Competitive Experimentation and Private Information

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Derivations for the Gamma-Exponential Model

Derivation of expected hazard rates. By Bayes rule, posterior beliefs are

$$\begin{aligned}
\pi_{t,t'}(\lambda|x,y) &= \pi\left(\lambda|z_i=x,z_j=y,t_i\geq t,t_j\geq t'\right)\\
&= \frac{\pi\left(\lambda\right)h\left(x|\lambda\right)h\left(y|\lambda\right)\left[1-F_i\left(t|\lambda\right)\right]\left[1-F_j\left(t'|\lambda\right)\right]}{\int_\Lambda \pi\left(\lambda'\right)h\left(x|\lambda'\right)h\left(y|\lambda'\right)\left[1-F_i\left(t|\lambda'\right)\right]\left[1-F_j\left(t'|\lambda'\right)\right]d\lambda'}\\
&= \frac{e^{-\alpha\lambda}\lambda^{\beta-1}\lambda e^{\lambda x}\lambda e^{\lambda y}e^{-(\zeta_i+\lambda)t}e^{-(\zeta_j+\lambda)t'}}{\int_\Lambda e^{-\alpha\lambda'}\lambda'^{\beta-1}\lambda' e^{\lambda' x}\lambda' e^{\lambda' y}e^{-(\zeta_i+\lambda)t}e^{-\lambda(\zeta_j+\lambda)t'}d\lambda'} = \frac{e^{-\lambda(\alpha-x-y+t+t')}\lambda^{\beta+1}}{\Gamma\left(\beta+2\right)\left(\alpha-x-y+t+t'\right)^{-\beta-2}},
\end{aligned}$$

$$\begin{aligned}
\pi_{t,t'}(\lambda|x,y+) &= \pi\left(\lambda|z_i=x,z_j\geq y,t_i\geq t,t_j\geq t'\right)\\
&= \frac{\pi\left(\lambda\right)h\left(x|\lambda\right)\left[1-H(y|\lambda)\right]\left[1-F_i\left(t|\lambda\right)\right]\left[1-F_j\left(t'|\lambda\right)\right]}{\int_\Lambda \pi\left(\lambda'\right)h\left(x|\lambda'\right)\left[1-H(y|\lambda')\right]\left[1-F_i\left(t|\lambda'\right)\right]\left[1-F_j\left(t'|\lambda'\right)\right]d\lambda'}\\
&= \frac{\lambda^\beta\left[e^{-\lambda(\alpha+t+t'-x)}-e^{-\lambda(\alpha+t+t'-x-y)}\right]}{\Gamma\left(\beta+1\right)\left[\left(\alpha+t+t'-x\right)^{-\beta-1}-\left(\alpha+t+t'-x-y\right)^{-\beta-1}\right]}.
\end{aligned}$$

Taking expectations of the hazard rate λ with respect to these three posteriors yields the following expressions:

$$\begin{aligned}
\mathbb{E}^i_{t,t'}\left[\lambda|\,x,y\right] &= \frac{\beta+2}{\alpha-x-y+t+t'},\\
\mathbb{E}^i_{t,t'}\left[\lambda|\,x,y-\right] &= \frac{\beta+1}{\alpha-x-y+t+t'},\\
\mathbb{E}^i_{t,t'}\left[\lambda|\,x,y+\right] &= \left(\beta+1\right)\frac{\left(\alpha+t+t'-x\right)^{-\beta-2}-\left(\alpha+t+t'-x-y\right)^{-\beta-2}}{\left(\alpha+t+t'-x\right)^{-\beta-1}-\left(\alpha+t+t'-x-y\right)^{-\beta-1}}.
\end{aligned}$$

Lemma A.3 *In the Gamma-exponential model, $E^i_{t,t}\left[\zeta_i+\lambda|\,x,y+\right]$ satisfies Assumptions 1 and 2.*

Proof. We use the following technical result.

Theorem A.1 *Let $q : \Lambda \times \Re^n \to \Re_+$, integrable in its first argument and differentiable in the remaining n arguments, with $\int_\Lambda q(\lambda,\vec{\theta})d\lambda \in (0,\infty)$. Then the c.d.f. defined by:*

$$\varphi(L,\vec{\theta}) \equiv \frac{\int_{\underline{\lambda}}^{L} q(\lambda,\vec{\theta})d\lambda}{\int_\Lambda q\left(\lambda',\vec{\theta}\right)d\lambda'}$$

for every $L \in \left(\underline{\lambda},\bar{\lambda}\right)$ is stochastically strictly increasing in every component of $\vec{\theta}$ if $q(\lambda,\vec{\theta})$ is log-supermodular in (λ,θ_i), i.e. if $\partial \log q(\lambda,\vec{\theta})/\partial\theta_i$ is strictly increasing in λ.

