INVERSE LIMITS WITH IRREDUCIBLE SET-VALUED FUNCTIONS

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Abstract. In this paper, we develop a definition for a class of set-valued functions which will be called irreducible functions. We show that these functions can be used to obtain an indecomposable continuum as an inverse limit, and we give sufficient conditions for two such inverse limits to be homeomorphic. This class of irreducible functions includes all open maps on [0, 1] which are not homeomorphisms. The inverse limits of piecewise linear open maps on [0, 1] were classified by William Watkins in 1982, and the results of this paper build on those results and expand the class of functions to which they apply.

1. Introduction

Inverse limits of maps have been studied for decades, both as a means of studying continua as well as dynamical systems. In 2004 and 2006, Mahavier and Ingram introduced the concept of inverse limits with upper semi-continuous (usc) set-valued functions [3, 5]. Since then, this idea has been studied extensively with much of the work focused on determining the conditions under which the theorems and methods used for inverse limits of maps can be extended to inverse limits with usc functions.

One property possessed by inverse limits of maps which does not always hold for inverse limits of usc functions is the full projection property, and this property seems to be crucial to obtaining an indecomposable continuum as an inverse limit. The author, as well as Ingram, Varagona, and Meddaugh have all written on indecomposability in inverse limits with usc functions (see [1, 4, 7]), and all have employed the full projection property to obtain their results.

In [4], a method was developed for constructing usc functions on [0, 1] whose inverse limits have the full projection property and are indecomposable continua. Such a function has the property that its inverse is the union of a collection of continuous, single-valued maps (with certain restrictions placed on these collections).

In this paper, we take the technique which was used in [4], and we apply it to irreducible continua in general. We show that functions constructed in this way may be used to obtain indecomposable inverse limits, and we give sufficient conditions for two sequences of these functions to have homeomorphic inverse limits.

2010 Mathematics Subject Classification. 54F15, 54D80, 54C60, 54H20.

Key words and phrases. inverse limits, upper semi-continuous, set-valued functions, indecomposability, full projection property.
More specifically, if $X$ is a continuum which is irreducible between $A$ and $B$, and $Y$ is a continuum which is irreducible between $C$ and $D$, we define what it means for a function $F: X \to 2^Y$ to be irreducible with respect to $A, B \subseteq X$ and $C, D \subseteq Y$. Such functions are defined in terms of their inverses which are unions of single-valued maps. The main theorem of Section 3.2 is the following.

**Theorem.** Let $\{X, F\}$ be an inverse sequence where for each $i \in \mathbb{N}$, $F_i: X_{i+1} \to 2^{X_i}$ is irreducible with respect to $A_{i+1}, B_{i+1} \subseteq X_{i+1}$ and $A_i, B_i \subseteq X_i$. Then $\lim \leftarrow F$ has the full projection property and is an indecomposable continuum.

When we restrict our attention to the case where irreducibility is with respect to points, we are able to define what it means for two irreducible functions to be consistent. This definition is described in Section 4.1 and culminates in this result.

**Theorem.** Let $\{X, F\}$ and $\{X, G\}$ be inverse sequences such that for each $i \in \mathbb{N}$, $F_i: X_{i+1} \to 2^{X_i}$ and $G_i: X_{i+1} \to 2^{X_i}$ are irreducible with respect to $a_{i+1}, b_{i+1} \in X_{i+1}$ and $a_i, b_i \in X_i$. If for each $i \in \mathbb{N}$, $F_i$ and $G_i$ are consistent, then $\lim \leftarrow F$ and $\lim \leftarrow G$ are homeomorphic.

We demonstrate the utility of this theorem through examples, and in Section 4.2 further implications of this theorem are discussed. In particular, the $n$th degree hat functions (which are discussed by Watkins in [9]) are irreducible functions. We will use this fact to expand the reach of Watkins' results to include not just the $n$th degree hat functions but any irreducible function whose inverse is the union of $n$ maps.

2. Preliminaries

A set $X$ is a *continuum* if it is a non-empty, compact, connected subset of a metric space. A subset of a continuum $X$ which is itself a continuum is called a *subcontinuum* of $X$. A continuum is called *decomposable* if it is the union of two proper subcontinua. A non-degenerate continuum which is not decomposable is called *indecomposable*.

If $X$ is a continuum, we denote by $2^X$ the set of all non-empty compact subsets of $X$. If $X$ and $Y$ are continua and $x \in X$, a function $F: X \to 2^Y$ is said to be *upper semi-continuous (usc)* at $x$ if for every open set $V \subseteq Y$ containing $F(x)$, there exists an open set $U \subseteq X$ containing $x$ such that $F(t) \subseteq V$ for all $t \in U$. $F$ is said to be *usc* if it is usc at each point of $X$. The *graph* of a function $F: X \to 2^Y$, denoted $\Gamma(F)$, is the subset of $X \times Y$ consisting of all points $(x, y)$ such that $y \in F(x)$. In [3], it was shown that if $X$ and $Y$ are continua, a function $F: X \to 2^Y$ is usc if and only if $\Gamma(F)$ is compact. This is typically easier to verify and will be treated as the definition of usc for the purposes of this paper.

Suppose $X = (X_i)_{i \in \mathbb{N}}$ is a sequence of continua, and $F = (F_i)_{i \in \mathbb{N}}$ is a sequence of usc functions such that for each $i \in \mathbb{N}$, $F_i: X_{i+1} \to 2^{X_i}$. Then the pair $\{X, F\}$ is called an *inverse sequence*, and
the inverse limit of that inverse sequence, denoted \( \lim^{-} F \), is the set

\[
\lim^{-} F = \{ x \in \prod_{i=1}^{\infty} X_i : x_i \in F_i(x_{i+1}) \text{ for all } i \in \mathbb{N} \}.
\]

(In this paper, sequences—both finite and infinite—will be written in bold, and their terms will be written in italics.) The continua, \( X_i \), are called the factor spaces of the inverse sequence; and the usc functions, \( F_i \), are called the bonding functions of the inverse sequence. Given any continuum \( X \) and a usc function \( F : X \to 2^X \), there is a naturally induced inverse sequence \( \{ X, F \} \) where for each \( i \in \mathbb{N} \), \( X_i = X \) and \( F_i = F \).

Given an inverse sequence \( \{ X, F \} \) and \( n \in \mathbb{N} \), we define the following two sets

\[
\Gamma_n = \{ x \in \prod_{i=1}^{\infty} X_i : x_i \in F_i(x_{i+1}) \text{ for all } 1 \leq i < n \},
\]

\[
\Gamma'_n = \{ x \in \prod_{i=1}^{n} X_i : x_i \in F_i(x_{i+1}) \text{ for all } 1 \leq i < n \}.
\]

Note that \( \lim^{-} F = \bigcap_{n \in \mathbb{N}} \Gamma_n \), so if each \( \Gamma_n \) is a continuum, so is \( \lim^{-} F \). Also, \( \Gamma_n = \Gamma'_n \times \prod_{i=n+1}^{\infty} X_i \), so if each \( \Gamma'_n \) is a continuum, then each \( \Gamma_n \) will be also, and thus so will \( \lim^{-} F \).

If \( X \) is a sequence of continua and \( j \in \mathbb{N} \), the projection maps

\[
\pi_j : \prod_{i=1}^{\infty} X_i \to X_j \text{ and } \pi_{[1,j]} : \prod_{i=1}^{\infty} X_i \to \prod_{i=1}^{j} X_i
\]

are defined by \( \pi_j(x) = x_j \), and \( \pi_{[1,j]}(x) = (x_1, \ldots, x_j) \). If \( \{ X, F \} \) is an inverse sequence, then we will typically consider these maps to have \( \lim^{-} F \) as their domain rather than writing \( \pi_j|_{\lim^{-} F} \) or \( \pi_{[1,j]}|_{\lim^{-} F} \). Also, if \( 1 \leq j \leq n \), we may also use the same notation to represent the projection maps whose domains are \( \prod_{i=1}^{n} X_i \) or a subset of \( \prod_{i=1}^{n} X_i \). In context, it should be clear what the intended domain is.

Of particular concern in this paper is the full projection property.

**Definition 2.1.** Let \( \{ X, F \} \) be an inverse sequence. We say that \( \lim^{-} F \) has the full projection property if for every proper subcontinuum \( K \) of \( \lim^{-} F \), \( \pi_i(K) = \pi_i(\lim^{-} F) \) for at most finitely many \( i \in \mathbb{N} \).

**Remark.** If for each \( i \in \mathbb{N} \), \( F_i : X_{i+1} \to 2^{X_i} \) is surjective, then for all \( i \in \mathbb{N} \), \( \pi_i(\lim^{-} F) = X_i \). Thus, \( \pi_i(\lim^{-} F) \) may be replaced with \( X_i \) in Definition 2.1. This is the definition that has typically been used in the past (see [1, 4, 7]).

In [4], the following definition and theorem were established to characterize indecomposability in inverse limits.
Definition 2.2. A usc function \( F : X \to 2^Y \) is indecomposable provided that for any pair of subcontinua \( A \) and \( B \) of \( \Gamma(F) \) with \( A \cup B = \Gamma(F) \), either \( \pi_2(A) = Y \) or \( \pi_2(B) = Y \).

