

# OPTIMAL DYNAMIC CONTESTS\*

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June 2011

## Abstract

We study equilibrium play and the design of optimal incentives in a two-player dynamic contest, with costly effort and one-dimensional observed achievement. Our game is a continuous version of Harris and Vickers (ReStud 1987)'s tug-of-war. We derive stronger results and many formulae, including total expected output and effort in the game, which allows to study mechanism design. In the unique Symmetric Markov Perfect Equilibrium of the contest, the laggard gets discouraged, and tries less hard than the winner. This in turn allows the winner to let up when his lead exceeds a known threshold, which still falls short of the finish line if and only if the final prize is large enough relative to effort productivity per unit of effort cost. Otherwise, a player's effort is globally increasing in his lead. The sum of players' efforts is highest at the onset, when the race is tied, and declines monotonically in the score gap between players. The value to a player of participating in the race is bounded above even for an infinite prize, and is non-monotonic in the productivity and cost scale of effort. The model is quite tractable, and is currently awaiting solution of an optimal scoring function that in which the winner is "taxed" and the loser "subsidized". This scoring function must trade off the ex ante incentives to become a leader against the ex post incentives of the laggard to give up. We conjecture that this scoring function is concave, so that the latter, ex post, consideration prevails. To study the efficiency of a race as an incentive device, we will compare the output performance of the race to that of two separate, individual incentive schemes of equal expected total cost to the principal.

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\*This material is based upon work supported by the National Science Foundation under Grant No. 0550043, that we gratefully acknowledge. We thank participants to several seminars and conferences. The usual disclaimer applies.

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# 1 Introduction

When an agent's performance is curbed by moral hazard, how should a principal optimally design incentives? This question has been analyzed extensively in the single-agent case both in the static and in the repeated contexts, as well as, in a static context, in a multi-agent framework. In the latter case, an important dimension of the problem is team production. If agents' efforts are not separable in the technology to produce the output that the principal cares about, then the principal gains additional latitude to provide incentives. But when technologies are independent, so each agent produces a performance that depends only on her own effort (and noise), should incentives also be independent across agents?

The literature on tournaments suggests a negative answer: setting up a direct competition among agents whose technologies are separate and independent can improve incentives. This literature is almost exclusively concerned with static situations. In reality, however, tournaments are almost always dynamic. Prominent examples are sport leagues and promotion policies within professional companies. The analysis of dynamic tournaments is scarce. The best known example is the race set up by Harris and Vickers (1987, Restud) either in one dimension, where only the difference in performances matter, or in two dimensions, where the scale of individual performances also matters to determine rewards. The existing analysis is exclusively positive. We know nothing about the optimal design of a dynamic tournament.

More generally, we know almost nothing about the design of optimal incentives for multiple players who interact repeatedly. How are the rules of a professional sports league optimally designed? Notice that the space of mechanisms is very wide. Even fully describing it appears to be a daunting task. The principal can choose handicaps, length of the tournament, monetary prizes or penalties at each stage and in the whole tournament, rules for the stage game as a function of the current state of the overall game, etc.

In this paper, we attempt a first attack to the problem by focusing attention on a restricted, yet natural and still fairly flexible class of mechanisms in a two-agent game. No rewards and penalties are paid during the game. Payoffs come only at the end. Players continuously exert costly efforts that produce outputs with some noise. The winner is the player who achieves first a pre-determined output difference. I.e. the winner is the first player who

outperforms the opponent by a certain margin. The winner gets a prize and the loser gets nothing, having spent effort costs to no avail. An example is tennis without a tie-breaker. The game can last for ever, although in equilibrium it must always end in finite time a.s. if the game is designed to maximize incentives. We consider a principal who chooses the finish line, the prize, and the instantaneous weighting of the performance difference to maximize expected total output, either gross or net of effort costs.

In the specific linear-quadratic example of this paper, a player produces effort at quadratic cost to control linearly the drift of a Brownian Motion, which is his cumulative output. The difference in cumulative outputs is also a Brownian Motion. Payoffs are undiscounted.

First, we focus on the choice of the finish line and of the optimal prize when the “score” to determine victory is in “natural scale,” namely the difference in cumulative outputs. The winner is the first player to drive this difference to a finish line in his favor. In the unique Symmetric Markov Perfect Equilibrium of this game, the equilibrium strategy has three properties. First, optimal effort is always strictly positive. Second, it is either increasing or hump-shaped in own score, so the initial lead encourages the early leader and discourages the laggard, and the leader slacks off only when his lead becomes sufficiently large and the prize is sufficiently high. Third, the leader always exerts pointwise more effort than the follower, whoever happens to be the leader. These results replicate exactly those obtained by Harris and Vickers (1987) in their one-dimensional tug-of-war, which is a discrete time, discrete state, exponential-chance version of this game. This suggests that these equilibrium properties transcend the specific setup.