Proof We want to show that for every $L \in \left(\underline{\lambda},\bar{\lambda}\right)$

$$0 > \frac{\partial}{\partial\theta_i}\frac{\int_{\underline{\lambda}}^{L} q(\lambda,\vec{\theta})d\lambda}{\int_\Lambda q(\lambda,\vec{\theta})d\lambda} = \frac{\int_{\underline{\lambda}}^{L}\frac{\partial q(\lambda,\vec{\theta})}{\partial\theta_i}d\lambda}{\int_\Lambda q(\lambda,\vec{\theta})d\lambda} - \frac{\int_{\underline{\lambda}}^{L} q(\lambda,\vec{\theta})d\lambda\int_\Lambda\frac{\partial q(\lambda,\vec{\theta})}{\partial\theta_i}d\lambda}{\left[\int_\Lambda q(\lambda,\vec{\theta})d\lambda\right]^2}$$

using

$$\int\frac{\partial q(\lambda,\vec{\theta})}{\partial\theta_i}d\lambda = \int\frac{\partial\log q(\lambda,\vec{\theta})}{\partial\theta_i}q(\lambda,\vec{\theta})d\lambda$$

the claim reads

$$\int_\Lambda\frac{\partial\log q(\lambda,\vec{\theta})}{\partial\theta_i}\frac{q(\lambda,\vec{\theta})}{\int_\Lambda q\left(\lambda',\vec{\theta}\right)d\lambda'}d\lambda > \int_{\underline{\lambda}}^{L}\frac{\partial\log q(\lambda,\vec{\theta})}{\partial\theta_i}\frac{q(\lambda,\vec{\theta})}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'}d\lambda$$

A sufficient condition for the latter inequality is that the RHS be strictly increasing in L. Since the RHS is differentiable in L, it suffices that

$$\begin{aligned}
0 &< \frac{\partial}{\partial L}\left[\int_{\underline{\lambda}}^{L}\frac{\partial\log q(\lambda,\vec{\theta})}{\partial\theta_i}\frac{q(\lambda,\vec{\theta})}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'}d\lambda\right] \\
&= \frac{\partial\log q(L,\vec{\theta})}{\partial\theta_i}\frac{q(L,\vec{\theta})}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'} - q(L,\vec{\theta})\int_{\underline{\lambda}}^{L}\frac{\partial\log q(\lambda,\vec{\theta})}{\partial\theta_i}\frac{q(\lambda,\vec{\theta})}{\left[\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'\right]^2}d\lambda
\end{aligned}$$

$$\text{or } \frac{\partial\log q(L,\vec{\theta})}{\partial\theta_i} > \int_{\underline{\lambda}}^{L}\frac{\partial\log q(\lambda,\vec{\theta})}{\partial\theta_i}\frac{q(\lambda,\vec{\theta})}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'}d\lambda.$$

But, since $\int_{\underline{\lambda}}^{L}\frac{q(\lambda,\vec{\theta})}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'}d\lambda = 1$, this follows from the assumption that the LHS is strictly increasing in λ. ■

It now suffices to show that for every x,y,t,t', the c.d.f. associated with the posterior beliefs $\pi_{t,t'}(\lambda|x,y+)$ are stochastically strictly decreasing in t and in t' and strictly increasing in x and in y. We prove all these results as corollaries of Theorem A1. Let $\vec{\theta} = (x,y,t,t')$,

$$q(\lambda,\vec{\theta}) = \pi(\lambda)\,h(x|\lambda)\left[1-H(y|\lambda)\right]\left[1-F(t|\lambda)\right]\left[1-F(t'|\lambda)\right],$$

$$\varphi(L,\vec{\theta}) = \int_{\underline{\lambda}}^{L} \pi_{t,t'}\left(\lambda|x,y+\right) d\lambda.$$