Theorem 2.3. [4, Theorem 19] Let \( \{X, F\} \) be an inverse sequence for which each \( F_i \) is an indecomposable usc function. If \( \lim_\leftarrow F \) is connected and has the full projection property, then \( \lim_\leftarrow F \) is an indecomposable continuum.

In order to employ Theorem 2.3 for any inverse sequence, we must first be able to show that its inverse limit has the full projection property. One way to approach this is through the notion of irreducibility.

Definition 2.4. Let \( A \) and \( B \) be closed sets. A continuum \( X \) is said to be irreducible between \( A \) and \( B \) if \( X \) intersects each set, but no proper subcontinuum of \( X \) does.

If \( A \) and \( B \) are singleton sets \( (A = \{a\}, B = \{b\}) \), then we may say that \( X \) is irreducible between \( a \) and \( b \).

We say that a continuum \( X \) is irreducible if there exist closed sets \( A \) and \( B \) such that \( X \) is irreducible between \( A \) and \( B \).

The following theorem is almost identical to a theorem which can be found in [4]. The only difference is that the theorem in [4] employs irreducibility between points whereas the following theorem uses irreducibility between closed sets. Since the theorems are not quite the same, a proof is included.

Theorem 2.5. Let \( \{X, F\} \) be an inverse sequence such that \( \lim_\leftarrow F \) is a continuum. If for each \( n \in \mathbb{N} \), there exist closed sets \( A, B \subseteq \pi_n(\lim_\leftarrow F) \), such that \( \Gamma'_n \) is irreducible between the sets
\[
\{x \in \Gamma'_n : x_n \in A\} \text{ and } \{x \in \Gamma'_n : x_n \in B\},
\]
then \( \lim_\leftarrow F \) has the full projection property.

Proof. Let \( K \) be a subcontinuum of \( \lim_\leftarrow F \) with \( \pi_i(K) = \pi_i(\lim_\leftarrow F) \) for infinitely many \( i \in \mathbb{N} \). Choose \( j \in \mathbb{N} \) such that \( \pi_j(K) = \pi_j(\lim_\leftarrow F) \) and choose \( A, B \subseteq \pi_j(\lim_\leftarrow F) \) such that \( \Gamma'_j \) is irreducible between the sets
\[
\{x \in \Gamma'_j : x_j \in A\} \text{ and } \{x \in \Gamma'_j : x_j \in B\}.
\]

Since \( \pi_j(K) = \pi_j(\lim_\leftarrow F) \), it contains both \( A \) and \( B \), so the continuum \( \pi_{[1,j]}(K) \) must intersect both
\[
\{x \in \Gamma'_j : x_j \in A\} \text{ and } \{x \in \Gamma'_j : x_j \in B\}.
\]

Since \( \Gamma'_j \) is irreducible between these sets, this means that \( \pi_{[1,j]}(K) = \Gamma'_j \). It follows that for all \( i \in \mathbb{N} \) with \( 1 \leq i \leq j \), \( \pi_{[1,i]}(K) = \Gamma'_i \).

Since there are infinitely many such \( j \in \mathbb{N} \), it follows that \( \pi_{[1,j]}(K) = \Gamma'_i \) for all \( i \in \mathbb{N} \). Therefore, \( K = \lim_\leftarrow F \). \( \square \)
3. Irreducible Set-valued Functions

In this section, we will define a type of usc function which will be called an irreducible function. The purpose of this definition will be realized in Theorem 3.10 where we state that sequences of irreducible functions may be used to yield an inverse limit which has the full projection property and is an indecomposable continuum. Towards this end, we will first show in Lemma 3.9 that irreducible functions are also indecomposable functions. Thus, once it has been established that the inverse limits described in Theorem 3.10 have the full projection property, the fact that they are also indecomposable continua will follow from Theorem 2.3.

3.1. Irreducible Collections of Maps. As has been indicated, the definition of an irreducible function is given in terms of the function’s inverse. Its inverse must be the union of a collection of single-valued maps. The criteria such a collection must meet are outlined in this next definition. In this definition as well as in the rest of this paper, the following notation will be used.

**Notation.** Given a subset Λ of the real numbers, Λ' refers to the set of limit points of Λ; and given an interval $I \subseteq \mathbb{R}$, $I_\Lambda$ refers to the intersection of $I$ with $\Lambda$.

**Definition 3.1.** Let $X$ and $Y$ be irreducible continua, and $\Lambda \subseteq [0, 1]$ be a closed set with $0, 1 \in \Lambda$ and $\Lambda \setminus \Lambda' = \Lambda$. A collection of maps $\{f_\lambda : Y \to X\}_{\lambda \in \Lambda}$ is called **irreducible with respect to** $a, b \in X$ and $c, d \in Y$ if $X$ is irreducible between $a$ and $b$, $Y$ is irreducible between $c$ and $d$, and the following hold:

1. $a \in f_\lambda(Y)$ if and only if $\lambda = 0$, and $b \in f_\lambda(Y)$ if and only if $\lambda = 1$.
2. If $0 \notin \Lambda'$, then $f_0^{-1}(a) = \{c\}$ or $f_0^{-1}(a) = \{d\}$.
3. If $1 \notin \Lambda'$, then $f_1^{-1}(b) = \{c\}$ or $f_1^{-1}(b) = \{d\}$.
4. If $\lambda, \mu \in \Lambda$ with $\lambda < \mu$, then $f_\lambda(y) \neq f_\mu(y)$ for all $y \notin \{c, d\}$, and $\Gamma(f_\lambda) \cap \Gamma(f_\mu) \neq \emptyset$ if and only if $(\lambda, \mu)_\Lambda = \emptyset$.
5. If $(\lambda_i)_{i \in \mathbb{N}}$ is a sequence of points in $\Lambda$ and $\lambda_i \to \lambda$ as $i \to \infty$, then $f_{\lambda_i} \to f_\lambda$ uniformly as $i \to \infty$.

When no ambiguity shall arise, or when mention of the points, $a, b \in X$ and $c, d \in Y$ is unnecessary, we will simply say that $\{f_\lambda : Y \to X\}_{\lambda \in \Lambda}$ is an irreducible collection of maps.

Figures 1 and 2 provide examples of the graphs of irreducible collections of maps. In Figure 1 is a collection $\{f_\lambda : [0, 1] \to [0, 1]\}_{\lambda \in \Lambda}$ where $\Lambda = \{0, 1/5, 2/5, 3/5, 4/5, 1\}$. $f_0$ is the bottom function and is the only function which takes on the value of $0$; $f_{1/5}$ is the function immediately above $f_0$ whose graph intersects the graph of $f_0$ only at $1$; and so on. Notice that since $\Lambda$ has no limit points, no function is a limit of other functions.
Figure 1. \{f_\lambda : [0, 1] \to [0, 1]\}_{\lambda \in \Lambda}

Figure 2. \{g_\omega : [0, 1] \to [0, 1]\}_{\omega \in \Omega}

In Figure 2 is a collection \{g_\omega : [0, 1] \to [0, 1]\}_{\omega \in \Omega}. Perhaps the simplest indexing set for this collection would be

\[\Omega = \left\{\frac{2^n - 1}{2^{n+1}} : n \in \mathbb{N}\right\} \cup \left\{\frac{2^n + 1}{2^{n+1}} : n \in \mathbb{N}\right\} \cup \left\{\frac{2^{n+1} - 1}{2^{n+1}} : n \in \mathbb{N}\right\} \cup \left\{\frac{1}{2}, 1\right\},\]

but any closed subset \(\Omega \subseteq [0, 1]\) could be used so long as 0 \(\in\) \(\Omega\), and \(\Omega\) has exactly two limit points—one of which is 1, and the other is a two-sided limit point which lies in (0, 1).

Another thing worth noting about the collection pictured in Figure 2 is that \(g_0^{-1}(1) = \left[0, \frac{1}{2}\right]\). This is allowed because 1 is a limit point of \(\Omega\). Since 0 is not a limit point of \(\Omega\), \(g_0^{-1}(0)\) must be a singleton subset of \(\{0, 1\}\). Specifically, in this case, \(g_0^{-1}(0) = \{0\}\).

This definition of an irreducible collection of maps can be generalized to the context of irreducibility with respect to closed sets in the following way.

