)\), we must fix \( \varepsilon > 0 \) and find some \( \delta > 0 \) such that for any sequence \( \xi \), if \( \xi \) is a \( \delta \)-pseudo-orbit on a thick set then \( \xi \) is \( \varepsilon \)-shadowed on a thick set. Our strategy is to find, for any such \( \xi \), a \( \delta \)-pseudo-orbit \( \eta \) that agrees with \( \xi \) on a thick set. By \((1)\), we can find a point that \( \varepsilon \)-shadows \( \eta \), and this point must also shadow \( \xi \) on a thick set, namely some superset of the set where \( \xi \) and \( \eta \) agree.

Let \( \varepsilon > 0 \). There is some \( \delta > 0 \) such that every \( \delta \)-pseudo-orbit in \( X \) can be \( \varepsilon \)-shadowed. Let \( \xi = \langle x_i : i \in \mathbb{N} \rangle \) be a \( \delta \)-pseudo-orbit on some thick set \( T \). Let \( K = k_\delta \), and let \( X_0, \ldots, X_{K-1} \) be the equivalence classes of \( \sim_\delta \). For convenience, we label these equivalence classes with elements of the group \( \mathbb{Z}_K \); thus, when we write \( X_{i+j} \) then the addition is understood to be taken modulo \( K \). We also label the equivalence classes so that \( f(X_i) = X_{i+1} \) for all \( i \in \mathbb{Z}_K \) (recall that \( f \) cyclically permutes the equivalence classes of \( \sim_\delta \)).

Let \( x \in X_i \) and \( y \in X_j \), and suppose that \( d(x, y) < \delta \). Since \( f \) cyclically permutes the \( X_k \), \( f^{-K}(X_i) = X_i \). Let \( z \in f^{-K}(x) \); then \( z \in X_i \). Furthermore, \( \langle z, f(z), \ldots, f^{K-1}(z), y \rangle \) is a \( \delta \)-chain in \( X \) of length \( K = k_\delta \) between \( z \) and \( y \). By definition, \( z \sim_\delta y \), which means \( y \in X_i \). Thus \( x \) and \( y \) are the same \( \sim_\delta \) equivalence class whenever \( d(x, y) < \delta \).

Define the relative modulus of \( i \in \mathbb{N} \), denoted \( m(i) \), to be the unique \( k \), \( 0 \leq k < K \), such that \( x_i \in X_{i+k} \). By the previous paragraph, if \( d(x_{i+1}, f(x_i)) < \delta \) then \( x_{i+1} \) and \( f(x_i) \) are in the same \( X_j \). In other words, if \( d(x_{i+1}, f(x_i)) < \delta \) (that is, if \( i \in T \)) then \( x_i \) and \( x_{i+1} \) have the same relative modulus. Let \( A = \{ i \in \mathbb{N} : m(i) = m(i+1) \} \). \( T \subseteq A \), so \( A \) is thick.

For \( k \in \mathbb{Z}_K \), let \( A_k = \{ i \in \mathbb{N} : m(i) = k \} \). By the definition of \( A \), every interval in \( \mathbb{N} \) that is a subset of \( A \) is also a subset of some \( A_k \). Since \( A \) is thick (i.e., contains arbitrarily long intervals) it follows that some \( A_k \) must also contain arbitrarily long intervals (i.e., it must be thick). We may assume, without loss of generality, that \( A_0 \) is thick.

By Lemma 2.3, there is some \( M \in \mathbb{N} \) such that, for any \( m \geq M \) and any \( x, y \in X_0 \), there is a \( \delta \)-chain in \( (X, f) \) of length \( mK \) from \( x \) to \( y \).