Since the expressions

$$\frac{\partial \log q(\lambda,\vec{\theta})}{\partial t} = -\frac{f\left(t|\lambda\right)}{1-F\left(t|\lambda\right)} = -\lambda, \quad \frac{\partial \log q(\lambda,\vec{\theta})}{\partial t'} = -\frac{f\left(t'|\lambda\right)}{1-F\left(t'|\lambda\right)} = -\lambda$$

$$\frac{\partial \log q(\lambda,\vec{\theta})}{\partial x} = \frac{h'\left(x|\lambda\right)}{h\left(x|\lambda\right)} = \lambda, \quad \frac{\partial \log q(\lambda,\vec{\theta})}{\partial y} = -\frac{h\left(y|\lambda\right)}{1-H\left(y|\lambda\right)} = \frac{\lambda e^{\lambda x}}{1-e^{\lambda x}}$$

are strictly monotonic in λ, all monotonicity results follow from Theorem A1. The limit $\lim_{t\to\infty} \mathbb{E}_{t,t'}\left[\lambda|\, x,y+\right] = 0$ follows from:

$$\lim_{\tau\to\infty} \pi_{\tau,\tau}\left(\lambda > \varepsilon|x,y+\right) = \lim_{\tau\to\infty} \frac{\int_{\varepsilon}^{\infty} e^{-\alpha\lambda}\lambda^{\beta-1}\lambda e^{\lambda x}\left(1-e^{\lambda y}\right)e^{-\lambda 2\tau}d\lambda}{\int_{0}^{\infty} e^{-\alpha\lambda'}\lambda'^{\beta-1}\lambda' e^{\lambda' x}\left(1-e^{\lambda' y}\right)e^{-\lambda' 2\tau}d\lambda'} = 0. \blacksquare$$

Lemma A.4 *In the Gamma-exponential model,* $E_{t,t}^{i}\left[\zeta_i+\lambda|\, x,y+\right]$ *satisfies Assumption 3.*

Proof. It is immediate to verify that $\frac{d}{dt}\mathbb{E}_{t,t'}^{i}\left[\zeta_i+\lambda|\, x,y+\right] = -2\frac{d}{dx}\mathbb{E}_{t,t'}^{i}\left[\zeta_i+\lambda|\, x,y+\right]$, as it can be appreciated with the change of variable $q=\alpha+2t-x$, and

$$\begin{aligned}
\phi\left(q,y,\beta\right) &= \left(\beta+1\right)\frac{q^{-\beta-2}-\left(q-y\right)^{-\beta-2}}{q^{-\beta-1}-\left(q-y\right)^{-\beta-1}} \\
\frac{d}{dt}\mathbb{E}_{t,t'}^{i}\left[\zeta_i+\lambda|\, x,y+\right] &= \frac{\partial\phi}{\partial q}\frac{\partial q}{\partial t} = 2\frac{\partial\phi}{\partial q}, \\
\frac{d}{dx}\mathbb{E}_{t,t'}^{i}\left[\zeta_i+\lambda|\, x,y+\right] &= \frac{\partial\phi}{\partial q}\frac{\partial q}{\partial x} = -\frac{\partial\phi}{\partial q}
\end{aligned}$$

Hence, setting $\overline{G}$ just a bit larger than 2 and thus $\overline{G}' < -1/2$ satisfies the inequality in Assumption 3. For the first inequality, we show that it is strict at $\overline{G}' = -1/2$ and therefore, by continuity, for $\overline{G}'$ slightly larger:

$$2\frac{d}{dy}\mathbb{E}_{t,t'}^{i}\left[\zeta_i+\lambda|\, x,y+\right] < -\frac{d}{dt}\mathbb{E}_{t,t'}^{i}\left[\zeta_i+\lambda|\, x,y+\right] = 2\frac{d}{dx}\mathbb{E}_{t,t'}^{i}\left[\zeta_i+\lambda|\, x,y+\right].$$