Our continuous setup affords a sharper characterization of the equilibrium than the discrete-state space of Harris and Vickers (1987). The leader’s effort peaks before the race ends if and only if the prize and productivity of effort are large enough and/or the noise and cost scale of effort are low enough, where the relevant threshold is computed explicitly. Effort at the beginning of a game is monotonically increasing in the prize, productivity of effort, and decreasing in noise and cost scale, but effort when about to win or lose the race are hump-shaped in those same parameters. The sum of players’ efforts is highest at the onset, when the race is tied, and declines monotonically in the score gap between players. The value to a player of participating in the race is bounded above even for an infinite prize, and is non-monotonic in the productivity and cost scale of effort.

We then turn to the mechanism design problem. This section is in progress. We find that the optimal finish line is unbounded. This appears to be an artifact of the linear-quadratic setup. Since this affords much tractability, we maintain it and focus on the choice of the optimal prize for exogenously given finish line. We compute explicitly the expected effort and output, gross and net of effort costs, at the beginning of the race when players play the SMPE. When the prize is very high, players engage in an initial “war” phase, where their efforts are very large. The reason is that the large prize justifies the high effort, and strategic complementarity reinforces this war effect. Large efforts nearly cancel out and the score is moved by randomness. Once a player attains a lead, he can first reinforce it and then maintain it using the threat that, should the opponent try to catch up, the game would enter again the “war” phase. A player can always react in time because the output difference evolves continuously. So both players maintain low efforts unless the score is almost a tie. This equilibrium pattern makes output rise modestly with the prize and is terrible for cost smoothing.

Finally, we ask which is the optimal distortion of the score that the principal can introduce. Rather than deciding the game based on output difference, the principal can choose some non-linear function of this difference. This corresponds to the notion of point-by-point re-weighting of the score. The choice of the optimal scoring rule is non-recursive, and does not lend itself to any standard method of dynamic (infinite-dimensional) optimization. Computing a discrete approximation of any desired accuracy to the optimal scoring rule is a finite problem, but it appears that it can be solved only by trying all possibilities.

We conjecture that the optimal scoring rule is concave: it is optimal to penalize the leader and help the follower as the game unfolds. Penalizing the leader, by making it harder and harder for him to move his score up, has undesirable *ex ante* incentives, because it reduces the returns to put in effort and gain a lead, but desirable *ex post* incentives, because it encourages the laggard to try harder and not to give up, as well as the leader to work really hard to preserve the acquired lead against the wind of the adverse scoring rule.

## 2 A Symmetric Linear-Quadratic Tug-of-War

### 2.1 Setup

Two agents  $i = A, B$  compete for a prize  $\pi > 0$ . Agent  $i$  chooses continuously  $n_{it}$  at quadratic cost  $c_0 n_{it}^2/2$ , where  $c_0 > 0$ , to produce flow output

$$dx_{it} = n_{it}\mu dt + \frac{\sigma}{\sqrt{2}}dW_{it}$$

where each  $W_{it}$  is Wiener and they are independent ( $W_B \perp W_A$ ). Let  $z_t = x_{Bt} - x_{At}$ , with

$$\begin{aligned} dz_t &= (n_{Bt} - n_{At}) \mu dt + \frac{\sigma}{\sqrt{2}}d(W_{Bt} - W_{At}) \\ &= (n_{Bt} - n_{At}) \mu dt + \sigma dW_t \end{aligned}$$

where  $W_t$  is Wiener. The rules of the competition are as follows: player  $B$  wins if and when  $z_t$  hits  $\omega > 0$ , and player  $A$  wins if and when  $z_t$  hits  $-\omega$ , with  $\omega > |z_0|$ . The winner gets  $\pi > 0$  and the loser gets nothing. Payoffs are undiscounted.

## 3 Equilibrium Analysis

### 3.1 Equilibrium Definition

We restrict attention to Symmetric Markov Perfect Equilibrium (SMPE), hereby defined on the one dimensional state space of score differences  $z$ .

Fix a pair of functions  $n_i : [-\omega, \omega] \rightarrow R_+$ ,  $i = A, B$  such that the following stochastic differential equation

$$dz_t = [n_A(z_t) - n_B(z_t)] \mu dt + \sigma dW_t$$

s.t.  $z_0 = 0$  has a unique solution  $z_t^{n_A, n_B, z_0}$ . We define any such pair of functions *admissible*.

En example is a pair of constant functions. Let

$$T_\omega(n_A, n_B, z_0) = \inf \{t \geq 0 : z_t^{n_A, n_B, z_0} = \omega\}$$

be the stopping time at which the process  $z_t^{n_A, n_B, z_0}$  hits  $\omega$ . A Markov Perfect Equilibrium is a pair of admissible functions  $\{n_i(z)\}_{i=A,B}$  such that for all  $z_0 \in (-\omega, \omega)$ ,  $n_i(z_0)$  maximizes the expected prize minus costs given that player  $i$  will follow the strategy  $n_i$  at all other

points and the opponent plays  $n_{-i}(z)$ .

$$n_B(z_0) = \arg \max_u \left\{ \pi \Pr(T_\omega(n_A, u, z_0) < T_{-\omega}(n_A, u, z_0)) \right. \\ \left. - E \left[ \int_0^{T_{-\omega}(n_A, u, z_0) \vee T_\omega(n_A, u, z_0)} c_0 \frac{u_B^2(z_s^{n_A, u, z_0})}{2} dz_s^{n_A, u, z_0} \right] \right\}$$

and similarly for player  $B$ , where  $\omega$  is replaced by  $-\omega$ .