We now define a \( \delta \)-chain in \((X, f)\) by recursion. Let \( y_0 \) be any element of \( X_0 \) and set \( N_0 = 0 \). Suppose now we have constructed a \( \delta \)-chain \( \langle y_i : i \leq KN_0 \rangle \) in \((X, f)\), where \( N_0 \in \mathbb{N} \). Furthermore, suppose that this \( \delta \)-chain agrees with the sequence \( \xi \) on intervals of size 1, 2, \ldots, \( n-1 \),
n and that \( y_{Kn_n} \in X_0 \). We will show how to extend this \( \delta \)-chain so that it agrees with \( \langle x_i : i \in \mathbb{N} \rangle \) on an interval of size \( n + 1 \) (this constitutes the inductive step of our recursive construction). Because \( A_0 \) is thick, there is some interval \( J_n \subseteq A_0 \) of length \( K + n \) such that the least element of \( J_n \), say \( j_n \), is at least \( KN_n + KM \). Let \( a_n \) be the least integer such that \( Ka_n \in J_n \). Because \( y_{Kn_n} \in X_0 \), there is a \( \delta \)-chain \( \langle z_i : KN_n \leq i \leq Ka_n \rangle \) in \( (X, f) \) such that \( z_{Kn_n} = y_{Kn_n} \) and \( z_{Ka_n} = x_{Ka_n} \). Let this \( \delta \)-chain determine \( y_i \) up to \( i = Ka_n \): that is, set \( y_i = z_i \) for \( KN_n \leq i \leq Ka_n \). Then let \( y_{Ka_n+i} = x_{Ka_n+i} \) for all \( 0 \leq i \leq n \) (in other words, we let the \( y_i \) follow the \( x_i \) on the next \( n \) members of \( J_n \)). This extends our length-\( KN_n \) \( \delta \)-chain to a longer \( \delta \)-chain containing an interval of size \( n + 1 \) on which it agrees with the sequence \( \langle x_i : i \in \mathbb{N} \rangle \). To finish the inductive step, let \( KN_{n+1} \) be the least multiple of \( K \) greater than \( Ka_n + n \), and set \( y_{Ka_n+n+i} = f^i(y_{Ka_n+n}) \) for \( 0 \leq i \leq KN_{n+1} - Ka_n + n \). This simply ensures that the length of our extended chain is a multiple of \( K \), which was one of our inductive hypotheses. Since \( i \in A_0 \), we also have \( y_{Ka_n+n} = x_{Ka_n+n} \in X_{Ka_n+n} = 0 \), which implies \( y_{Kn_{n+1}} \in X_0 \). This completes the inductive step.

The procedure described in the previous paragraph defines a \( \delta \)-pseudo-orbit \( \eta = \langle y_i : i \in \mathbb{N} \rangle \) in \( X \), and \( \eta \) agrees with \( \xi \) on a thick set (i.e., \( \{ i \in \mathbb{N} : y_i = x_i \} \) is thick). Since \( X \) has shadowing, there is a point \( y \in X \) that \( \varepsilon \)-shadows the \( \delta \)-pseudo-orbit \( \eta \). Because \( \eta \) agrees with \( \xi \) on a thick set, \( y \) also shadows \( \langle x_i : i \in \mathbb{N} \rangle \) on a thick set. This proves \((3)\).

\[ \quad \]

Notice that there are chain transitive systems that do not have any of the above shadowing properties (for example, the identity map on \([0, 1]\)) and there are chain transitive systems that do have them (the shift map on \(2^\mathbb{N}\)). In fact, these examples are both chain mixing, so even this stronger property does not determine shadowing. Any adding machine does have all of these properties: this can be proved by choosing a “nice” metric on the adding machine and then applying Proposition 1.2. However, the property that \((X, f)\) factors over an adding machine is not strong enough to prove shadowing (consider an adding machine \((Y, g)\) and then set \( X = Y \times [0, 1], f = g \times \text{id}_{[0, 1]} \)).

We say that a family \( \mathcal{F} \) has **partition regularity** or the **Ramsey Property** if, whenever \( A \in \mathcal{F} \) and \( A = \bigcup_{0 \leq i < n} A_i \), there is some \( i \) such that \( A_i \in \mathcal{F} \). The field of Ramsey theory involves a deep study of this property, both for families of subsets of \( \mathbb{N} \) and in other contexts. Examples of families with the Ramsey property include \( \mathcal{F}_d \), any ultrafilter \( \mathcal{F} \), the family of piecewise syndetic sets (this is a strengthening
of van der Waerden’s Theorem; see [4]), and the family of IP sets (this is Hindman’s Theorem; see [8]).

**Theorem 2.5.** Suppose \((X,f)\) is chain transitive and has shadowing, and let \(\mathcal{F}\) be any family with partition regularity. Then \(X\) has \((\mathcal{P}(\mathbb{N}),\mathcal{F})\)-shadowing. That is, any sequence of points can be \(\varepsilon\)-shadowed (for any fixed \(\varepsilon > 0\)) on a set in \(\mathcal{F}\).

*Proof.* Let \(\xi = \langle x_i : i \in \mathbb{N} \rangle\) be any sequence of points and let \(\varepsilon > 0\).

As with filters, let us say that a family \(\mathcal{F}\) is *principal* if there is a finite \(A \in \mathcal{F}\). If this is the case then, writing \(A = \{a_i : 0 \leq i < n\}\), we have \(A = \bigcup_{0 \leq i < n} \{a_i\}\) and, by partition regularity, \(\{a_i\} \in \mathcal{F}\) for some \(i\). If this is the case then our theorem is trivial: if \(x \in f^{-a_i}(x_{a_i})\) then \(x\) \(\varepsilon\)-shadows \(\xi\) on \(\{a_i\}\). Let us assume, therefore, that \(\mathcal{F}\) is not principal. It follows that, for any \(A \in \mathcal{F}\) and \(N > 0\), \(A \setminus \{0,1,\ldots,N-1\} \in \mathcal{F}\) (to see this, write \(A = (A \setminus \{0,1,\ldots,N\}) \cup \bigcup_{n<N,n \in A} \{n\}\) and apply partition regularity).