Algebra yields

$$\frac{d}{dx}\mathbb{E}_{t,t'}^{i}\left[\zeta_i+\lambda|\, x,y+\right] = \left(\beta+1\right)\frac{\left[\begin{array}{c}\left(\beta+2\right)\left[\left(\alpha+2t-x\right)^{-\beta-3}-\left(\alpha+2t-x-y\right)^{-\beta-3}\right]\cdot \\ \cdot\left[\left(\alpha+2t-x\right)^{-\beta-1}-\left(\alpha+2t-x-y\right)^{-\beta-1}\right] \\ -\left(\beta+1\right)\left[-\left(\alpha+2t-x\right)^{-\beta-2}+\left(\alpha+2t-x-y\right)^{-\beta-2}\right]^2\end{array}\right]}{\left[\left(\alpha+2t-x\right)^{-\beta-1}-\left(\alpha+2t-x-y\right)^{-\beta-1}\right]^2}$$

$$\frac{d}{dy}\mathbb{E}^i_{t,t'}\left[\zeta_i+\lambda|\, x,y+\right]$$

$$= (\beta+1)\frac{\begin{bmatrix} -(\beta+2)(\alpha+2t-x-y)^{-\beta-3}\left[(\alpha+2t-x)^{-\beta-1}-(\alpha+2t-x-y)^{-\beta-1}\right] \\ +(\beta+1)(\alpha+2t-x-y)^{-\beta-2}\left[(\alpha+2t-x)^{-\beta-2}-(\alpha+2t-x-y)^{-\beta-2}\right]\end{bmatrix}}{\left[(\alpha+2t-x)^{-\beta-1}-(\alpha+2t-x-y)^{-\beta-1}\right]^2}.$$

Hence,

$$-\frac{d}{dx}\mathbb{E}^i_{t,t'}\left[\zeta_i+\lambda|\, x,y+\right]+\frac{d}{dy}\mathbb{E}^i_{t,t'}\left[\zeta_i+\lambda|\, x,y+\right]$$

$$= -(\beta+1)\frac{\begin{bmatrix} (\beta+2)\left[(\alpha+2t-x)^{-\beta-3}\right]\left[(\alpha+2t-x)^{-\beta-1}-(\alpha+2t-x-y)^{-\beta-1}\right] \\ -(\beta+1)\left[(\alpha+2t-x)^{-\beta-2}\right]\left[(\alpha+2t-x)^{-\beta-2}-(\alpha+2t-x-y)^{-\beta-2}\right]\end{bmatrix}}{\left[(\alpha+2t-x)^{-\beta-1}-(\alpha+2t-x-y)^{-\beta-1}\right]^2}$$

$$\propto (\beta+1)\frac{(\alpha+t+t'-x)^{-\beta-2}}{(\alpha+t+t'-x)^{-\beta-1}-(\alpha+t+t'-x-y)^{-\beta-1}}$$

$$-(\beta+2)\frac{(\alpha+t+t'-x)^{-\beta-3}}{(\alpha+t+t'-x)^{-\beta-2}-(\alpha+t+t'-x-y)^{-\beta-2}}$$

$$= (\beta+1)\frac{q^{-\beta-2}}{q^{-\beta-1}-(q-y)^{-\beta-1}}-(\beta+2)\frac{q^{-\beta-3}}{q^{-\beta-2}-(q-y)^{-\beta-2}}$$

We want this quantity to be strictly negative. This is so if the following derivative is strictly positive for all β, y:

$$\frac{d}{d\beta}\left(\beta\frac{q^{-\beta-1}}{q^{-\beta}-(q-y)^{-\beta}}\right)$$

$$= \frac{q^{-\beta-1}}{q^{-\beta}-(q-y)^{-\beta}}+\beta\frac{-q^{-\beta-1}\left[q^{-\beta}-(q-y)^{-\beta}\right]\ln q+\left[q^{-\beta}\ln q-(q-y)^{-\beta}\ln(q-y)\right]q^{-\beta-1}}{\left[q^{-\beta}-(q-y)^{-\beta}\right]^2}$$

$$\propto q^{-\beta}-(q-y)^{-\beta}+\beta\left[-\left[q^{-\beta}-(q-y)^{-\beta}\right]\ln q+\left[q^{-\beta}\ln q-(q-y)^{-\beta}\ln(q-y)\right]\right]$$

$$= q^{-\beta}-(q-y)^{-\beta}+(q-y)^{-\beta}\beta\left(\ln q-\ln(q-y)\right)$$

$$: = \chi(\beta,y)$$

Note that for $y<0$

$$\frac{d\chi(\beta,y)}{dy}=\beta^2\frac{\ln q-\ln(q-y)}{(q-y)^{\beta+1}}-\frac{\beta}{(y-q)(q-y)^{\beta}}-\frac{\beta}{(q-y)^{\beta+1}}=-\frac{(\ln(q-y)-\ln q)\beta^2}{(q-y)^{\beta+1}}<0,$$

while $\chi(\beta,0) = q^{-\beta} - q^{-\beta} + q^{-\beta}\beta(\ln q - \ln(q)) = 0$, $\chi(\beta,-\infty) = q^{-\beta} - \frac{1+\beta(\ln(q-y)-\ln q)}{(q-y)^{\beta}} = q^{-\beta} + \frac{\beta\frac{1}{q-y}}{(q-y)^{\beta}} = q^{-\beta} > 0$. So indeed $\chi(\beta,0) > 0$ for all β, y. ■