We now impose symmetry, and look for a Symmetric MPE, a function  $n$  such that

$$n_B(z) = n_A(-z) := n(z).$$

Assume that a SMPE exists and players play it. Let  $\bar{V}_i(z)$  the continuation value to player  $i$  of the game starting from  $z$  in the SMPE path. By symmetry, we study the function

$$\bar{V}_B(z) = \bar{V}_A(-z) := \bar{V}(z).$$

**Lemma 1**  $\bar{V} \in \mathbb{C}^2$ .

**Proof.** To be added.

Therefore,  $\bar{V}$  solves the HJB equation

$$0 = \sup_u -c_0 \frac{u^2}{2} + [u - n(-z)]\mu \bar{V}'(z) + \frac{\sigma^2}{2} \bar{V}''(z)$$

s.t.  $\bar{V}(\omega) = \pi$  and  $\bar{V}(-\omega) = 0$ .

The NFOC for player B is

$$u(z) = \frac{\mu}{c_0} \bar{V}'(z) := n(z)$$

and for player A

$$n_A(z) = -\frac{\mu}{c_0} \bar{V}'_A(z) = \frac{\mu}{c_0} \bar{V}'(-z).$$

These NFOC yield the two-boundary value *Delayed Differential Equation* problem:

$$0 = \frac{[\mu \bar{V}'(z)]^2}{2c_0} - \frac{\mu^2}{c_0} \bar{V}'(-z) \bar{V}'(z) + \frac{\sigma^2}{2} \bar{V}''(z) \quad (1)$$

or

$$\bar{V}''(z) = \frac{\mu^2}{c_0 \sigma^2} \bar{V}'(z) [2\bar{V}'(-z) - \bar{V}'(z)] \equiv \delta \bar{V}'(z) [2\bar{V}'(-z) - \bar{V}'(z)] \quad (2)$$

s.t.  $\bar{V}(-\omega) = 0$ ,  $\bar{V}(\omega) = \pi$ , hereby defining

$$\delta \equiv \frac{\mu^2}{c_0 \sigma^2}.$$

### 3.2 The Normalized Game

We can rescale the problem. Let

$$V(s) \equiv V(z/\omega) \equiv \delta \bar{V}(z)$$

hereby defining  $s = z/\omega \in (0, 1)$ . Then (2) becomes

$$V''(s) = V'(s)[2V'(-s) - V'(s)]. \quad (3)$$

with boundary conditions

$$V(-1) = 0, V(1) = \delta\pi.$$

Therefore, WLOG, we can find the equilibrium value  $V$  for  $\omega = 1$ , and then recover the value of the original problem at score  $z$  by

$$\bar{V}(z) = \frac{1}{\delta} V\left(\frac{z}{\omega}\right)$$

and, from there, the SMPE strategies of the original game with finish line  $\omega$ :

$$n_\omega(z) = \frac{\mu}{c_0} \bar{V}'(z) = \frac{\mu}{c_0 \delta} \frac{1}{\omega} V'\left(\frac{z}{\omega}\right). \quad (4)$$

Notice that doubling  $\omega$  and  $z$  halves effort  $n$ . That is, for every  $z$  and  $s = z/\omega \in (-1, 1)$

$$\omega n_\omega(\omega s) = \frac{\mu}{c_0 \delta} V'(s) = n_1(s)$$

so

$$n_\omega(z) = \frac{1}{\omega} n_1\left(\frac{z}{\omega}\right). \quad (5)$$

When the finish line doubles, players spend exactly half the effort and double the distance from the starting point  $z_0 = 0$ .

From now, WLOG we set  $\omega = 1$  and solve the corresponding normalized problem where the only parameter is  $\delta\pi$ . Then we recover  $n_\omega(z)$  from (5).

### 3.3 The Key Differential Equations

We can reduce the normalized problem to a *first*-order delayed differential equation in the effort best response, by letting  $m(z) = V'(z)$ , so that from (3)

$$m'(z) = m(z)[2m(-z) - m(z)] \quad (6)$$

subject to

$$\int_{-1}^1 m(z) dz = \delta\pi. \quad (7)$$

where we used  $V(-1) = 0$  and  $V(1) = \delta\pi$ .