Fix \(\delta > 0\) such that every \(\delta\)-pseudo-orbit in \(X\) can be \(\varepsilon\)-shadowed. Let \(K = k_\delta\), and let \(X_0,\ldots,X_{K-1}\) be the equivalence classes of \(\sim_\delta\). As in the proof of Theorem 2.4, we label these equivalence classes with elements of the group \(\mathbb{Z}_K\) and so that \(f(X_i) = X_{i+1}\) for all \(i \in \mathbb{Z}_K\).

Again as in the proof of Theorem 2.4, define the *relative modulus* of \(i \in \mathbb{N}\), denoted \(m(i)\), to be the unique \(k\), \(0 \leq k < K\), such that \(x_i \in X_{i+k}\). By Lemma 2.3, there is some \(M \in \mathbb{N}\) such that, for any \(m \geq M\) and any \(x,y \in X\), there is a \(\delta\)-chain in \((X,f)\) of length \(mK\) from \(x\) to \(y\).

For \(0 \leq k < K\) and \(0 \leq m < KM\), let

\[
A_{k,m} = \{i \in \mathbb{N} : m(i) = k \text{ and } i \equiv m \pmod{KM}\}.
\]

Because \(\mathcal{F}\) has partition regularity, there are some \(k\) and \(m\) such that \(A_{k,m} \in \mathcal{F}\). Simply by relabelling the \(X_i\), we may assume that \(k = 0\). Let \(A = A_{k,m} \setminus \{0,\ldots,KM-1\}\); because \(\mathcal{F}\) is not principal, \(A \in \mathcal{F}\).

Our strategy now is to find a \(\delta\)-pseudo-orbit \(\langle \xi_i : i \in \mathbb{N} \rangle\) in \(X\) such that \(\langle x_i : i \in \mathbb{N} \rangle\) agrees with \(\langle \xi_i : i \in \mathbb{N} \rangle\) on a superset of \(A\). We do this by recursion. Let \(\xi_0 \in X_0\) be arbitrary and let \(A = \{a_0,a_1,a_2,\ldots\}\).

By our definition of \(A\), \(a_0 \geq KM\). By our choice of \(M\), there is a \(\delta\)-chain \(\langle \xi_i : 0 \leq i \leq KM \rangle\) such that \(\xi_{KM} \in f^{KM-a_0}(x_{a_0})\). We then define \(\xi_{KM+i} = f^i(\xi_{KM})\) for \(KM \leq i \leq a_0\). Notice that \(\xi_{a_0} = x_{a_0}\), as desired. Now assume that we have defined \(\xi_i\) for all \(0 \leq i \leq a_n\), and that \(\xi_{a_n} = x_{a_n}\) for all \(0 \leq i \leq n\). By our definition of \(A\), there is some \(p\) such that \(a_{n+1} = a_n + pKM\). By our choice of \(M\), there is a \(\delta\)-chain \(\langle \xi_i : a_n \leq i \leq a_n + pKM \rangle\) in \(X\) such that \(\xi_{a_n+pKM} = \xi_{a_{n+1}} = x_{a_{n+1}}\). This shows that we can extend the \(\delta\)-chain \(\langle \xi_i : 0 \leq i \leq a_n \rangle\) to a \(\delta\)-chain.


\( \langle \xi_i : 0 \leq i \leq a_{n+1} \rangle \) of length \( a_{n+1} \) so that \( \xi_{ai} = x_{ai} \) for all \( 0 \leq i \leq n+1 \).

By recursion, we obtain an infinite \( \delta \)-pseudo-orbit \( \langle \xi_i : i \in \mathbb{N} \rangle \) such that \( \xi_{an} = x_{an} \) for all \( n \in \mathbb{N} \).

Since \( \langle \xi_i : i \in \mathbb{N} \rangle \) is a \( \delta \)-pseudo-orbit in \( X \), there is a point \( \xi \in X \) that \( \varepsilon \)-shadows \( \langle \xi_i : i \in \mathbb{N} \rangle \). Since \( \xi_i = x_i \) whenever \( i \in A \), we have

\[ \{ i \in \mathbb{N} : d(f^i(\xi), x_i) < \varepsilon \} \supseteq A \in \mathcal{F}. \]

This implies that \( \{ i \in \mathbb{N} : d(f^i(\xi), x_i) < \varepsilon \} \in \mathcal{F} \), as desired. \( \square \)

3. SHADOWING VERSUS THICK SHADOWING IN GENERAL

We have seen that shadowing and thick shadowing are equivalent under the assumption of chain transitivity. The following example shows that they are not equivalent in general.

