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**Theorem 1.** *If $n$ is odd, then $n^2+19$ has at least 6 divisors.*

*Proof.* Let $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $n^2+19 = 4(k^2+k+5)$ has at least six factors 1, 2, 4, $\frac{n^2+19}{4}$, $\frac{n^2+19}{2}$, and $n^2+19$. To see that these factors are distinct, suppose $4 \geq \frac{n^2+19}{4}$. Then $n^2 \leq -3$, a contradiction. Hence we have $1 < 2 < 4 < \frac{n^2+19}{4} < \frac{n^2+19}{2} < n^2+19$. ∎

**Theorem 2.** *If $n$ is odd, then $n^2+119$ has at least 8 divisors.*

*Proof.* Let $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $n^2+119 = 4[k(k+1)+30]$. Moreover the quantity $k(k+1)+30$ is always even, say $2m$ for $m \in \mathbb{Z}$. Hence $n^2+119 = 8m$ and $n^2+119$ is divisible by at least eight numbers 1, 2, 4, 8, $\frac{n^2+119}{8}$, $\frac{n^2+119}{4}$, $\frac{n^2+119}{2}$, and $n^2+119$. Suppose $8 \geq \frac{n^2+119}{8}$. Then $n^2 \leq -55$, a contradiction. So we must have $8 < \frac{n^2+119}{8}$, hence the divisors are distinct. ∎

**Theorem 3.** *If $n$ is odd, then 11 is the smallest number $a$ such that $n^2+a$ has at least 6 divisors.*

*Proof.* Let $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $n^2+11 = 4(k^2+k+3)$. Hence $n^2+11$ is divisible by at least six numbers 1, 2, 4, $\frac{n^2+11}{4}$, $\frac{n^2+11}{2}$, and $n^2+11$. Suppose $2 \geq \frac{n^2+11}{4}$. Then $n^2 \leq -3$, a contradiction. Hence $1 < 2 < \frac{n^2+11}{4} < \frac{n^2+11}{2} < n^2+11$.

Now we show that 4 is not equal to any of these five numbers. Suppose $4 = \frac{n^2+11}{4}$. Then $n^2 = 5$, a contradiction. Hence $4 \neq \frac{n^2+11}{4}$. Now suppose $4 \geq \frac{n^2+11}{2}$. Then $n^2 \leq -3$, a contradiction. Hence $4 < \frac{n^2+11}{2}$, and 4 is a distinct sixth divisor.

Suppose there is an $a < 11$ such that $n^2+a$ has at least six divisors. When $n = 1$, then $1+a < 12$ has at least six divisors. But 12 is the smallest number with six divisors. Therefore 11 is the smallest number $a$ for which the claim holds. ∎

**Theorem 4.** *If $n$ is odd, then 47 is the smallest number $a$ such that $n^2+a$ has at least 8 divisors.*

*Proof.* Let $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $n^2+47 = 4[k(k+1)+12]$. Moreover the quantity $k(k+1)+12$ is always even, say $2m$ for $m \in \mathbb{Z}$. Hence $n^2+47 = 8m$ and $n^2+47$ is divisible by at least eight numbers 1, 2, 4, 8, $\frac{n^2+47}{8}$, $\frac{n^2+47}{4}$, $\frac{n^2+47}{2}$, and $n^2+47$. Suppose $4 \geq \frac{n^2+47}{8}$. Then $n^2 \leq -15$, a contradiction. So we must have $4 < \frac{n^2+47}{8}$, hence the divisors are distinct.

Now we show that 8 is not equal to any of these seven numbers. Suppose $8 = \frac{n^2+47}{8}$. Then $n^2 = 17$, a contradiction. So we must have $8 \neq \frac{n^2+47}{8}$. Now suppose $8 \geq \frac{n^2+47}{4}$. Then $n^2 \leq -31$, a contradiction. So $8 < \frac{n^2+47}{4}$, and 8 is a distinct eighth divisor.

Suppose there is an $a < 47$ such that $n^2+a$ has at least eight divisors. When $n = 1$, then $1+a < 48$ has at least eight divisors. There are five numbers below 48 with eight divisors: 24, 30, 36, 40, and 42. Hence $a$ can only be 23, 29, 35, 39, and 41. When $a = 23$, 29, 35, and 41, $3^2 + a$ has only six, four, six, and six divisors, respectively. When $a = 39$, $5^2 + a$ has only seven divisors. Therefore 47 is the smallest number $a$ for which the claim holds. ■