Homework 1.

(1) (a) Let $X = \xi_1 + \cdots + \xi_n$, where ξ_i are independent indicator random variables which equal 1 with probability 1/2. What is $\mathbf{P}(X = k)$. Compare $\mathbf{P}(X = k)$ and $\mathbf{P}(X = m)$ for all pairs of numbers (k, m). Which value of k maximizes $\mathbf{P}(X = k)$? Compute the value of $\mathbf{P}(|X - n| \le 2\sqrt{n})$ approximately (assuming that n is large).

(b) What are the answers if we replace 1/2 by a constant $0 < p < 1$? (again assuming that n is very large and does not depend on p).

Hint: Recall the central limit theorem.

(2) Prove that the function $\binom{y}{n}$ is convex. Read the proof in page 7, Chapter 1.

(3) Let S be a set of n elements (n even). Choose a random subset A of S by selecting each element with probability 1/2. Repeat the process to choose another random subset B. What is the expectation and variance of $|A \cap B|$? What happens if each element of B is chosen with probability 1/3 ? What are the expectation and variance of $|A \cup B|$ in each case ?

(4) Problem 1, page 10.

(5) Problem 2, page 10.

(6) Problem 4, page 11.

Homework 2.

(1) Let σ be a random permutation from S_n. Let X be the number of fixed points of σ.

(a) Compute the expectation and variance of X.

(b) Prove that probability that $X = 0$ tends to $1/e$ as n tends to infinity.

(2) Compute the probability that X contains a cycle of length more than $n/2$ (assuming that n is even).

(3) Problem 7, page 21.

(4) Problem 2, page 21.

(5) Problem 9, page 21.

Homework 3.

(1) Prove, without using the prime number theorem, that $\pi(n) = \Theta(n/\log n)$, where $\pi(n)$ is the number of primes between 1 and n.

(2) Let X be the number of K_4 in $G(n,p)$. Compute the variance of X.

(3) Problem (2), page 58.

(4) Problem (4), page 59.

(5) Problem (5), page 59.

Homework 4. Due Wednesday the week after the break

(1) Prove the central limit theorem for the number of H in $G(n,p)$, where H is a fixed balanced graph, and p is a constant. Find out if the proof works for $p = n^{-c}$ for some constant $c > 0$. What the the largest c can you deduce ?

(2) Find the smallest unbalanced graph. Call this H. What is the threshold function for the appearance of H in $G(n,p)$?

(3) Section 5.8, exercise 3.

(4) Section 5.8, exercise 4.

(5) Read the proof of decomposable covering in 5.4. Prove the following statement: A set of n balls in R^3 partition R^3 in at most $O(n^3)$ connected components. What happens in R^d for a general d (we think of d fixed and n large). What happens if we replace ball by planes in R^3 (and hyperplanes in R^d) ?

Homework 5. Due last week of the semester

(1) Prove that if A is increasing and B is decreasing, then A and B are negatively correlated. Derive a corollary for the intersection of two family of subsets.

(2) Consider $G(n,1/2)$ and let A_n be the event that there is at least $\frac{1}{16}\binom{n}{3}$ triangles, and B_n be the event that there is at least $\frac{1}{128}\binom{n}{4}$ K_4. Prove that $\limsup \mathbf{P}(A_n \cap B_n) \geq \frac{1}{4}$.

(3) Derive a concentration result for the number of K_4 in $G(n,1/2)$.

(4) Let M_n be an $n \times n$ matrix whose entries are random ± 1 (with probability 1/2). Prove that for any fixed unit vector v, $\|M_n v\| \leq 10\sqrt{n}$ with probability $1 - o(1)$.

(5) The sum of $n+1$ binomial coefficients is 2^n. Show that the sum of the middle $100\sqrt{n}$coefficients is already at least $.99 \times 2^n$.