**Example 3.1.** Let \( X = [0, 2] \) and let \( f : X \to X \) be defined by

\[ f(x) = x + \frac{1}{4} |\sin(\pi x)|. \]

It is easy to check that \( f \) is a homeomorphism \( X \to X \). We endow \( X \) with its usual metric as a subset of \( \mathbb{R} \). The map \( f \) fixes the points 0, 1, and 2, and it is strictly increasing for points in \((0, 1) \) and \((1, 2) \).

Fix \( 0 < \varepsilon < 1 \). For any \( \delta > 0 \), set \( x_0 = 0 \) and, for all \( n > 0 \), set \( x_n = \max \{ f(x_{n-1}) + \frac{\delta}{2}, 2 \} \). This defines an \( \delta \)-pseudo-orbit in \( X \), and it is not hard to check that, for some \( N \in \mathbb{N} \), \( x_N = 2 \). If \( \xi \) is any orbit in \( X \), then either \( \xi \subseteq [0, 1] \) or \( \xi \subseteq [1, 2] \), so \( \xi \) cannot \( \varepsilon \)-shadow \( \langle x_i : i \in \mathbb{N} \rangle \).

This shows that \( X \) does not have the shadowing property.

Nonetheless, we claim that \( X \) has thick shadowing (i.e., \( (D, T) \)-shadowing). In fact, we will show that \( X \) has \( (T, T) \)-shadowing, which is stronger than thick shadowing by Lemma 1.1. To see this, let \( \varepsilon > 0 \).

Because \( f \) is increasing on \( S = [\varepsilon, 1 - \varepsilon] \cup [1 + \varepsilon, 2 - \varepsilon] \), there is a \( \delta > 0 \) such that, whenever \( x \in S \), \( [x, x + \delta] \cap f([x, x + \delta]) = \emptyset \). Let \( \xi = \langle x_i : i \in \mathbb{N} \rangle \) be a \( \delta \)-pseudo-orbit on some thick set \( T \). By our choice of \( \delta \), there is some \( M \in \mathbb{N} \) such that any \( \delta \)-chain in \( X \) can have at most \( M \) members in \( S \). If \( m \geq M \), then any finite \( \delta \)-chain of length \( m \) must intersect one of \( [0, \varepsilon) \), \((1 - \varepsilon, 1 + \varepsilon)\), or \((2 - \varepsilon, 2] \) on an interval of length at least \( \frac{1}{3}(m - M) \). Since \( \xi \) contains arbitrarily long \( \delta \)-chains, it follows that \( \xi \) contains arbitrarily long segments in \( [0, \varepsilon) \), \((1 - \varepsilon, 1 + \varepsilon) \), or \((2 - \varepsilon, 2] \). Thus, for \( i \) equal to either 0, 1, or 2, the orbit \( \langle i, i, \ldots, i, \ldots \rangle \) \( \varepsilon \)-shadows \( \xi \) on a thick set.

Setting \( \varepsilon = \frac{1}{4} \), we claim that there is no \( \varepsilon \)-chain \( \xi = \langle x_i : 0 \leq i \leq n \rangle \) on \( 1 \) to 0. Let \( m \) be the least element of \( \{0, 1, \ldots, n\} \) such that \( x_m < \frac{1}{2} \). Because \( f \) is increasing on \((0, 1) \) and \( \varepsilon = \frac{1}{4} \), we must have
$f(x_{m-1}) < \frac{3}{4}$. An easy computation shows that then $x_{m-1} < \frac{1}{2}$, contradicting our choice of $m$. This shows directly that $(X, f)$ is not chain transitive (Theorem 2.4, together with the previous two paragraphs, show it indirectly).

Corollary 3.2. Shadowing and thick shadowing are not equivalent in arbitrary dynamical systems. In other words, Theorem 2.4 cannot be strengthened by dropping the condition of chain transitivity.

Our example gives a system that has thick shadowing but not shadowing. Naturally, the authors also searched for the complementary example: a system that has shadowing but not thick shadowing. We were unable to find such an example, so the following question must for now remain open:

Question 3.3. Does shadowing imply thick shadowing?

Of course, it is possible that the answer to Question 3.3 might be negative, in which case it would be of interest to determine which additional hypotheses are necessary to give the desired implication. As demonstrated in the previous section, chain transitivity is one such additional condition. However, as this condition actually results in equivalence of shadowing and thick shadowing, it seems likely that a weaker condition will suffice.

REFERENCES


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