**Definition 3.2.** Let \(X\) and \(Y\) be irreducible continua, and \(\Lambda \subseteq [0, 1]\) be a closed set with 0, 1 \(\in\) \(\Lambda\) and \(\Lambda \setminus \Lambda' = \Lambda\). A collection of maps \(\{f_\lambda : Y \to X\}_{\lambda \in \Lambda}\) is called **irreducible with respect to** \(A, B \subseteq X\) and \(C, D \subseteq Y\) if \(X\) is irreducible between the sets \(A\) and \(B\), \(Y\) is irreducible between the sets \(C\) and \(D\), and the following hold:

1. \(A \cap f_\lambda(Y) \neq \emptyset\) if and only if \(\lambda = 0\), and \(B \cap f_\lambda(Y) \neq \emptyset\) if and only if \(\lambda = 1\).
2. If \(0 \notin \Lambda'\), then \(f_0^{-1}(A) \subseteq C\) or \(f_0^{-1}(A) \subseteq D\).
3. If \(1 \notin \Lambda'\), then \(f_1^{-1}(B) \subseteq C\) or \(f_1^{-1}(B) \subseteq D\).
4. (a) If \(\lambda, \mu \in \Lambda\) with \(\lambda < \mu\), then \(f_\lambda(y) \neq f_\mu(y)\) for all \(y \notin C \cup D\), and \(\Gamma(f_\lambda) \cap \Gamma(f_\mu) = \emptyset\) if and only if \((\lambda, \mu)_\Lambda = \emptyset\).
   (b) If \(\lambda, \mu \in \Lambda\), and \(L \in \{C, D\}\), then \(\Gamma(f_\lambda|_L) \cap \Gamma(f_\mu|_L) = \emptyset\) implies that \(\Gamma(f_\lambda|_L) \cap \Gamma(f_\sigma|_L) = \emptyset\) for all \(\sigma \in \Lambda \setminus \{\lambda, \mu\}\).
   (c) If \(L \in \{C, D\}\) and \(A \cap f_0(L) \neq \emptyset\), then \(\Gamma(f_0|_L) \cap \Gamma(f_\lambda|_L) = \emptyset\) for all \(\lambda \in \Lambda \setminus \{0\}\); and if \(B \cap f_1(L) \neq \emptyset\), then \(\Gamma(f_1|_L) \cap \Gamma(f_\lambda|_L) = \emptyset\) for all \(\lambda \in \Lambda \setminus \{1\}\).
(5) If \((\lambda_i)_{i \in \mathbb{N}}\) is a sequence of points in \(\Lambda\) and \(\lambda_i \to \lambda\) as \(i \to \infty\), then \(f_{\lambda_i} \to f_\lambda\) uniformly as \(i \to \infty\).

When no ambiguity shall arise, or when mention of the sets, \(A, B \subseteq X\) and \(C, D \subseteq Y\) is unnecessary, we will simply say that \(\{f_\lambda : Y \to X\}_{\lambda \in \Lambda}\) is an irreducible collection of maps.

Note that if \(A, B, C, \) and \(D\) are singleton sets, then Definition 3.2 is equivalent to Definition 3.1.

Figures 3 and 4 give an example of an irreducible collection of maps on an irreducible set other than \([0, 1]\). The continuum \(X\) is pictured in Figure 3, and Figure 4 pictures four maps \(f_0, f_\frac{1}{3}, f_\frac{2}{3}, f_1 : X \to X\) which could satisfy Definition 3.2. Since it is impossible to show the graphs of these functions with only two dimensions, only their images are shown, but one can easily imagine a collection of functions with the given images that satisfy Definition 3.2.

**Lemma 3.3.** Let \(\{f_\lambda : Y \to X\}_{\lambda \in \Lambda}\) be an irreducible collection of maps. Then \(\bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda)\) is a continuum.

**Proof.** Let \(K = \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda)\). First, to show that \(K\) is compact, define a function \(H : \Lambda \times Y \to Y \times X\) by \(H(\lambda, y) = (y, f_\lambda(y))\). From Property 5 of Definition 3.2, it follows that \(H\) is continuous. Since \(\Lambda \times Y\) is compact, \(H(\Lambda \times Y) = K\) is compact.
Next, suppose that \( K \) is not connected. Then there exist non-empty, closed, disjoint sets \( A, B \subseteq K \) with \( K = A \cup B \). Since each \( f_\lambda \) is a continuous function, its graph is connected, so either \( \Gamma(f_\lambda) \subseteq A \) or \( \Gamma(f_\lambda) \subseteq B \). Let \( \mathcal{A} = \{ \lambda \in \Lambda : \Gamma(f_\lambda) \subseteq A \} \) and \( \mathcal{B} = \{ \lambda \in \Lambda : \Gamma(f_\lambda) \subseteq B \} \).

Since \( A \) and \( B \) are both non-empty, \( \mathcal{A} \) and \( \mathcal{B} \) are both non-empty. Without loss of generality, suppose that \( 1 \in \mathcal{B} \), and let \( \alpha = \max \mathcal{A} \). Then \([\alpha, 1] \cap \mathcal{B}\) is a closed set, so it has a minimal element \( \beta \). Since \( \alpha \notin \mathcal{B} \), \( \beta \neq \alpha \), so \( \beta > \alpha \). In particular, \( \beta \) is the smallest element of \( \Lambda \) greater than \( \alpha \). This means that \( (\alpha, \beta) \subseteq K = \emptyset \), so by Property 4 of Definition 3.2, we have that \( \Gamma(f_\alpha) \cap \Gamma(f_\beta) \neq \emptyset \). This is a contradiction since \( \Gamma(f_\alpha) \subseteq A \), \( \Gamma(f_\beta) \subseteq B \), and \( A \) and \( B \) are disjoint.

Therefore, \( K \) must be connected and is thus a continuum. \( \square \)

**Corollary 3.4.** Let \( \{f_\lambda : Y \to X\}_{\lambda \in \Lambda} \) be a collection of maps irreducible with respect to \( A, B \subseteq X \) and \( C, D \subseteq Y \). Then \( \bigcup_{\lambda \in \Lambda} f_\lambda(Y) = X \).

**Proof.** Let \( K = \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda) \). Then \( \bigcup_{\lambda \in \Lambda} f_\lambda(Y) = \pi_2(K) \) where \( \pi_2 : Y \times X \to X \). From Lemma 3.3, \( K \) is a continuum, so since \( \pi_2 \) is continuous, \( \bigcup_{\lambda \in \Lambda} f_\lambda(Y) \) is a continuum. Also, since \( A \cap f_0(Y) \neq \emptyset \) and \( B \cap f_1(Y) \neq \emptyset \), \( \bigcup_{\lambda \in \Lambda} f_\lambda(Y) \) is a subcontinuum of \( X \) which intersects both \( A \) and \( B \). Since \( X \) is irreducible between \( A \) and \( B \), it follows that \( \bigcup_{\lambda \in \Lambda} f_\lambda(Y) = X \). \( \square \)

This next theorem appears in [6, p. 72] and will be useful in proving Lemma 3.6

**Theorem 3.5** (Cut-wire Theorem). Let \( X \) be a compact metric space, and let \( A \) and \( B \) be closed subsets of \( X \). If no component of \( X \) intersects both \( A \) and \( B \), then \( X = X_1 \cup X_2 \) where \( X_1 \) and \( X_2 \) are disjoint closed subsets of \( X \) with \( A \subseteq X_1 \) and \( B \subseteq X_2 \).

Lemma 3.6 below should begin to make apparent the purpose of each element of Definition 3.2 as well as why the word “irreducible” was chosen to describe these collections of maps.

**Lemma 3.6.** Suppose \( \{f_\lambda : Y \to X\}_{\lambda \in \Lambda} \) is irreducible with respect to \( A, B \subseteq X \) and \( C, D \subseteq Y \). Then the continuum \( \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda) \) is irreducible between the sets \( Y \times A \) and \( Y \times B \).

**Proof.** Suppose that \( K \) is a subcontinuum of \( \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda) \) which contains a point \( (y_1, a) \) and a point \( (y_2, b) \) where \( a \in A \) and \( b \in B \). From Definition 3.2, \( (y_1, a) \in \Gamma(f_0) \) and \( (y_2, b) \in \Gamma(f_1) \). We will show that this implies that for all \( \lambda \in \Lambda \), \( \Gamma(f_\lambda) \subseteq K \). For each \( \lambda \in \Lambda \), let \( C_\lambda = \Gamma(f_\lambda\vert C) \) and \( D_\lambda = \Gamma(f_\lambda\vert D) \). Also, let \( H : \Lambda \times Y \to Y \times X \) be defined by \( H(\lambda, y) = (y, f_\lambda(y)) \).

Case 1: Let \( \lambda_0 \in \Lambda \setminus \Lambda' \).

Sub-case (i): Suppose \( \lambda_0 \notin \{0, 1\} \). Then there exist \( \lambda_1, \lambda_2 \in \Lambda \) with \( C_{\lambda_1} \cap C_{\lambda_0} \neq \emptyset \) and \( D_{\lambda_0} \cap D_{\lambda_0} \neq \emptyset \). Without loss of generality, we may suppose that \( \lambda_1 < \lambda_0 < \lambda_2 \). Since \( \lambda_0 \notin \Lambda' \), no point of \( \Gamma(f_{\lambda_0}) \setminus (C_{\lambda_0} \cup D_{\lambda_0}) \) is a limit point of \( \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda) \setminus \Gamma(f_{\lambda_0}) \). Thus \( \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda) \setminus C_{\lambda_0} \) is
not connected. In fact, \( \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda) \setminus C_{\lambda_0} = U_1 \cup V_1 \) where
\[
U_1 = H \left( [0, \lambda_1] \times Y \right) \setminus H(\{\lambda_0\} \times C), \quad \text{and}
\]
\[
V_1 = H \left( [\lambda_0, 1] \times Y \right) \setminus H(\{\lambda_0\} \times C).
\]
Note that \((y_1, a) \in U_1\) and \((y_2, b) \in V_1\). Thus \(K\) intersects both \(U_1\) and \(V_1\), so since \(K\) is connected, it must not be a subset of \(U_1 \cup V_1\). Therefore, \(C_{\lambda_0} \cap K \neq \emptyset\).

