Integers love (i) being ordered, (ii) induction, and (iii) modular arithmetic.

**Modular arithmetic:** We say “a is congruent to b mod n” to mean that a and b have the same remainder when divided by n. We write this as $a \equiv b \pmod{n}$.

- E.g., $18 \equiv 2 \pmod{4}$ and $13 \equiv 7 \pmod{6}$.
- If $a \equiv b \pmod{n}$ then $a + c \equiv b + c \pmod{n}$ and also $ac \equiv bc \pmod{n}$. You can treat “mod” like equals, but just be super careful not to divide!
- **Fermat’s little theorem:** If $p$ is a prime number, then $a^p \equiv a \pmod{p}$.
- If $p$ is a prime, and $n$ a positive integer, then the exponent of $p$ in $n!$ is $$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots,$$
  where $\lfloor x \rfloor$ (the “floor of $x$”) is $x$ rounded down.

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1. Show $10x^3 + 4x + 3 = 16y$ has no integer solutions.
2. Show every odd square is 1 more than a multiple of 8.
3. What is the remainder of $10^{50}$ when you divide by 9?
4. Are there two powers of 2 with the same number of digits so that when you rearrange the digits of one, you can obtain the other?
5. Prove no polynomial with integer coefficients satisfies $p(7) = 5$ and $p(15) = 9$.
6. Find all positive integers $n$ such that $n!$ ends in exactly 1000 zeros.
7. Is there a polynomial $f(x)$ with integer coefficients such that $f(n)$ is prime for every integer $n$?
8. Prove that in the product $P = 1! \cdot 2! \cdot 3! \cdots 100!$, one of the factors can be erased so that the remaining product is a perfect square.
9. Prove that if $n \geq 3$ prime numbers form an arithmetic progression, then the common difference of the progression is divisible by every prime less than $n$.
10. Prove that if $1 < n$, then $n$ doesn’t divide $2^n - 1$.

*Some material taken from *Putnam and Beyond* by Razvan Gelca and Titu Andreescu and also *Mathematical Puzzles: A Connoisseur’s Collection* by Peter Winkler.*
11. The Fibonacci numbers are defined as \( F_1 = F_2 = 1 \) and for all \( n \geq 1, F_n + F_{n+1} = F_{n+2} \). Find a Fibonacci number that is divisible by 20.

12. Prove \( x^2 = y^3 + 7 \) has no integer solutions.

13. Prove that among any three distinct integers we can find two, say \( a \) and \( b \), such that the number \( a^3b - ab^3 \) is a multiple of 10.

14. Show that for any positive integers \( a \) and \( b \), the product \((36a + b)(a + 36b)\) is never a power of 2. (Asia-Pacific math olympiad 1998)

15. For any prime number \( p > 17 \), show that \( p^{32} - 1 \) is divisible by 16320.

16. Find the integers \( n \) for which \((n^3 - 3n^2 + 4)/(2n - 1)\) is an integer.

17. Prove that the expression

\[ \frac{\gcd(m, n)}{n} \binom{n}{m} \]

is an integer for all \( n \geq m \geq 1 \).

18. Find all functions \( f : \mathbb{N} \to \mathbb{N} \) satisfying

\[ f(n) + 2f(f(n)) = 3n + 5, \quad \text{for all } n \in \mathbb{N}. \]

19. Suppose \( p > 5 \) is a prime and that \( n \) has exactly \( p - 1 \) of its digits equal to 1. Prove that \( n \) is divisible by \( p \).

20. Prove that the sequence \( 2^n - 3 \) has an infinite subsequence whose terms are all pairwise relatively prime.

21. Let \( n > 1 \) be a positive integer. Prove that \((x+1)^n - x^n = ny \) has no positive integer solutions.

22. Let \( k \) and \( n \) be integers with \( 0 \leq k \leq n^2/4 \). Assume \( k \) has no prime divisor greater than \( n \). Prove that \( n! \) is divisible by \( k \).

23. Show that each positive integer can be written as the difference of two positive integers having the same number of prime factors.

24. Let \( n > 2 \). Show that \( n(n-1)^4 + 1 \) isn’t prime.

25. Find all positive integer solutions to \( 2^x \cdot 3^y = 1 + 5^z \).

26. Let \( x_n \) be some sequence satisfying the recurrence \( x_{n+1} = 5x_n - 6x_{n-1} \). Prove infinitely many terms of the sequence are composite.

27. Prove that if \( n \geq 3 \) prime numbers form an arithmetic progression, then the common difference of the progression is divisible by every prime less than \( n \).

28. Find all prime numbers \( p \) such that when divided by every prime number \( q < p \), the remainder is always square-free.\(^1\)

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\(^1\)An integer is square-free iff 1 is the only square dividing it.