**Lemma 2** *The marginal value  $m(z) = V'(z)$  obeys the second order ODE*

$$2m(z)m''(z) = 3(m'(z))^2 - 4m'(z)m(z)^2 - 3m(z)^4 \quad (8)$$

*Proof:* Take another derivative in (6)

$$m''(z) = 2\{m'(z)[m(-z) - m(z)] - m(z)m'(-z)\}. \quad (9)$$

Equation (6) allows us to eliminate both  $m(-z)$  and  $m'(-z)$  from (9) using  $2m(-z) = m'(z)/m(z) + m(z)$  and  $m'(-z) = m(-z)[2m(z) - m(-z)]$ , and deduce

$$\begin{aligned} m''(z) &= 2m'(z)[m(-z) - m(z)] - 2m(z)m'(-z) \\ &= 2m'(z) \left[ m(-z) - \frac{1}{2}m(z) - \frac{1}{2}m(z) \right] - 2m(z)m(-z)[2m(z) - m(-z)] \\ &= 2m'(z) \left[ \frac{m'(z)}{2m(z)} - \frac{1}{2}m(z) \right] - 2m(z) \left[ \frac{m'(z)}{2m(z)} + \frac{1}{2}m(z) \right] \left[ \frac{3}{2}m(z) - \frac{m'(z)}{2m(z)} \right] \\ &= \frac{3[m'(z)]^2 - 4m'(z)m^2(z) - 3m^4(z)}{2m(z)} \end{aligned}$$

Altogether this yields (8), as required. ■

If we can find a solution  $m(z)$  to (8) subject to (7), we can recover the equilibrium strategy of the original game with  $\omega = 1$  combining (4) and the definition  $m = V'$ :

$$n_1(z) = \frac{\sigma^2}{\mu} m(z) \quad (10)$$

From this function, we can recover the SMPE strategy of the original unscaled game with finish line  $\omega$

$$n_\omega(z) = \frac{\sigma^2}{\mu\omega} m\left(\frac{z}{\omega}\right).$$

### 3.4 Characterization of the Equilibrium Strategy

We verify that our SMPE has the same qualitative properties as the equilibrium strategies of the Harris and Vickers (1987) tug-of-war, which is cast in discrete time and state space, with exponential chance of success in the stage game.



**Lemma 3** *The SMPE strategy is strictly positive for all  $z \in (-\omega, \omega)$ :*

$$n_\omega(z) = \frac{\sigma^2}{\mu\omega} V' \left( \frac{z}{\omega} \right) > 0$$

**Proof.** The normalized HJB equation (3) reveals that if ever  $V'(z) = m(z) = 0$ , then we must have  $V''(z) = m'(z) = 0$ . But (3) can be differentiated as many times as desired, and each summand has a lower order derivative factor of  $V$  at each stage, all derivatives of  $V$  vanish at  $z$ . For instance,

$$V'''(s) = V''(s)[2V'(-s) - V'(s)] + V'(s)[-2V''(-s) - V''(s)]. \quad (11)$$

But then  $V$  is constant in a neighborhood of  $s$ . Extending this,  $V$  is constant on the entire domain, contrary to the boundary conditions  $V(-1) = 0 < \pi = V(1)$ . ■

**Lemma 4 (Slope of Individual Effort in Score)** *The SMPE strategy is either globally increasing or rising and then falling. In particular, it is increasing at  $z = -\omega$ .*

**Proof.** Equations (6) and (11) respectively yield

$$m'(0) = m^2(0) \quad \text{and} \quad m''(0) = -2m^3(0) < 0.$$

So at the beginning of the game, when it is tied at  $z_0 = 0$ , equilibrium effort is positive, and locally increasing and concave in the score. Since  $m(z)$  is increasing at  $z = 0$ , it cannot be globally declining. Modifying (8), we find the equivalent second order nonlinear ODE

$$m''(z) = \frac{3(m'(z))^2}{2m(z)} - 2m'(z)m(z) - \frac{3}{2}m^3(z). \quad (12)$$

Assuming that  $m'(z) = 0$ , we find  $m''(z) = -3m^3(z)/2 < 0$  because  $m(z) > 0$ . So either  $m(z)$  is monotonic increasing or it has a unique maximum. ■

**Lemma 5 (Follow the Leader)** *The leader tries harder than the follower.*

**Proof.** Define  $q(z) = m(z) - m(-z)$ , with  $q(0) = 0$ . We claim that  $q(z) \geq 0$  as  $z \geq 0$ .

$$\begin{aligned} q'(z) &= m'(z) + m'(-z) \\ &= m(z)[2m(-z) - m(z)] + m(-z)[2m(z) - m(-z)] \\ &= 4m(z)m(-z) - m^2(z) - m^2(-z) \end{aligned}$$

So  $q'(0) = 2m^2(0) > 0$ . Let  $z'$  be the least  $z' \in (0, 1)$  with  $q(z') = 0$ . Then  $m(z') = m(-z')$ , and

$$q'(z') = 2m^2(z') > 0.$$

But then there is a smaller zero  $z'' \in (0, z')$  of  $q(z)$ , contradicting the choice of  $z'$ . ■

Notice that, using (6), the sum of efforts satisfies

$$\frac{d}{dz} [m(z) + m(-z)] = m'(z) - m'(-z) = m^2(-z) - m^2(z)$$

which has the same sign as  $z$  when  $z > 0$  and vice versa when  $z < 0$ . Therefore, the sum of efforts declines as the leader's lead widens, and increases as the race gets tighter. A competitive situation makes for a more interesting race.

