UNDERGRADUATE PROJECT: STABILITY PROPERTIES OF WATER WAVES

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Introduction

The general goal of the project is to study equations which describe the behavior of water waves. One of the most well-known equations describing water waves is the *Korteweg-de-Vries equation* (KdV). However, there are other water wave models and the project will deal with the KdV and another related equation: the Degasperis-Processi. These equations are differential equations, that is they are equations involving derivatives. The **solutions** to these equations represent behaviors for the waves.

Not all the solutions can be realized experimentally. As explained below, only **stable** solutions can be observed. The main goal of the project is to develop methods to study the stability properties of the solutions of the KdV and related equations.

The proposed research can be summarized as follows:

- Develop a method to study the stability of certain solutions in the context of water waves.
- Apply these methods to the case of KdV.
- Extend the study to models that include physical effects not taken into account by the KdV.

Stability

Stability is a fundamental concept in physics. An illustration of the concept is given by trying to make a pencil stand on its lead. In theory, it is possible but, in practice, because it is such an unstable state, it cannot be done. In the example just described, the study of stability is very simple and there is no need for a mathematical analysis to prove or disprove the stability of the system. The concept of stability carries over to solutions of differential equations such as the ones forming the Degasperis-Processi equation. In the context, the concept of stability takes an abstract form and its study often involves some sophisticated mathematical tools. However, such a study is fundamental because only stable solutions can be seen experimentally, i.e. seen in a real-life event. Unstable solutions, just like in the case of the pencil explained above, although exist in theory, will never be observed. Hence, with stability studies, one can distinguish the solutions that can be seen experimentally from the ones that cannot.

Integrability

Integrability is a property that characterizes systems for which one can make long term predictions. For example, the equations describing the motion of a satellite around a planet are integrable because one can use them to predict with accuracy where the satellite will be at any point in the future. Integrability can be seen as the complete opposite of what is called chaos. Chaos characterizes the systems which are extra sensitive to initial conditions, and thus long term predictions are practically impossible. The *butterfly effect* is a popular

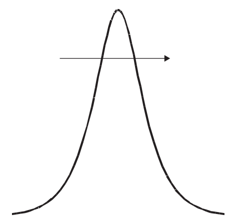

FIGURE 1

notion which illustrates what chaos means in the context of the weather system. It can be stated as follows: a butterfly's wings in Brazil might create tiny changes in the atmosphere that ultimately cause a tornado to appear in Texas.

The great advantage of studying the integrable systems, is that, due to their high degree of regularity, it is usually possible to perform a mathematical analysis to fully understand their behavior. However, integrable systems are usually idealized and it is often necessary to alter an integrable equation in order to be able to take into account all the effects encountered in a given experiment. In the case of a satellite orbiting around a planet for example, the effect that would break integrability can take the form of the friction between the satellite and the atmosphere. The study of the idealized systems is nevertheless still very important in the context of applications. The main reason is that the study of the idealized integrable models gives a lot of insight on the behavior of the more realistic and complex models.

The models the project is dealing with take the form of differential equations. In the context, the immediate consequence of integrability is the presence of *soliton solutions* which we describe below.

Soliton Solutions

Soliton solutions (also called solitary wave solutions) were first observed as water waves. In the context, they take the form of a single wave that propagates at a certain speed. What distinguishes them from other common waves is that they occur as one solitary wave that is very persistent and, unlike other waves that dissipate rapidly, they can be seen traveling on large distances. The first sighting of a soliton was made by J.S. Russell in 1834 who followed for miles the strangely persistent solitary wave that had formed in a canal in Scotland [3]. Ther observation gave rise, several years later, to the mathematical field of integrability in which the concept of soliton plays a major role. Another example of a soliton is given by the tsunamis that hit Japan last year. Solitons are now known to appear in a wide variety of contexts. A simple sketch of a soliton solution is illustrated in Figure 1 which a single wave propagating to the right.

Objectives of the project

As mentioned before, the first equation to be studied is the KdV equation which takes the form

$$u_t + u_{xxx} + uu_x = 0.$$

The KdV is integrable and as such is an idealized system which models water waves under very particular conditions (for example, the height of the wave it describes must be large in comparison with the depth of the water). The KdV is a well-known and well-studied system and thus nothing new will be obtained. However, studying it and reproducing known results will give an undergraduate a chance to learn the basic tools needed to work on other more complex and much less well-known cases.

There are other models which are in some sense generalizations of the KdV. While they are not as mathematically nice and are more challenging to study, they do incorporate physical effects that the KdV ignores. In other words, they offer a more reliable description of certain physical phenomena. For the project, I have one particular case in mind, it is the Degasperis-Processi equation. It does admit soliton-type solutions and several of its solutions have not been studied from the point of view of stability [5].

The main goal of the project is to develop mathematical methods to study solitons and their stability properties in the context of the equations mentioned above. In order to study the stability of a solution, it is necessary to perform a linearization of the equation about the solution. This gives rise to the equality

$$\mathcal{L}\, w = \lambda\, w,$$

where $\mathcal{L}$ is a matrix linear differential operator and λ and w are, respectively, the eigenvalue and eigenvector associated to the problem. The set of values the eigenvalue λ can take tells us if a solution is stable or not. In the case where the solution is a soliton, the mathematical objects that are the *Evans function* [4] and the *Lax pair* can be used to find information about the eigenvalue and thus about stability (there is a great amount of literature on the subject; see [6] for a relatively recent review).

The KdV is integrable and thus, as mentioned before, it is possible to use mathematics to understand it quite well. As a consequence, the stability of large classes of its solutions are well-understood from the point of view of their stability [1, 2]. However the equations to be studied (which is of Degasperis-Processi -type) have solutions for which the stability properties are not known [5]. We plan to use the KdV as a way to learn the tools to be used. The original results will be obtained for the Degasperis-Processi.

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