Lemma A.5 *In the Gamma-exponential model, there exists a unique monotonic differentiable equilibrium.*

Proof. Proceeding by contradiction, suppose that for some x, y there exist two optimal stopping times as first quitter. We can rephrase this as follows: for some τ_1, there exist signals $(x,y) \neq (x',y')$ such that $q_A = \mathbb{E}_{\tau_1,\tau_1}[\lambda|\, x, y+]$, $q_B = \mathbb{E}_{\tau_1,\tau_1}[\lambda|\, y, x+]$, $q_A = \mathbb{E}_{\tau_1,\tau_1}[\lambda|\, x', y'+]$, $q_B = \mathbb{E}_{\tau_1,\tau_1}[\lambda|\, y', x'+]$.

Note that we can take $(x,y) < 0$, and $(x',y') < 0$ by continuity of $dV^i_{1,t}(\tau_1|x)/d\tau_1\big|_{t=\tau_1}$ and $dV^i_{1,t}(\tau_1|y)/d\tau_1\big|_{t=\tau_1}$. Simple manipulations of Eq. (6) then yield:

$$\begin{aligned} & q_A^{-1}(\beta+1)(\alpha+2\tau_1-x-y))^{-(\beta+2)} - (\alpha+2\tau_1-x-y))^{-(\beta+1)} = \\ = \; & q_A^{-1}(\beta+1)(\alpha+2\tau_1-x)^{-(\beta+2)} - (\alpha+2\tau_1-x)^{-(\beta+1)} \end{aligned}$$

$$\begin{aligned} & q_B^{-1}(\beta+1)(\alpha+2\tau_1-x-y))^{-(\beta+2)} - (\alpha+2\tau_1-x-y))^{-(\beta+1)} = \\ = \; & q_B^{-1}(\beta+1)(\alpha+2\tau_1-y)^{-(\beta+2)} - (\alpha+2\tau_1-y)^{-(\beta+1)}. \end{aligned}$$

Consider the functions

$$\varphi_i(\xi) = q_i^{-1}(\beta+1)(\alpha+2\tau-\xi)^{-(\beta+2)} - (\alpha+2\tau-\xi)^{-(\beta+1)}$$

for $i = A, B$. Note that

$$\varphi_i'(\xi) \propto q_i^{-1}(\beta+2) + \xi - \alpha - 2\tau,$$

is continuous in ξ, has at most one zero, $\xi_0 = \alpha + 2\tau - q_i(\beta+2)$, and has the same sign as $\xi - \xi_0$. So, suppose that $x' < x$. The conditions $\varphi_A(x) = \varphi_A(x+y)$ and $\varphi_A(x') = \varphi_A(x'+y')$ require $x'+y' > x+y$, and hence $y' > y$. But then, the condition $\varphi_B(x) = \varphi_B(x+y)$ and $\varphi_B(x') = \varphi_B(x'+y')$ require that $x'+y' < x+y$, and we have a contradiction. A similar contradiction is obtained when supposing that $x' > x$. ■

The Binary Prior Model

Suppose that the prize arrives according to an exponential process of intensity λ, i.e. with $F_i(t_i|\lambda) = 1 - e^{-\lambda t_i}$. The prior is concentrated on two points:

$$p_0 = \Pr(\lambda_1) = 1 - \Pr(\lambda_0)$$

where $\lambda_1 > \lambda_0 > 0$. Conditional on λ, the private signal is distributed as $\Pr(z_i \leq z|\lambda) = e^{\lambda z}$ for $z \leq 0$. Therefore the posterior belief of λ_1 is