Similarly, \( \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda) \setminus D_{\lambda_0} = U_2 \cup V_2 \) where
\[
U_2 = H \left( [0, \lambda_0] \times Y \right) \setminus H(\{\lambda_0\} \times D), \quad \text{and}
\]
\[
V_2 = H \left( [\lambda_2, 1] \times Y \right) \setminus H(\{\lambda_0\} \times D).
\]
Then since \(K\) intersects both \(U_2\) and \(V_2\), it must not be contained in their union, so \(D_{\lambda_0} \cap K \neq \emptyset\). Therefore, we have that \(K\) intersects both \(C_{\lambda_0}\) and \(D_{\lambda_0}\).

Suppose that there exists a continuum \(L\) in \(K \cap \Gamma(f_{\lambda_0})\) which intersects both \(C_{\lambda_0}\) and \(D_{\lambda_0}\). Then since \(\Gamma(f_{\lambda_0})\) is irreducible between these sets \(L = \Gamma(f_{\lambda_0})\), so since \(L \subseteq K \cap \Gamma(f_{\lambda_0})\), in particular, \(\Gamma(f_{\lambda_0}) \subseteq K\).

If there does not exist any such continuum, then by the Cut-wire Theorem, there exist disjoint closed sets \(M\) and \(N\) such that \(C_{\lambda_0} \cap K \subseteq M\), \(D_{\lambda_0} \cap K \subseteq N\), and \(M \cup N = K \cap \Gamma(f_{\lambda_0})\). Then if \(\tilde{U} = U_1 \cup M\) and \(\tilde{V} = V_2 \cup N\) (where \(U_1\) and \(V_2\) are as defined above), then \(\tilde{U}\) and \(\tilde{V}\) separate \(K\). This is a contradiction since \(K\) is connected. Therefore \(K \cap \Gamma(f_{\lambda_0})\) does contain a continuum \(L\) which intersects both \(C_{\lambda_0}\) and \(D_{\lambda_0}\), and this implies that \(L = \Gamma(f_{\lambda_0}) \subseteq K\).

Sub-case (ii): Suppose \(\lambda_0 \in \{0, 1\}\). If \(\lambda_0 = 0\), then \(0 \notin \Lambda\), so from Definition 3.2, we know that \(f_0^{-1}(a) \subseteq C\) or \(f_0^{-1}(a) \subseteq D\). If \(f_0^{-1}(a) \subseteq C\), then \((y_1, a) \in C_0\), so \(C \cap K \neq \emptyset\). Then we may choose \(\lambda_3\) to be the smallest element of \(\Lambda \setminus \{0\}\). Since \(f_0^{-1}(A) \subseteq C\), we know that \(D_0 \cap D_{\lambda_3} \neq \emptyset\). We may construct sets in a way similar to the construction of \(U\) and \(V\) above to separate \(\bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda) \setminus D_0\).

Thus, since \(K\) is connected, we may conclude that \(D_0 \cap K \neq \emptyset\).

Just as in sub-case (i), if there is continuum in \(K \cap \Gamma(f_0)\) which intersects both \(C_0\) and \(D_0\), then that continuum must be \(\Gamma(f_0)\), so \(\Gamma(f_0) \subseteq K\). Otherwise, the Cut-wire Theorem gives us a separation of \(K \cap \Gamma(f_0)\) by closed sets \(S\) and \(T\) such that \(K \cap C_0 \subseteq S\) and \(K \cap D_0 \subseteq T\). Then, if \(\tilde{U} = S\), and \(\tilde{V} = T \cup H([\lambda_3, 1] \times Y)\), then \(\tilde{U}\) and \(\tilde{V}\) separate \(K\). This is a contradiction, so we have that \(\Gamma(f_0) \subseteq K\).

If \(f_0^{-1}(a) \subseteq D\) then the exact same argument holds by swapping the roles of \(C\) and \(D\), and if \(\lambda_0 = 1\), then a similar argument shows that \(\Gamma(f_1) \subseteq K\). This concludes Case 1.

Case 2: Let \(\lambda_0 \in \Lambda\). It must be shown that \(\Gamma(f_{\lambda_0}) \subseteq K\), or in particular, that \((y, f_{\lambda_0}(y)) \in K\) for all \(y \in Y\).

Since \(\Lambda \setminus \Lambda'\) is dense in \(\Lambda\), there exists a sequence \((\lambda_i)_{i \in \mathbb{N}} \subseteq \Lambda \setminus \Lambda'\) whose limit is \(\lambda_0\). From Case 1, we have that if \(y \in Y\), the sequence \((y, f_{\lambda_i}(y))_{i \in \mathbb{N}}\) is in \(K\), and from Property 5 of Definition 3.2,
(y, f_\lambda(y)) \to (y, f_{\lambda_0}(y)) as i \to \infty. Therefore, since K is closed, (y, f_{\lambda_0}(y)) \in K. Since this holds for all y \in Y, we have that \Gamma(f_{\lambda_0}) \subseteq K.

From these two cases, it follows that \Gamma(f_{\lambda}) \subseteq K for all \lambda \in \Lambda, and therefore, \bigcup_{\lambda \in \Lambda} \Gamma(f_{\lambda}) \subseteq K. Hence \bigcup_{\lambda \in \Lambda} \Gamma(f_{\lambda}) is irreducible between the sets Y \times A and Y \times B.

\section{Irreducible Functions}

We are now ready to define the term “irreducible function.”

\textbf{Definition 3.7.} A function \( F : X \to 2^Y \) is called \textit{irreducible with respect to} \( A, B \subseteq X \) and \( C, D \subseteq Y \), if there exists a collection of maps \( \{ f_\lambda : Y \to X \}_{\lambda \in \Lambda} \) which is irreducible with respect to \( A, B \subseteq X \) and \( C, D \subseteq Y \) such that for all \( x \in X \),

\[ F(x) = \bigcup_{\lambda \in \Lambda} f_\lambda^{-1}(x). \]

When no ambiguity shall arise, or when mention of the sets, \( A, B \subseteq X \) and \( C, D \subseteq Y \) is unnecessary, we will simply say that \( F \) is an irreducible function.

Corollary 3.4 stated that given an irreducible collection of maps \( \{ f_\lambda : Y \to X \}_{\lambda \in \Lambda} \), the union of their images \( \bigcup_{\lambda \in \Lambda} f_\lambda(Y) \) is equal to \( X \). Thus, the maps in the collection can be inverted to yield an irreducible function \( F : X \to 2^Y \). Therefore any of the irreducible collections from Figures 1, 2, or 4 correspond to irreducible functions.

In particular, Figure 5 shows an irreducible collection of maps \( \{ g_\omega : [0,1] \to [0,1] \}_{\omega \in \Omega} \), and Figure 6 shows its corresponding irreducible function \( G : [0,1] \to 2^{[0,1]} \).

\textbf{Lemma 3.8.} If \( F : X \to 2^Y \) is an irreducible function, then \( F \) is usc, and \( \Gamma(F) \) is a continuum.

\textit{Proof.} Let \( \{ f_\lambda : Y \to X \}_{\lambda \in \Lambda} \) be the irreducible collection corresponding to \( F \). then \( \Gamma(F) = \bigcup_{\lambda \in \Lambda} \Gamma(f^{-1}_\lambda) \) which is homeomorphic to \( \bigcup_{\lambda \in \Lambda} \Gamma(f_{\lambda}) \). Therefore, by Lemma 3.3, \( \Gamma(F) \) is a continuum. In particular, \( \Gamma(F) \) is also compact, so \( F \) is usc. \qed
Lemma 3.9. Every irreducible set-valued function is an indecomposable function.

Proof. Let $F : X \to 2^Y$ be an irreducible function with the corresponding collection $\{f_\lambda : Y \to X\}_{\lambda \in \Lambda}$ irreducible with respect to $A, B \subseteq X$ and $C, D \subseteq Y$. Suppose that $K$ and $L$ are subcontinua of $\Gamma(F)$ with $K \cup L = \Gamma(F)$.

Case 1: Suppose that there exists $\lambda \in \Lambda$ such that $\Gamma(f_\lambda^{-1}) \cap K = \emptyset$. Then $\Gamma(f_\lambda^{-1}) \subseteq L$, so $\pi_2(L) \supseteq \pi_2[\Gamma(f_\lambda^{-1})] = Y$. Similarly, if there exists $\lambda \in \Lambda$ such that $\Gamma(f_\lambda^{-1}) \cap L = \emptyset$, then $\pi_2(K) = Y$.

Case 2: Suppose that for all $\lambda \in \Lambda$, $\Gamma(f_\lambda^{-1}) \cap K$ and $\Gamma(f_\lambda^{-1}) \cap L$ are both nonempty. Let $\lambda_0, \lambda_1$ be adjacent elements of $\Lambda$. Without loss of generality, suppose that $\Gamma(f_{\lambda_0}|_C) \cap \Gamma(f_{\lambda_1}|_C) \neq \emptyset$. Let $H : \Lambda \times Y \to X \times Y$ be defined by $(f_\lambda(y), y)$.