### 3.5 The Key Change of Variable

Although the first order DDE boundary value problem (6)-(7) and equivalent second order ODE boundary value problem (12)-(7) are not immediately soluble, a key change of variable affords great tractability and a virtually complete characterization of equilibrium behavior.

Let

$$\rho(z) := \frac{m'(z)}{m^2(z)} = 2 \frac{m(-z)}{m(z)} - 1.$$

This function plays a key role in our analysis. To gain an intuition, notice that

$$\rho(z) = -\frac{d}{dz} \frac{1}{m(z)} = -\frac{d}{dz} \frac{1}{V'(z)} = \frac{V''(z)}{m(z)V'(z)}$$

so this variable is a measure of curvature of the value function.

We now put the transformation  $z \mapsto \rho$  to good use. From (6), for all  $z$  we have the identity  $[1 + \rho(z)][1 + \rho(-z)] = 4$ , or

$$\rho(-z) = \frac{3 - \rho(z)}{1 + \rho(z)}. \tag{13}$$

We now look for a relationship between this function and the level of equilibrium effort.

From Lemma (5), for  $z > 0$  we have  $m(-z) < m(z)$  and, so clearly  $\rho(z) \in (-1, 1)$  with

$\rho(0) = 1$  and  $\rho(z) > -1$ . For  $z < 0$  we have  $\rho(z) \in (-1, \infty)$ . Next, using (12)

$$\begin{aligned}
\rho'(z) &= \frac{d}{dz} \left( \frac{m'(z)}{m(z)^2} \right) \\
&= \frac{m''(z)m(z) - 2m'(z)^2}{m(z)^3} \\
&= m(z) \frac{-m'(z)^2 - 4m'(z)m(z)^2 - 3m(z)^4}{2m(z)^4} \\
&= -\frac{m(z)}{2}(\rho(z) + 1)(\rho(z) + 3) < 0.
\end{aligned}$$

This implies that  $\rho(-1)$  is the maximum value of  $\rho$ ,  $\rho(1)$  is the minimum value, and we can change variable from  $z$  to  $r = \rho(z)$ .

Let effort as a function of the new variable

$$\widehat{m}(r) = m(\rho^{-1}(r))$$

with slope

$$\begin{aligned}
\widehat{m}'(r) &= \frac{m'(\rho^{-1}(r))}{\rho'(\rho^{-1}(r))} = -\frac{2m'(\rho^{-1}(r))}{m(\rho^{-1}(r))(r+1)(r+3)} \\
&= -\frac{2\rho(\rho^{-1}(r))m(\rho^{-1}(r))}{(r+1)(r+3)} = -\frac{2r\widehat{m}(r)}{(r+1)(r+3)}
\end{aligned}$$

Integrating this differential equation we obtain

$$\widehat{m}(r) = C \frac{1+r}{(3+r)^3} \quad (14)$$

which, at  $r = 1 = \rho(0)$ , yields a constant of integration  $C = 32\widehat{m}(1) = 32m(0)$ . Here  $m(0)$  is still unknown. Notice that effort is increasing in the value-curvature measure  $r$  when  $r < 0$ , i.e. when effort is falling, which happens only for the leader, and decreasing when  $r > 0$  and effort is rising.

Using this expression for  $\widehat{m}(r)$ , we complete the change of variable:

$$\begin{aligned}
\rho'(z) &= -\frac{m(z)}{2}(\rho(z) + 1)(\rho(z) + 3) \\
&= -\frac{\widehat{m}(\rho(z))}{2}(\rho(z) + 1)(\rho(z) + 3) \\
&= -C \frac{1 + \rho(z)}{2(3 + \rho(z))^3}(\rho(z) + 1)(\rho(z) + 3) \\
&= -C \frac{1 + \rho(z)}{2(3 + \rho(z))^3}(\rho(z) + 1)(\rho(z) + 3) \\
&= -\frac{C}{2} \left( \frac{1 + \rho(z)}{3 + \rho(z)} \right)^2
\end{aligned}$$

So for  $r = \rho(z)$  we finally have

$$dz = -\frac{2}{C} \left( \frac{3+r}{1+r} \right)^2 dr. \quad (15)$$

We can also integrate the ODE for  $\rho(z)$  with  $C = 32m(0)$  to obtain an implicit solution

$$-\frac{1}{(1+\rho(z))4m(0)} + \frac{1}{4m(0)} \log(1+\rho(z)) + \frac{1}{16m(0)}\rho(z) + z = K.$$

To pin down the constant  $K$ , let  $z = 0$ ,  $\rho(0) = 1$ , so

$$K = -\frac{1}{8m(0)} + \frac{\log 2}{4m(0)} + \frac{1}{16m(0)} + 0 = \frac{\log 16 - 1}{16m(0)}.$$

Replacing and simplifying  $1/m(0)$ , we finally find an implicit equation for  $\rho(z)$  given  $m(0)$ :

$$-\frac{4}{1+\rho(z)} + 4 \log(1+\rho(z)) + \rho(z) + 16m(0)z = \log 16 - 1. \quad (16)$$