$$\begin{aligned} \pi_{t,t'}(x,y) &= \frac{p_0 e^{-\lambda_1 t} e^{-\lambda_1 t'} \lambda_1 e^{\lambda_1 x} \lambda_1 e^{\lambda_1 y}}{(1-p_0) e^{-\lambda_0 t} e^{-\lambda_0 t'} \lambda_1 e^{\lambda_0 x} \lambda_0 e^{\lambda_0 y} + p_0 e^{-\lambda_1 t} e^{-\lambda_1 t'} \lambda_1 e^{\lambda_1 x} \lambda_1 e^{\lambda_1 y}} \\ &= [1 + q(p_0, \lambda_0, \lambda_1, t, t', x, y)]^{-1} \end{aligned}$$

where

$$q\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)=\frac{1-p_0}{p_0}e^{(\lambda_1-\lambda_0)(t+t'-x-y)}\left(\frac{\lambda_0}{\lambda_1}\right)^2$$

So

$$\mathbb{E}^i_{t,t'}\left[\rho_i\left(\tau_i|\lambda\right)|\,x,y\right]=\frac{\lambda_0 q\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)+\lambda_1}{q\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)+1}.$$

This is differentiable in x,y,t,t'. Since q is decreasing in x,y and increasing in t,t' and the expected hazard rate is decreasing in q, then the expected hazard rate is increasing in x,y and decreasing in t,t'.

Next

$$\mathbb{E}^i_{t,t'}\left[\rho_i\left(\tau_i|\lambda\right)|\,x,y-\right]=\frac{\lambda_0 q_-\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)+\lambda_1}{q_-\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)+1}$$

where

$$q_-\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)=\frac{1-p_0}{p_0}e^{(\lambda_1-\lambda_0)(t+t'-x-y)}\left(\frac{\lambda_0}{\lambda_1}\right)$$

This has the same properties as $\mathbb{E}^i_{t,t'}\left[\rho_i\left(\tau_i|\lambda\right)|\,x,y\right]$ w.r. to $x,\ y,\ t,\ t'$.

Finally

$$\mathbb{E}^i_{t,t'}\left[\rho_i\left(\tau_i|\lambda\right)|\,x,y+\right]=\frac{\lambda_0 q_+\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)+\lambda_1}{q_+\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)+1}.$$

where

$$q_+\left(p_0,\lambda_0,\lambda_1,t,t',x,y\right)=\frac{1-p_0}{p_0}e^{(\lambda_1-\lambda_0)(t+t'-x)}\frac{1-e^{\lambda_0 y}}{1-e^{\lambda_1 y}}\left(\frac{\lambda_0}{\lambda_1}\right)$$

This has the same properties as $\mathbb{E}^i_{t,t'}\left[\rho_i\left(\tau_i|\lambda\right)|\,x,y\right]$ w.r. to $x,\ t,\ t'$. Because

$$\frac{d}{dq}\left(\frac{\lambda_0 q+\lambda_1}{q+1}\right)=-\frac{\lambda_1-\lambda_0}{\left(q+1\right)^2}<0,$$

$e^{(\lambda_1-\lambda_0)(t+t'-x-y)}$ decreases in x and y and increases in t, because

$$\begin{aligned}\frac{d}{dy}\left(\frac{1-e^{\lambda_0 y}}{1-e^{\lambda_1 y}}\right) &= \frac{\lambda_1 e^{y\lambda_1}\left(1-e^{y\lambda_0}\right)-\lambda_0 e^{y\lambda_0}(1-e^{y\lambda_1})}{\left(1-e^{y\lambda_1}\right)^2}\\ &\propto \frac{\lambda_1 e^{y\lambda_1}}{1-e^{y\lambda_1}}-\frac{\lambda_0 e^{y\lambda_0}}{1-e^{y\lambda_0}},\end{aligned}$$

and $\frac{d}{d\lambda}\left(\frac{\lambda e^{y\lambda}}{1-e^{y\lambda}}\right)=\frac{e^{y\lambda}\left(1-e^{y\lambda}+y\lambda\right)}{\left(1-e^{y\lambda}\right)^2}>0$, it follows that Assumptions 1 and 2 are satisfied. Because

$$2\frac{d}{dx}E^i_{t,t}\left[\rho_i(t|\lambda)|\,x,y+\right]=-\frac{d}{dt}E^i_{t,t}\left[\rho_i(t|\lambda)|\,x,y+\right],$$

setting $\overline{G}'=-1/2$, the inequalities in Assumption 3 are satisfied again if:

$$0<\frac{d}{dy}E^i_{t,t}\left[\rho_i(t|\lambda)|\,x,y+\right]<\frac{d}{dx}E^i_{t,t}\left[\rho_i(t|\lambda)|\,x,y+\right]$$