Similarly to the proof of Lemma 3.6, $\Gamma(F) \setminus \Gamma[(f_{\lambda_0}|_C)^{-1}]$ can be separated by the sets $U$ and $V$ where if $\lambda_1 < \lambda_0$,

$$U = H([0, \lambda_1]_\Lambda \times Y) \setminus H([\lambda_0] \times C),$$

and if $\lambda_0 < \lambda_1$,

$$V = H([\lambda_0, 1]_\Lambda \times Y) \setminus H([\lambda_0] \times C),$$

and $\pi_2(K) = Y$.

We are now ready to prove the main result of this section.

Theorem 3.10. Let $\{X, F\}$ be an inverse sequence where for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ is irreducible with respect to $A_{i+1}, B_{i+1} \subseteq X_{i+1}$ and $A_i, B_i \subseteq X_i$. Then $\lim \downarrow F$ has the full projection property and is an indecomposable continuum.

Proof. Since each $F_i$ is an irreducible function, for each $i \in \mathbb{N}$ there is a corresponding collection $\{f_\lambda^{(i)} : X_i \to X_{i+1}\}_{\lambda \in \Lambda_i}$ which is irreducible with respect to $A_{i+1}, B_{i+1} \subseteq X_{i+1}$ and $A_i, B_i \subseteq X_i$. From Lemma 3.3, we have that $\Gamma'_2 = \bigcup_{\lambda \in \Lambda_1} \Gamma(f_\lambda^{(1)})$ is connected, and from Lemma 3.6, $\Gamma'_2$ is irreducible between the sets $X_1 \times A_2$ and $X_1 \times B_2$. 

Now suppose that for some $n \in \mathbb{N}$, $\Gamma'_{n}$ is a continuum and is irreducible between the sets

$$\mathcal{A} = \{ x \in \Gamma'_{n} : x_n \in A_n \} \text{ and } \mathcal{B} = \{ x \in \Gamma'_{n} : x_n \in B_n \}.$$ 

For each $\lambda \in \Lambda_n$, define a function $h_{\lambda} : \Gamma'_{n} \to X_{n+1}$ by $h_{\lambda}(x_1, \ldots, x_n) = f_{\lambda}^{(n)}(x_n)$. Then the collection of maps $\{ h_{\lambda} : \Gamma'_{n} \to X_{n+1} \}_{\lambda \in \Lambda_n}$ is irreducible with respect to $A_{n+1}, B_{n+1} \subseteq X_{n+1}$ and $\mathcal{A}, \mathcal{B} \subseteq \Gamma'_{n}$. Also $\Gamma'_{n+1} = \bigcup_{\lambda \in \Lambda} \Gamma(h_{\lambda})$. By Lemma 3.3, $\bigcup_{\lambda \in \Lambda} \Gamma(h_{\lambda})$ is a continuum, and by Lemma 3.6, it is irreducible between the sets $\Gamma'_{n} \times A_{n+1}$ and $\Gamma'_{n} \times B_{n+1}$.

By induction we can say that for each $n \in \mathbb{N}$, $\Gamma'_{n}$ is a continuum which is irreducible between the sets $\Gamma'_{n-1} \times A_{n}$ and $\Gamma'_{n-1} \times B_{n}$. Therefore, $\varprojlim \mathbf{F}$ is a continuum, and by Theorem 2.5, $\varprojlim \mathbf{F}$ has the full projection property.

Finally, by Lemma 3.9, each $F_i$ is an indecomposable function, so by Theorem 2.3, $\varprojlim \mathbf{F}$ is an indecomposable continuum. □

4. Homeomorphisms between Inverse Limits of Irreducible Functions

Now that we have established a method for constructing inverse sequences whose inverse limits are indecomposable, it is natural to ask what more we know about these inverse limits and how they are related to each other. In Section 4.1 we will establish sufficient conditions for two inverse sequences of irreducible functions to have homeomorphic inverse limits. We will establish our conditions first for sequences of functions which are irreducible with respect to points. This will lead to one of our main results, Theorem 4.6.

Next, in the context of irreducibility with respect to sets, the conditions will be more restrictive, but we will be able to establish conditions under which two inverse sequences of irreducible functions will have homeomorphic inverse limits. This result will be stated in Theorem 4.11.

In Section 4.2, we will discuss some applications of Theorem 4.6. Specifically, we will focus on the case where all of our factor spaces are $[0, 1]$, and all of the bonding functions are the same irreducible function whose corresponding irreducible collection is finite. Functions of this type will have Knaster continua as their inverse limits, and they may be classified by the number of maps in their corresponding irreducible collections.

4.1. Consistent Irreducible Functions. We begin with the following definition and lemma which will be applied extensively in this section.

**Definition 4.1.** Let $\{ \mathbf{X}, \mathbf{F} \}$ be an inverse sequence where for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ is an irreducible function with the associated irreducible collection $\{ f_{\lambda}^{(i)} : X_i \to X_{i+1} \}_{\lambda \in \Lambda_i}$. Define the *itinerary map* for $\{ \mathbf{X}, \mathbf{F} \}$ to be the function $\mathcal{F} : X_1 \times \prod_{i=1}^{\infty} \Lambda_i \to \varprojlim \mathbf{F}$ given by $\mathcal{F}(x, \lambda_1, \lambda_2, \ldots) = y$ where $y_1 = x$ and $y_{i+1} = f_{\lambda_i}^{(i)}(y_i)$ for $i \in \mathbb{N}$. 
Lemma 4.2. Let \( \{X, F\} \) be an inverse sequence where for each \( i \in \mathbb{N} \), \( F_i : X_{i+1} \to 2^{X_i} \) is an irreducible function with the associated irreducible collection \( \{f^{(i)}_\lambda : X_i \to X_{i+1}\}_{\lambda \in \Lambda_i} \). Then the itinerary map \( F \) for this inverse sequence is continuous and a closed map.

Proof. \( F \) is clearly continuous in its first coordinate, and its continuity in all other coordinates follows from Property 5 of Definition 3.2. Then, since its domain is compact and its range is Hausdorff, \( F \) is a closed map. \( \square \)

Definition 4.3. Let \( X \) and \( Y \) be irreducible continua, and let \( \{f_\lambda : Y \to X\}_{\lambda \in \Lambda} \) and \( \{g_\lambda : Y \to X\}_{\lambda \in \Lambda} \) each be irreducible with respect to \( a, b \in X \) and \( c, d \in Y \). Let these collections have the additional property that each of \( f^{-1}_0(a), f^{-1}_1(b), g^{-1}_0(a) \), and \( g^{-1}_1(b) \) is either a subset of \( \{c, d\} \) or is equal to \( Y \). We say that \( \{f_\lambda : Y \to X\}_{\lambda \in \Lambda} \) and \( \{g_\lambda : Y \to X\}_{\lambda \in \Lambda} \) are consistent if the following hold:

1. \( f^{-1}_0(a) = g^{-1}_0(a) \) and \( f^{-1}_1(b) = g^{-1}_1(b) \), and

2. for each \( \lambda, \mu \in \Lambda \),

   \[ \{y \in Y : f_\lambda(y) = f_\mu(y)\} = \{y \in Y : g_\lambda(y) = g_\mu(y)\} \].

Two irreducible functions are said to be consistent if their corresponding irreducible collections are consistent.

The following terminology will be useful, particularly in the proof of Lemma 4.5. If \( \{f_\lambda : Y \to X\}_{\lambda \in \Lambda} \) and \( \{g_\lambda : Y \to X\}_{\lambda \in \Lambda} \) are consistent, then given a pair \( (\alpha, l) \in \{(0, a), (1, b)\} \), we say that \( (\alpha, l) \) is Type I if \( f^{-1}_\alpha(l) = g^{-1}_\alpha(l) \subseteq \{c, d\} \), and we say that \( (\alpha, l) \) is Type II if \( f^{-1}_\alpha(l) = g^{-1}_\alpha(l) = Y \).

Example 4.4. The irreducible collection \( \{f_\lambda : [0, 1] \to [0, 1]\}_{\lambda \in \Lambda} \) pictured in Figure 7 and the irreducible collection \( \{g_\omega : [0, 1] \to [0, 1]\}_{\omega \in \Omega} \) pictured in Figure 8 are consistent.
Proof. We will use the set $\Lambda = \{ \ldots, 1/16, 1/8, 1/4, 1/2, 3/4, 7/8, 15/16 \ldots \} \cup \{0, 1\}$ as the indexing set. $f_0^{-1}(0) = \{0, 1\} = g_0^{-1}(0)$, and $f_1^{-1}(1) = [0, 1] = g_1^{-1}(1)$, so in particular, $f_0^{-1}(0) = g_0^{-1}(0)$ and $f_1^{-1}(1) = g_1^{-1}(1)$ which satisfies Property 1 of Definition 4.3. In addition, we have that the pair $(0, 0)$ is Type I, and $(1, 1)$ is Type II. For Property 2 of the definition to be met, we must have that for $\lambda, \mu \in \Lambda$, $f_\lambda(x) = f_\mu(x)$ if and only if $g_\lambda(x) = g_\mu(x)$.

Note that there are infinitely many ways that $\Lambda$ can be used to index the collection of maps pictured Figure 7 so that it fits the definition of an irreducible collection. Likewise for the collection pictured in Figure 8. Specifically, we may say that $f_{3/4}$ is the function which goes from $(0, 3/4)$ to $(1, 7/8)$ and that $g_{3/4}$ is the function which goes from $(0, 7/8)$ to $(1, 15/16)$. We may then index the rest of the functions accordingly. This insures that $f_\lambda(x) = x$ if and only if $g_\lambda(x) = g_\mu(x)$.