Next, we turn our attention to the boundary condition. Using (14) and (15),

$$m(z)dz = \widehat{m}(r)dr = C \frac{1+r}{(3+r)^3} \left[ -\frac{2}{C} \left( \frac{3+r}{1+r} \right)^2 dr \right] = -\frac{2}{(3+r)(1+r)} dr.$$

So the boundary condition (7) becomes

$$\begin{aligned} \delta\pi &= \int_{-1}^1 m(z)dz = \int_{\rho(-1)}^{\rho(1)} -\frac{2}{(3+r)(1+r)} dr \\ &= \int_{\rho(1)}^{\rho(-1)} \frac{2}{(3+r)(1+r)} dr = \log \left( \frac{1+\rho(-1)}{1+\rho(1)} \frac{3+\rho(1)}{3+\rho(-1)} \right) \end{aligned}$$

or

$$\frac{1+\rho(-1)}{1+\rho(1)} \frac{3+\rho(1)}{3+\rho(-1)} = e^{\delta\pi}.$$

Using (13) at  $z = 1$  to replace for  $\rho(-1)$  in terms of  $\rho(1)$ , we can solve for

$$e^{\delta\pi} = \frac{1 + \frac{3-\rho(1)}{1+\rho(1)}}{1+\rho(1)} \frac{3+\rho(1)}{3 + \frac{3-\rho(1)}{1+\rho(1)}} = \frac{\frac{4}{1+\rho(1)}}{1+\rho(1)} \frac{3+\rho(1)}{\frac{6+2\rho(1)}{1+\rho(1)}} = \frac{2}{1+\rho(1)}$$

and finally

$$\rho(1) = 2e^{-\delta\pi} - 1 \quad (17)$$

$$\rho(-1) = 2e^{\delta\pi} - 1. \quad (18)$$

So  $\rho(-1)$  is increasing in  $\delta\pi$  and  $\rho(1)$  is decreasing. The intuition is simple. Notice that when  $\delta\pi = (\mu/\sigma)^2 (\pi/c_0) = 0$ , i.e. either effort is unproductive ( $\mu = 0$ ) or noise dominates ( $\sigma = \infty$ ) or costs are prohibitive ( $c_0 = \infty$ ) or there is no prize ( $\pi = 0$ ), then  $\rho(-1) = \rho(1) = 1$ . Players put no effort. Conversely, as  $\delta\pi \rightarrow \infty$ ,  $\rho(-1) \rightarrow \infty$  and  $\rho(1) \rightarrow -1$ .

Using (16) at  $z = 1$

$$-\frac{4}{1 + \rho(1)} + 4 \log(1 + \rho(1)) + \rho(1) + 16m(0) = \log 16 - 1$$

and (17) for  $\rho(1)$

$$-\frac{4}{2e^{-\delta\pi}} + 4 \log(2e^{-\delta\pi}) + 2e^{-\delta\pi} - 1 + 16m(0) = \log 16 - 1$$

and finally we obtain effort at the onset of the race as a function of parameters:

$$m(0) = \frac{e^{\delta\pi} - e^{-\delta\pi}}{8} + \frac{\delta\pi}{4}. \quad (19)$$

This initial effort level is clearly increasing in  $\delta\pi$ . When  $\delta\pi$  is small, effort is not worth its cost. As the normalized prize  $\delta\pi$  rises, so does initial effort.

Next, we study the effort level at the boundaries  $z = -1, 1$ . By definition of  $\rho(z)$

$$m'(z) = \rho(z)m(z)^2 = r\hat{m}(r)^2$$

using the change of variable (15) and the expression (14) for  $\hat{m}(r)$

$$\begin{aligned} m'(z)dz &= r\hat{m}(r)^2 \left[ -\frac{2}{C} \left( \frac{3+r}{1+r} \right)^2 \right] dr \\ &= r \left[ C \frac{1+r}{(3+r)^3} \right]^2 \left[ -\frac{2}{C} \left( \frac{3+r}{1+r} \right)^2 \right] dr \\ &= -\frac{2rC}{(3+r)^4} dr \end{aligned}$$

We can then integrate, using  $\rho(0) = 1$  and

$$C = 32m(0) = 4(e^{\delta\pi} - e^{-\delta\pi}) + 8\delta\pi$$

to obtain an exact relationship between effort  $m(z)$  and  $\rho(z)$  at any score  $z$ :

$$\begin{aligned}
m(z) &= m(0) + \int_0^z m'(s)ds = m(0) + \int_{\rho(0)}^{\rho(z)} -\frac{2rC}{(3+r)^4}dr \\
&= m(0) + 64m(0) \int_{\rho(z)}^1 \frac{r}{(3+r)^4}dr \\
&= m(0) + 64m(0) \left[ -\frac{1+r}{2(3+r)^3} \right]_{\rho(z)}^1 \\
&= m(0) + 64m(0) \left\{ -\frac{1}{64} + \frac{1+\rho(z)}{2(3+\rho(z))^3} \right\} \\
&= 32m(0) \frac{1+\rho(z)}{(3+\rho(z))^3}
\end{aligned}$$

and finally, using (19) for  $m(0)$  and simplifying

$$m(z) = 4(e^{\delta\pi} - e^{-\delta\pi} + 2\delta\pi) \frac{1+\rho(z)}{(3+\rho(z))^3}. \quad (20)$$

