Fancier Stuff with $\binom{n}{k}$

**Example 1** Suppose we have $a$ indistinguishable sticks that each look like $|$, and we have $b$ indistinguishable stones that each look like $\ast$. How many ways are there for us to arrange all of these objects in a row? (e.g., something like $||\ast|\ast\ast|\ast|\ast$)

Sometimes, we can convert one problem into another by cooking up a *bijection*!

**Example 2** Suppose we have 5 different pirates and $b$ indistinguishable pieces of gold that each look like $\ast$. How many ways are there for us to divvy up all the gold among these pirates?
Example 3  How many ways can we put \( b \) indistinguishable objects into \( k \) distinguishable boxes?

Answer is \( \binom{k + b - 1}{b} \).

Example 4  The pirates couldn’t figure out what to do with the gold, so they bring it back to their pirate queen. There are now a total of 6 pirates (5 regular pirates, and one pirate queen).

(a) How many ways are there to divide 20 pieces of gold among the 6 pirates? (Assuming they won’t throw any of the gold away)

(b) How many ways are there to divide 20 total pieces of gold among the 6 pirates if the pirate queen demands everyone gets at least 1 piece of gold? [Which is a good strategy to avoid any mutinies]

(c) How many ways are there to divide 20 total pieces of gold among the 6 pirates if the pirate queen demands she gets at least 4 pieces of gold and everyone else must get at least one piece of gold?

Example 5  Three students find a free pile of 7 pencils (which are all indistinguishable). They consider taking some.

(a) If they decide they will collectively take all 7 pencils, how many ways are there for them to do this?

(b) The students decide they want to leave at least one pencil behind (taking the last one is rude). How many total ways are there for them to divide up the pencils such that they leave at least one behind? (e.g., perhaps they take none, perhaps one student grabs 4 and nobody else takes any, ...)
Some More Cool Stuff with \( \binom{n}{k} \)

**Example 6** Here’s a cool fact: \( \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k} \).

*Proof:*

**Example 7** Now time for some “totally unrelated” algebra... Please fill in the blanks with the correct numbers

\[
\begin{align*}
(x + y)^0 &= - \\
(x + y)^1 &= - x^1 + y^1 \\
(x + y)^2 &= - x^2 + x^1 y^1 + y^2 \\
(x + y)^3 &= - x^3 + x^2 y^1 + x^1 y^2 + y^3 \\
(x + y)^4 &= - x^4 + x^3 y^1 + x^2 y^2 + x^1 y^3 + y^4
\end{align*}
\]
If you write down your answers for the numbers involved in these “binomial expansions” in a sort of table, you get this triangle of stuff!

\[
\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & 1 & & 1 \\
 & & & 1 & & 2 & & 1 \\
 & & 1 & & 3 & & 3 & & 1 \\
 & 1 & & 4 & & 6 & & 4 & & 1 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\end{array}
\]

Figure 1: The first few rows of “Pascal’s Triangle”

It’s called “Pascal’s triangle” named after the seventeenth French century mathematician Blaise Pascal, but it was actually known to other parts of the world for centuries before this!

Figure 2: Depictions of this triangle from modern-day India (left) and modern-day China (right) each predating Pascal by hundreds of years

**Question 8** What are some things you notice about this triangle? (There are LOTS of cool patterns)
The numbers in the $n$th row of Pascal’s triangle are the coefficients in the expansion of the binomial $(x + y)^n$

- Each row of Pascal’s triangle starts and ends with the number 1
- Each entry of the triangle is the sum of the two entries above it

In fact, we have the following big theorem!

**Theorem 9 (The Binomial Theorem)** Let $n \geq 0$ be any integer. Then we have

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n} x^n y^0.$$

In other words, the number in row $n$ column $k$ of Pascal’s triangle is $\binom{n}{k}$. How cool! This is why the numbers $\binom{n}{k}$ are called the **binomial coefficients**.

**Proof of the binomial theorem:**
Example 10 Use Pascal’s triangle to expand \((x + y)^7\).

Example 11 Use the binomial theorem to simplify \(\sum_{k=0}^{n} \binom{n}{k}\), the sum of the \(n^{th}\) row of Pascal’s triangle.

Open question: Other than the number 1, are there any numbers that appear in Pascal’s triangle more than 10 times?

Surprisingly, nobody has been able to figure out the answer to this yet! Maybe you can help!1

---

1You can google “Singmaster’s conjecture” to learn more.