Thus, these collections meet the conditions of Definition 4.3, so they are consistent. \hfill \Box

Lemma 4.5. Let $\{X, F\}$ and $\{X, G\}$ be inverse sequences such that for each $i \in \mathbb{N}$, $F_i : X_{i+1} \rightarrow 2^{X_i}$ and $G_i : X_{i+1} \rightarrow 2^{X_i}$ are irreducible with respect to $a_{i+1}, b_{i+1} \in X_{i+1}$ and $a_i, b_i \in X_i$. Let $F$ be an itinerary map for $\{X, F\}$, and let $G$ be an itinerary map for $\{X, G\}$. If for each $i \in \mathbb{N}$, $F_i$ and $G_i$ are consistent, then the composition $G \circ F^{-1}$ is a well-defined function from $\lim F$ to $\lim G$.

Proof. For each $i \in \mathbb{N}$, let $\{f_\lambda^{(i)} : X_i \rightarrow X_{i+1} \}_{\lambda \in \Lambda_i}$ and $\{g_\lambda^{(i)} : X_i \rightarrow X_{i+1} \}_{\lambda \in \Lambda_i}$ be the irreducible collections associated with $F_i$ and $G_i$.

Let $x \in \lim F$, let $(x_1, \lambda_1, \lambda_2, \ldots), (x_1, \mu_1, \mu_2, \ldots) \in F^{-1}(x)$, and let $y = G(x_1, \lambda_1, \lambda_2, \ldots)$ and $z = G(x_1, \mu_1, \mu_2, \ldots)$. To show that $G \circ F^{-1}$ is well-defined, we must show that $y = z$. By the definitions of $F$ and $G$, $z_1 = y_1 = x_1$.

Proceeding by induction, suppose that for some $n_0 \in \mathbb{N}$, $y_i = z_i$ for all $i \leq n_0$. If $\lambda_{n_0} = \mu_{n_0}$, then $g_{\lambda_{n_0}}^{(n_0)}(y_{n_0}) = g_{\mu_{n_0}}^{(n_0)}(z_{n_0})$, so $y_{n_0+1} = z_{n_0+1}$. If $\lambda_{n_0} \neq \mu_{n_0}$, we will show that $x_{n_0} = y_{n_0} = z_{n_0}$.

Then since $f_{\lambda_{n_0}}^{(n_0)}(x_{n_0}) = x_{n_0+1} = f_{\mu_{n_0}}^{(n_0)}(x_{n_0})$, it will follow from Property 2 of Definition 4.3 that $g_{\lambda_{n_0}}^{(n_0)}(y_{n_0}) = g_{\mu_{n_0}}^{(n_0)}(z_{n_0})$, and therefore $y_{n_0+1} = z_{n_0+1}$.

Towards this end, note that since $f_{\lambda_{n_0}}^{(n_0)}(x_{n_0}) = f_{\mu_{n_0}}^{(n_0)}(x_{n_0})$, it follows from the fact that $\{f_\lambda^{(i)} : X_{n_0} \rightarrow X_{n_0+1} \}_{\lambda \in \Lambda_{n_0}}$ is an irreducible collection that $x_{n_0} \in \{a_{n_0}, b_{n_0}\}$. This means that $\lambda_{n_0-1} = \mu_{n_0-1} \in \{0, 1\}$. If $(\lambda_{n_0-1}, x_{n_0}) \in \{(0, a_{n_0}), (1, b_{n_0})\}$ is Type I, then we also have that $x_{n_0-1} \in \{a_{n_0-1}, b_{n_0-1}\}$, and thus $\lambda_{n_0-2} = \mu_{n_0-2} \in \{0, 1\}$. Then if we supposed that $(\lambda_{n_0-2}, x_{n_0-1})$ was also Type I, then we could continue on in this manner. This leads us to Case 1.

Case 1: Suppose that for all $j \in \mathbb{N}$ with $1 < j \leq n_0$, $\lambda_{j-1} = \mu_{j-1} \in \{0, 1\}$, $x_j \in \{a_j, b_j\}$, and the pair $(\lambda_{j-1}, x_j)$ is Type I. Then since $y_1 = z_1 = x_1$, it follows that $y_j = z_j = x_j$ for all $j \leq n_0$, and in particular, $y_{n_0} = z_{n_0} = x_{n_0}$.

Case 2: Suppose that for some $1 < j \leq n_0$, $\lambda_{j-1} = \mu_{j-1} \in \{0, 1\}$, $x_j \in \{a_j, b_j\}$, and the pair $(\lambda_{j-1}, x_j)$ is Type II. Then let $k$ be the largest integer less than or equal to $n_0$ such that $(\lambda_{k-1}, x_k)$ is Type II. Then from the definition of Type II, $(f_{\lambda_{k-1}}^{(k-1)})^{-1}(x_k) = (g_{\lambda_{k-1}}^{(k-1)})^{-1}(x_k) = X_{k-1}$ which
means that \( y_k = f^{(k-1)}(y_{k-1}) = x_k \). Since \( \mu_{k-1} = \lambda_{k-1} \), we can similarly show that \( z_k = x_k \). Thus, we in fact have that \( z_k = y_k = x_k \). If \( k = n_0 \), then we have our result. If not, then by assumption, \((\lambda_{j-1}, x_j)\) is Type I for all \( k < j \leq n_0 \), so from the same argument used in Case 1, it follows that \( x_{n_0} = y_{n_0} = z_{n_0} \).

In either case, we have that \( x_{n_0} = y_{n_0} = z_{n_0} \), so as already noted, this implies that \( y_{n_0+1} = z_{n_0+1} \). Therefore, by induction, \( y_i = z_i \) for all \( i \in \mathbb{N} \), so \( y = z \). Therefore \( G \circ F^{-1} \) is well-defined.

**Theorem 4.6.** Let \( \{X, F\} \) and \( \{X, G\} \) be inverse sequences such that for each \( i \in \mathbb{N} \), \( F_i : X_{i+1} \to 2^{X_i} \) and \( G_i : X_{i+1} \to 2^{X_i} \) are irreducible with respect to \( a_{i+1}, b_{i+1} \in X_{i+1} \) and \( a_i, b_i \in X_i \). If for each \( i \in \mathbb{N} \), \( F_i \) and \( G_i \) are consistent, then \( \overleftarrow{\lim F} \) and \( \overleftarrow{\lim G} \) are homeomorphic.

**Proof.** Let \( F \) and \( G \) be itinerary maps for \( \{X, F\} \) and \( \{X, G\} \) respectively. By Lemma 4.5, the composition \( G \circ F^{-1} \) is a well-defined function from \( \overleftarrow{\lim F} \) to \( \overleftarrow{\lim G} \). Let \( \Phi = G \circ F^{-1} \). Lemma 4.5 also gives us that \( F \circ G^{-1} \) is well-defined, so \( \Phi \) must be invertible and is thus bijective. Therefore, since \( \overleftarrow{\lim F} \) is compact and \( \overleftarrow{\lim G} \) is Hausdorff, to show that \( \Phi \) is a homeomorphism, we need only show that \( \Phi \) is continuous.

From Lemma 4.2, we have that \( G \) is continuous and \( F \) is a closed map. Therefore given a closed set \( A \subseteq \overleftarrow{\lim G} \), \( \Phi^{-1}(A) = F(G^{-1}(A)) \) is closed. Hence \( \Phi \) continuous and thus a homeomorphism between \( \overleftarrow{\lim F} \) and \( \overleftarrow{\lim G} \). \( \square \)

**Example 4.7.** Let \( F \) and \( G \) be the irreducible functions pictured in Figures 9 and 10 respectively. Then \( \overleftarrow{\lim F} \) is homeomorphic to \( \overleftarrow{\lim G} \).

**Proof.** The function \( F \) corresponds to the irreducible collection \( \{f_\lambda : [0, 1] \to [0, 1]\}_{\lambda \in \Lambda} \) pictured in Figure 7, and the function \( G \) corresponds to the collection \( \{g_\lambda : [0, 1] \to [0, 1]\}_{\lambda \in \Lambda} \) pictured in Figure 8. From Example 4.4, we have that these collections are consistent, so by definition, \( F \) and \( G \) are consistent. Therefore, by Theorem 4.6, their inverse limits are homeomorphic. \( \square \)
Example 4.8. Let $\Lambda$ be the set consisting of the standard Cantor set along with one point from each removed interval. Let $F, G, \tilde{F},$ and $\tilde{G}$ each be irreducible functions, as pictured in Figures 11, 12, 13, and 14, where each of their corresponding irreducible collections is indexed by $\Lambda$. Then $\lim\downarrow F$, $\lim\downarrow G$, $\lim\downarrow \tilde{F}$, and $\lim\downarrow \tilde{G}$ are all homeomorphic.

Proof. First note that $F$ and $G$ are consistent, as are $\tilde{F}$ and $\tilde{G}$. Thus Theorem 4.6 gives us that $\lim\downarrow F$ is homeomorphic to $\lim\downarrow G$, and $\lim\downarrow \tilde{F}$ is homeomorphic to $\lim\downarrow \tilde{G}$.