Therefore the minimal effort spent by a player in the game, which occurs when one is about to lose, equals

$$m(-1) = 32m(0) \frac{1+\rho(-1)}{(3+\rho(-1))^3} = \frac{(e^{\delta\pi} - e^{-\delta\pi} + 2\delta\pi) e^{\delta\pi}}{(1+e^{\delta\pi})^3}$$

This function is hump-shaped in  $\delta\pi$ , and peaks at  $\delta\pi = 0.98325$ . Similarly, effort when crossing the finish line as a winner equals

$$m(1) = \frac{(e^{\delta\pi} - e^{-\delta\pi} + 2\delta\pi) e^{-\delta\pi}}{(1+e^{-\delta\pi})^3}.$$

This function is hump-shaped in  $\delta\pi$ , and peaks at  $\delta\pi = 2.9211$ . Again, the intuition is simple. When  $\delta\pi$  is small, effort is not worth its cost. As the normalized prize  $\delta\pi$  rises, both extremal effort levels rise. But eventually the race becomes too harsh at the beginning, at  $z = 0$ , so the loser starts giving up and  $m(-1)$  declines. As the prize rises further, the leader can also relax and  $m(1)$  declines.

Since  $\rho(z)$  has the same sign as  $m'(z)$ , effort peaks just at  $z = 1$  when  $m'(1) = 0$ , or  $\rho(1) = 0 = 2e^{-\delta\pi} - 1$ . When  $m(z)$  peaks at some interior  $z^* < 1$ ,  $m'(z^*) = \rho(z^*) = 0$ , so from (20)  $m(z^*) = m(0) (32/27)$ . Therefore, equilibrium effort peaks at some interior  $z^* \in (-1, 1)$  if and only if

$$\delta\pi > \log 2$$

i.e. the winner's prize and/or the productivity of effort are large enough and the noise in output and/or the scale of the effort cost function are small enough. At the peak, effort takes value

$$\max_z m(z) = m(z^*) = \frac{32}{27}m(0).$$

The point  $z^*$  where  $m$  peaks solves  $\rho(z^*) = 0$ , so from (16) and (19) we can solve for the score where the leader's effort peaks

$$z^* = \frac{\log 16 + 3}{2(e^{\delta\pi} - e^{-\delta\pi}) + 4\delta\pi}.$$

It is easy to verify that indeed  $z^* = 1$  when  $\delta\pi = \log 2$ . Otherwise,  $m$  is globally increasing and peaks at  $z = 1$ , with value

$$\max_z m(z) = m(1) = \frac{1 - e^{-2\delta\pi} + 2\delta\pi e^{-\delta\pi}}{(1 + e^{-\delta\pi})^3}.$$

Given the expression (19) for effort  $m(0)$  at the onset of the game, (16) is an implicit solution for  $\rho(z)$  in terms of  $\delta\pi$  at all points  $z$  in the domain. This equation, along with (20) implicitly defines the equilibrium strategy  $m(z)$  in terms of  $\delta\pi$  only. For example, we could solve the cubic (20) to express  $\rho(z)$  as a function of  $m(z)$ , and plug it into (16) to obtain an implicit equation in  $m(z)$  alone.

Next, using  $V'(z) = m(z)$ , the change of variable,  $\rho(1) = 0$  and (18) we can recover the normalized value for the player at the onset of the game

$$\begin{aligned} V(0) &= V(-1) + \int_{-1}^0 V'(z)dz = 0 + \int_{-1}^0 m(z)dz \\ &= \int_{\rho(-1)}^{\rho(0)} \frac{2}{(3+r)(1+r)} dr = \int_1^{\rho(-1)} \frac{2}{(3+r)(1+r)} dr \\ &= \log \left( \frac{1 + \rho(-1)}{2} \frac{4}{3 + \rho(-1)} \right) = \log \left( \frac{2e^{\delta\pi}}{1 + e^{\delta\pi}} \right) \end{aligned}$$

which has the same sign as  $\delta\pi$ , is less than  $\delta\pi/2$ , is increasing in  $\delta\pi$  and asymptotes to  $\log 2$  as  $\delta\pi \rightarrow \infty$ . An infinite prize makes for such a hard-fought race that most rents are dissipated by effort costs. In terms of true non-normalized payoffs  $\bar{V}(0) = \delta^{-1}V(0)$ , the value of participating in the race equals

$$\bar{V}(0) = c_0 \left( \frac{\sigma}{\mu} \right)^2 \log \left( \frac{2e^{\left(\frac{\mu}{\sigma}\right)^2 \frac{\pi}{c_0}}}{1 + e^{\left(\frac{\mu}{\sigma}\right)^2 \frac{\pi}{c_0}}} \right) \quad (21)$$

which has the same properties as  $V$  when the prize  $\pi$  changes, but is non-monotonic in effort productivity  $\mu$ , noise  $\sigma$  and cost scale  $c_0$ . The reason is that when  $\mu$  rises the temptation to take advantage of this better technology triggers more competition and more rent dissipation. Against this effect is the cost saving associated to larger effort productivity.