The first inequality is already verified. The second inequality follows because:

$$\begin{aligned}
& \frac{d}{dx}\left(e^{(\lambda_1-\lambda_0)(2t-x)}\frac{1-e^{\lambda_0 y}}{1-e^{\lambda_1 y}}\right)-\frac{d}{dy}\left(e^{(\lambda_1-\lambda_0)(2t-x)}\frac{1-e^{\lambda_0 y}}{1-e^{\lambda_1 y}}\right) \\
= & -(\lambda_1-\lambda_0)\frac{1-e^{\lambda_0 y}}{1-e^{\lambda_1 y}}e^{(\lambda_1-\lambda_0)(2t-x)}-\left(-\frac{\lambda_0 e^{y\lambda_0}-\lambda_1 e^{y\lambda_1}-\lambda_0 e^{y\lambda_0}e^{y\lambda_1}+\lambda_1 e^{y\lambda_0}e^{y\lambda_1}}{\left(1-e^{\lambda_1 y}\right)^2}e^{(\lambda_1-\lambda_0)(2t-x)}\right) \\
= & -\frac{\lambda_1\left(1-e^{y\lambda_0}\right)-\lambda_0(1-e^{y\lambda_1})}{\left(1-e^{y\lambda_1}\right)^2}<0
\end{aligned}$$

Finally, the following

$$\lambda_0<\frac{c_i}{b_i}<p_0\lambda_1+(1-p_0)\,\lambda_0$$

is sufficient but not even necessary to satisfy Assumption 4.

Specializing Propositions 2 and 3, player i, observing signal x and upon observing that the opponent has left at time τ, will leave the race at time:

$$\sigma_{2,i}^{*}(x,\tau)=\max\left\{\tau,x+g_j^{*}(\tau)-\tau+K+\frac{\log\left(\frac{\lambda_1}{\lambda_0}\right)}{\lambda_1-\lambda_0}\right\}$$

and

$$\sigma_{2,i}^{*}(x,0)=\max\left\{0,x+\underline{x}_j^{*}+K\right\}$$

where

$$K=\frac{1}{\lambda_1-\lambda_0}\log\left(\frac{\lambda_1-\frac{c_i}{b_i}}{\frac{c_i}{b_i}-\lambda_0}\frac{p_0}{1-p_0}\frac{\lambda_1}{\lambda_0}\right)$$

is a measure of the prior bias in favor of λ_1.

From Proposition 5, we obtain the stopping time of a player i with signal x, conditional on the opponent j still being in the game is:

$$\sigma_{1,i}^{*}(x)=\max\left\{0,\frac{1}{2}\left[x+K+(\lambda_1-\lambda_0)^{-1}\log\left(\frac{1-e^{\lambda_1\sigma_{1,j}^{*-1}\left(\sigma_{1,i}^{*}(x)\right)}}{1-e^{\lambda_0\sigma_{1,j}^{*-1}\left(\sigma_{1,i}^{*}(x)\right)}}\right)\right]\right\}$$

In the symmetric payoff model $g_j^{*}\left(\sigma_{1,i}^{*}(x)\right)=\sigma_1^{*-1}\left(\sigma_1^{*}(x)\right)=x$, so we obtain a closed-form solution for the first quitting time:

$$\sigma_1^{*}(x)=\max\left\{0,\frac{1}{2}\left[x+K+(\lambda_1-\lambda_0)^{-1}\log\left(\frac{1-e^{\lambda_1 x}}{1-e^{\lambda_0 x}}\right)\right]\right\}.$$

Finally, from Proposition 7, we obtain that in any equilibrium of the public information game, for every pair of signals x,y, the first firm quits the race at time

$$T_1(x,y)=\frac{1}{2}(x+y+Q)$$

and

$$T_2(x,y)=x+y+Q-T_1,$$

where

$$Q=\frac{1}{(\lambda_1-\lambda_0)}\log\left(\frac{\lambda_1-c_2/b_2}{c_2/b_2-\lambda_0}\frac{p_0}{1-p_0}\left(\frac{\lambda_1}{\lambda_0}\right)^2\right).$$