Hence, to show that they are all homeomorphic it suffices to show that $\lim\downarrow G$ is homeomorphic to $\lim\downarrow \tilde{G}$. Towards this end, note that for each $x \in [0, 1]$, $G(1 - x) = 1 - G(x) = \tilde{G}(x)$ (where by $1 - G(x)$ we mean the set $\{1 - y : y \in G(x)\}$).

Claim: This property implies that $\lim\downarrow G$ and $\lim\downarrow \tilde{G}$ are homeomorphic.

To prove this claim, define a function $\phi : \lim\downarrow G \to \prod_{i=1}^{\infty} [0, 1]$ by

$$\phi(x_1, x_2, x_3, \ldots) = (x_1, 1 - x_2, x_3, 1 - x_4, \ldots).$$
This function is clearly continuous and injective, so we need only check that \( \phi(\lim \overleftarrow{G}) = \lim \overleftarrow{\tilde{G}} \).

First, to show that \( \phi(\lim \overleftarrow{G}) \subseteq \lim \overleftarrow{\tilde{G}} \), let \( x \in \lim \overleftarrow{G} \) and \( y = \phi(x) \). For even \( n \), \( y_n = 1 - x_n \), and for odd \( n \), \( y_n = x_n \). Let \( n \in \mathbb{N} \) be even. Then \( G(y_n) = \tilde{G}(1 - x_n) = G(x_n) \supset x_{n-1} = y_{n-1} \), so \( y_{n-1} \in \tilde{G}(y_n) \). Now let \( n \in \mathbb{N} \) be odd. Then \( \tilde{G}(y_n) = \tilde{G}(x_n) = (1 - G(x_n)) \supset (1 - x_{n-1}) = y_{n-1} \), so again, \( y_{n-1} \in \tilde{G}(y_n) \).

Therefore \( y \in \lim \overleftarrow{\tilde{G}} \), so \( \phi(\lim \overleftarrow{G}) \subseteq \lim \overleftarrow{\tilde{G}} \). Now, given \( y \in \lim \overleftarrow{\tilde{G}} \), the same argument which was just presented will show that the point \((y_1, 1 - y_2, y_3, 1 - y_4, \ldots) \in \lim \overleftarrow{G} \) and \( y \) is the image of this point. This means that \( \phi(\lim \overleftarrow{G}) \supseteq \lim \overleftarrow{\tilde{G}} \), and \( \phi \) is a homeomorphism. \( \square \)

This example is interesting because \( \overleftarrow{F} \) and \( \overleftarrow{\tilde{F}} \) do not satisfy the property that \( F(1 - x) = 1 - F(x) = \overleftarrow{\tilde{F}}(x) \), nor are they consistent. Thus it would not be immediately clear that \( \lim \overleftarrow{F} \) and \( \lim \overleftarrow{\tilde{F}} \) were homeomorphic if it were not for \( G \) and \( \overleftarrow{\tilde{G}} \) acting as intermediaries.

Obtaining a result such as Theorem 4.6 for functions which are irreducible with respect to sets is a bit more complicated. We would like to define the term “consistent” in the context of irreducibility with respect to sets, and we would like to do it in a way so that the inverse limits of consistent functions are homeomorphic (as in Theorem 4.6).

In order to do this, we will have to make the definition in this context more stringent than in Definition 4.3.

**Definition 4.9.** Let \( \{X, F\} \) and \( \{X, G\} \) be inverse sequences where for each \( i \in \mathbb{N} \), \( F_i : X_{i+1} \to 2^{X_i} \) and \( G_i : X_{i+1} \to 2^{X_i} \) are irreducible with respect to \( A_{i+1}, B_{i+1} \subseteq X_{i+1} \) and \( A_i, B_i \subseteq X_i \). Let \( \{f^{(i)}_\lambda : X_i \to X_{i+1}\}_{\lambda \in \Lambda_i} \) and \( \{g^{(i)}_\lambda : X_i \to X_{i+1}\}_{\lambda \in \Lambda_i} \) be the irreducible collections corresponding to \( F_i \) and \( G_i \) respectively. We say that these inverse sequences are **consistent** if for each \( i \in \mathbb{N} \) and \( \lambda, \mu \in \Lambda_i \),

\[
\{y \in X_i : f^{(i)}_\lambda(y) = f^{(i)}_\mu(y)\} = \{y \in X_i : g^{(i)}_\lambda(y) = g^{(i)}_\mu(y)\}
\]

and either of the following hold:

1. For all \( i \in \mathbb{N} \), if \( (\alpha, L_{i+1}) \in \{(0, A_{i+1}), (1, B_{i+1})\} \), then

\[
(f^{(i)}_\alpha)^{-1}(L_{i+1}) = (g^{(i)}_\alpha)^{-1}(L_{i+1}) \subseteq A_i \cup B_i;
\]

and

\[
(f^{(i)}_\alpha)^{-1}(L_{i+1}) = g^{(i)}_\alpha|_{(g^{(i)}_\alpha)^{-1}(L_{i+1})};
\]

2. For all \( i \in \mathbb{N} \),

\[
(f^{(i)}_0)^{-1}(A_{i+1}) = (g^{(i)}_0)^{-1}(A_{i+1}) = (f^{(i)}_1)^{-1}(B_{i+1}) = (g^{(i)}_1)^{-1}(B_{i+1}) = X_i,
\]

and if \( L_i \in \{A_i, B_i\} \), then whenever \( \lambda, \mu \in \Lambda_i \) with \( \Gamma(f^{(i)}_\lambda|_{L_i}) \cap \Gamma(f^{(i)}_\mu|_{L_i}) \neq \emptyset \), it follows that \( f^{(i)}_\lambda|_{L_i} = f^{(i)}_\mu|_{L_i} \).
Lemma 4.10. Let \( \{X, F\} \) and \( \{X, G\} \) be inverse sequences such that for each \( i \in \mathbb{N} \), \( F_i : X_{i+1} \to 2^{X_i} \) and \( G_i : X_{i+1} \to 2^{X_i} \) are irreducible with respect to \( A_{i+1}, B_{i+1} \subseteq X_{i+1} \) and \( A_i, B_i \subseteq X_i \). Let \( F \) be an itinerary map for \( \{X, F\} \), and let \( G \) be an itinerary map for \( \{X, G\} \). If \( \{X, F\} \) and \( \{X, G\} \) are consistent, then the composition \( G \circ F^{-1} \) is a well-defined function from \( \varprojlim F \) to \( \varprojlim G \).

Proof. Let \( x \in \varprojlim F \), let \( (x_1, \lambda_1, \lambda_2, \ldots), (x_1, \mu_1, \mu_2, \ldots) \in F^{-1}(x) \), and let \( y = G(x_1, \lambda_1, \lambda_2, \ldots) \) and \( z = G(x_1, \mu_1, \mu_2, \ldots) \). We must show that \( y = z \). This will be done by induction.

By the definition of \( G \), \( y_1 = z_1 = x_1 \). Now suppose that for some \( n_0 \in \mathbb{N} \), \( y_i = z_i \) for all \( i \leq n_0 \). We want to show that this implies that \( y_{n_0+1} = z_{n_0+1} \). If \( \lambda_{n_0} = \mu_{n_0} \), then this will clearly hold, since \( y_{n_0+1} = g_{\lambda_{n_0}}(y_{n_0}) \) and \( z_{n_0+1} = g_{\mu_{n_0}}(z_{n_0}) \). Suppose then that \( \lambda_{n_0} \neq \mu_{n_0} \).

Case 1: Suppose that \( \{X, F\} \) and \( \{X, G\} \) satisfy Property (1) of Definition 4.9. Since \( f_{\lambda_{n_0}}^{(n_0)}(x_{n_0}) = f_{\mu_{n_0}}^{(n_0)}(x_{n_0}) \), we must have that \( x_{n_0} \in A_{n_0} \cup B_{n_0} \). We have that \( (f_0^{(i)})^{-1}(A_{i+1}) \subseteq A_i \cup B_i \) and \( (f_0^{(i)})^{-1}(B_{i+1}) \subseteq A_i \cup B_i \) for all \( i \in \mathbb{N} \). Another way of saying this is to say that \( F_i(A_{i+1}) \) and \( F_i(B_{i+1}) \) are subsets of \( A_i \cup B_i \) for all \( i \in \mathbb{N} \). Thus, \( x_i \in A_i \cup B_i \) for all \( i \leq n_0 \).

This also means that \( \lambda_i = \mu_i \in \{0, 1\} \) for all \( i < n_0 \) and that \( x_i \in (f_0^{(i)})^{-1}(A_{i+1}) \cup (f_0^{(i)})^{-1}(B_{i+1}) \) for all \( i < n_0 \). Now specifically, we know that \( x_1 = y_1 = z_1 \) and that this point is either an element of \( (f_0^{(1)})^{-1}(A_2) \) or of \( (f_0^{(1)})^{-1}(B_2) \). Because we are supposing that Property (1) holds, we know that \( f_0 \) and \( g_0 \) are equal when restricted to \( (f_0^{(1)})^{-1}(A_2) \), and we know that \( f_1 \) and \( g_1 \) are equal when restricted to \( (f_0^{(1)})^{-1}(B_2) \). This means that \( x_2 = y_2 = z_2 \). Continuing on in this manner, we conclude that \( x_i = y_i = z_i \) for all \( i \leq n_0 \).

Therefore, since \( f_{\lambda_{n_0}}^{(n_0)}(x_{n_0}) = f_{\mu_{n_0}}^{(n_0)}(x_{n_0}) \), it must follow that \( g_{\lambda_{n_0}}^{(n_0)}(y_{n_0}) = g_{\mu_{n_0}}^{(n_0)}(z_{n_0}) \), so \( y_{n_0+1} = z_{n_0+1} \).

Case 2: Suppose that \( \{X, F\} \) and \( \{X, G\} \) satisfy Property (2) of Definition 4.9. Again, since \( f_{\lambda_{n_0}}^{(n_0)}(x_{n_0}) = f_{\mu_{n_0}}^{(n_0)}(x_{n_0}) \), we must have that \( x_{n_0} \in A_{n_0} \cup B_{n_0} \). In this case, we may only infer that \( \lambda_{n_0-1} = \mu_{n_0-1} \in \{0, 1\} \). Let \( L_{n_0} \in \{A_{n_0}, B_{n_0}\} \) be the set containing \( x_{n_0} \). Because there is some value of \( L_{n_0} \) at which \( f_{\lambda_{n_0}} \) and \( f_{\mu_{n_0}} \) are equal (specifically \( x_{n_0} \)), we have that the equality \( f_{\lambda_{n_0}}^{(n_0)}(x) = f_{\mu_{n_0}}^{(n_0)}(x) \) must hold for all \( x \in L_{n_0} \). It follows that \( g_{\lambda_{n_0}}^{(n_0)}(x) = g_{\mu_{n_0}}^{(n_0)}(x) \) for all \( x \in L_{n_0} \). Also, since we are supposing that Property (2) holds, we know that

\[
(g_{\lambda_{n_0}}^{(n_0)-1})^{-1}(L_{n_0}) = (g_{\mu_{n_0}}^{(n_0)-1})^{-1}(L_{n_0}) = (f_{\lambda_{n_0}}^{(n_0)-1})^{-1}(L_{n_0}) = X_{n_0-1}.
\]

This means that \( y_{n_0} = z_{n_0} \in L_{n_0} \). Then, since it has already been established that \( g_{\lambda_{n_0}}^{(n_0)} \) and \( g_{\mu_{n_0}}^{(n_0)} \) are equal when restricted to \( L_{n_0} \), it follows that \( g_{\lambda_{n_0}}^{(n_0)}(y_{n_0}) = g_{\mu_{n_0}}^{(n_0)}(z_{n_0}) \).

In either case, \( y_{n_0+1} = z_{n_0+1} \), and hence, by induction, \( z_i = y_i \) for all \( i \in \mathbb{N} \), and \( y = z \). \( \Box \)

Just as in Theorem 4.6, once it has been established that \( G \circ F^{-1} \) is well-defined, it follows easily that it is in fact a homeomorphism.
Theorem 4.11. Let \( \{X, F\} \) and \( \{X, G\} \) be inverse sequences such that for each \( i \in \mathbb{N} \), \( F_i : X_{i+1} \to 2^{X_i} \) and \( G_i : X_{i+1} \to 2^{X_i} \) are irreducible with respect to \( A_{i+1}, B_{i+1} \subseteq X_{i+1} \) and \( A_i, B_i \subseteq X_i \). If \( \{X, F\} \) and \( \{X, G\} \) are consistent, then \( \mathop{\lim}\limits_\leftarrow F \) and \( \mathop{\lim}\limits_\leftarrow G \) are homeomorphic.

4.2. Applications. Below is a corollary to Theorem 4.6, and it deals with the case of an irreducible function on \([0, 1]\) whose corresponding irreducible collection is finite. In proving this corollary, the following definition and theorem will be helpful.

Definition 4.12. Let \( X \) be a continuum. Given two usc functions \( F : X \to 2^X \) and \( G : X \to 2^X \), we say that \( F \) and \( G \) are topologically conjugate if there exists a surjective homeomorphism \( \phi : X \to X \) such that \( \phi \circ F = G \circ \phi \).

Theorem 4.13 (Ingram, Mahavier [3]). If \( F : X \to 2^X \) and \( G : X \to 2^X \) are topologically conjugate usc functions, then \( \mathop{\lim}\limits_\leftarrow F \) is homeomorphic to \( \mathop{\lim}\limits_\leftarrow G \).

Corollary 4.14. Let \( F, G : [0, 1] \to 2^{[0,1]} \) be irreducible functions. If their corresponding irreducible collections are each finite and contain the same number of maps, then \( \mathop{\lim}\limits_\leftarrow F \) is homeomorphic to \( \mathop{\lim}\limits_\leftarrow G \).

Proof. Let \( k \) be the cardinality of the irreducible collections corresponding to \( F \) and \( G \). Let \( h : [0, 1] \to [0, 1] \) be the map consisting of \( k \) straight lines—the first from \((0, 0)\) to \((1/k, 1)\), the second from \((1/k, 1)\) to \((2/k, 0)\), and so on. Notice that \( h \) and \( 1 - h \) are both irreducible functions.

Claim: \( \mathop{\lim}\limits_\leftarrow h \) is homeomorphic to \( \mathop{\lim}\limits_\leftarrow (1 - h) \).

To see that this is true, notice that if \( k \) is odd, then \( h(1 - x) = 1 - h(x) \). Thus, in this case, just as in Example 4.8, \( \mathop{\lim}\limits_\leftarrow h \) is homeomorphic to \( \mathop{\lim}\limits_\leftarrow (1 - h) \). If \( k \) is even, then if \( \phi : [0, 1] \to [0, 1] \) is defined by \( \phi(x) = 1 - x \) we have that \( \phi \circ h = (1 - h) \circ \phi \). Therefore, in this case, \( h \) and \( 1 - h \) are topologically conjugate, so by Theorem 4.13, \( \mathop{\lim}\limits_\leftarrow h \) is homeomorphic to \( \mathop{\lim}\limits_\leftarrow (1 - h) \).

Since \( F^{-1} \) is the union of \( k \) maps, it will be consistent with either \( h \) or \( 1 - h \). Since \( \mathop{\lim}\limits_\leftarrow h \) and \( \mathop{\lim}\limits_\leftarrow (1 - h) \) are homeomorphic though, in either case, \( \mathop{\lim}\limits_\leftarrow F \) is homeomorphic to \( \mathop{\lim}\limits_\leftarrow h \). Similarly, \( \mathop{\lim}\limits_\leftarrow G \) is homeomorphic to \( \mathop{\lim}\limits_\leftarrow h \), so \( \mathop{\lim}\limits_\leftarrow F \) and \( \mathop{\lim}\limits_\leftarrow G \) are homeomorphic to each other. \( \square \)

In [9], Watkins discusses functions such as \( h \) from the above proof. He would call \( h \) the \( k \)th degree hat function. More specifically, given \( n \in \mathbb{N} \), the \( n \)th degree hat function is a map on \([0, 1]\) whose graph consists of \( n \) straight lines—the first from \((0, 0)\) to \((1/n, 1)\), the second from \((1/n, 1)\) to \((2/n, 0)\), and so on. If \( n \) is even, the last line is from \((n - 1)/n, 1)\) to \((1, 0)\). If \( n \) is odd, the last line is from \((n - 1)/n, 0)\) to \((1, 1)\). The inverse limits of these functions are the class of continua known as the Knaster continua. The main theorem of [9] is the following:

Theorem 4.15 (Watkins). Let \( n, m \in \mathbb{N} \). If \( f : [0, 1] \to [0, 1] \) is the \( n \)th degree hat function and \( g : [0, 1] \to [0, 1] \) is the \( m \)th degree hat function, then \( \mathop{\lim}\limits_\leftarrow f \) is homeomorphic to \( \mathop{\lim}\limits_\leftarrow g \) if and only if \( n \) and \( m \) have the same prime factors.
In light of Corollary 4.14, we can generalize this theorem in the following way.

**Theorem 4.16.** Let \( n, m \in \mathbb{N} \). Suppose \( F : [0, 1] \to 2^{[0,1]} \) is an irreducible function whose inverse is the union of \( n \) maps, and \( G : [0, 1] \to 2^{[0,1]} \) is an irreducible function whose inverse is the union of \( m \) maps. Then \( \limleftarrow F \) is homeomorphic to \( \limleftarrow G \) if and only if \( n \) and \( m \) have the same prime factors.

Both Ingram and Varagona have studied the function whose graph is pictured in Figure 15. In [2], Ingram proved that its inverse limit was an indecomposable continuum with two endpoints such that every proper subcontinuum was an arc, and he speculated that it would be homeomorphic to the inverse limit of the 3rd degree hat function. This is in fact true and can be viewed as an application of Theorem 4.16.

The fact that Ingram’s speculation concerning this function was correct is not original to this paper however. This was already shown by Varagona in [8]. In fact, it was Varagona’s results concerning such “N-shaped” functions in [8] along with his results concerning what he called “steeples” in [7] that made the author originally consider whether a theorem such as Theorem 4.16 might be true. In addition, the methods Varagona employed in obtaining his results (for instance the use of itineraries to characterize points in the inverse limit space) provided much of the inspiration for the methods used in this section.

**References**


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