Finally, we consider again the function  $q(z) = m(z) - m(-z)$ , the difference in efforts which determines the speed at which the contest progresses. Using our change of variable

$$q(z) = m(z) - m(-z) = 32m(0) \frac{1 - \rho^2(z)}{(3 + \rho(z))^3}$$

so, after some algebra and using the fact that  $\rho$  is decreasing:

$$q'(z) = -32m(0) \frac{6\rho(z) + \rho^2(z) + 1}{(3 + \rho(z))^4} \rho'(z) > 0.$$

As the leader gains, not only he tries harder than the follower,  $q(z) > 0$  as  $\rho(z) < 1$  for  $z > 0$ , but this gap in efforts widens too. The contest accelerates on average towards the end. As a function of the normalized prize, the difference in efforts

$$\frac{dq(z)}{d(\delta\pi)} = 32 \frac{dm(0)}{d(\delta\pi)} \frac{1 - \rho^2(z)}{(3 + \rho(z))^3} - 32m(0) \frac{6\rho(z) + \rho^2(z) + 1}{(3 + \rho(z))^4} \frac{d\rho(z)}{d(\delta\pi)}$$

The first term on the RHS is positive by (19). For the second term, from (16) and the Implicit Function Theorem

$$\frac{d\rho(z)}{d(\delta\pi)} = \frac{-16 \frac{dm(0)}{d(\delta\pi)} z}{\left(1 + \frac{2}{1+\rho(z)}\right)^2}$$

which is negative for  $z > 0$ . Therefore, the difference in efforts between leader and follower widens with the stakes. Near the initial tie  $z = 0$ , both players put more effort, enticed by the higher prize, but the leader outs even more.

We summarize our findings in the following:

**Proposition 6 (Characterization of Equilibrium Behavior)** *In the unique Symmetric Markov Perfect Nash Equilibrium of the race:*

- *the effort strategy  $n_\omega$  is globally positive if the parameters: productivity of effort  $\mu$ , noise in output  $\sigma$ , scale of effort cost  $c_0$ , prize  $\pi$ , and finish line  $\omega$  are all positive and finite, and it is identically zero if  $\mu = 0$ ,  $\pi = 0$ ,  $\sigma = \infty$ ,  $c_0 = \infty$ , or  $\omega = \infty$ .*



- the effort strategy  $n_\omega$  is “scalable” in the finish line  $\omega$  according to (5), where  $n_1$  is defined by (10) in terms of the function  $m$ , and  $m$  itself solves either the boundary value problem (7)-(12) or the equivalent algebraic problem (16)-(20);
- the other parameters  $\mu, \sigma, c_0, \pi$  matter for the normalized strategy  $m$  only through the composite parameter “normalized prize”  $\delta\pi = \mu^2\pi / (c_0\sigma^2)$ ;
- the follower’s effort declines with the score gap:  $m'(z) > 0$  and thus  $\frac{d}{dz}n_\omega(-z) < 0$  for all  $z \in (-1, 0)$ ;
- if and only if  $\delta\pi > \log 2$  the effort strategy is hump-shaped in the score  $z \in (-1, 1)$  and peaks at score

$$z^* = \frac{\log 16 + 3}{2(e^{\delta\pi} - e^{-\delta\pi}) + 4\delta\pi} \in (0, 1)$$

with maximal (normalized) value

$$m(z^*) = \frac{4(e^{\delta\pi} - e^{-\delta\pi}) + 8\delta\pi}{27}$$

which is increasing in the (normalized) prize  $\delta\pi$ ;

- if and only if  $\delta\pi \leq \log 2$  the effort strategy is globally increasing in the score  $z \in (-1, 1)$  and peaks at the finish line  $z = 1$ , with maximal (normalized) value

$$m(1) = \frac{1 - e^{-2\delta\pi} + 2\delta\pi e^{-\delta\pi}}{(1 + e^{-\delta\pi})^3}$$

which is hump-shaped in the (normalized) prize  $\delta\pi$ ;

- the leader always exerts more effort than the follower, who always puts minimal (normalized) effort when about to lose:

$$m(-1) = \frac{(e^{\delta\pi} - e^{-\delta\pi} + 2\delta\pi) e^{\delta\pi}}{(1 + e^{\delta\pi})^3}$$

which is hump-shaped in the (normalized) prize  $\delta\pi$ ;

- initial effort  $m(0)$  is  $27/32$  times the maximal effort  $m(z^*)$  and thus is increasing in the (normalized) prize  $\delta\pi$ ;

- the difference in efforts by the two players  $m(z) - m(-z)$  increases monotonically in the score gap  $2z > 0$  between the leader and the follower, hence it is lowest and equal to zero when the score is tied at  $z = 0$  and, from there, the speed of the score increases on average towards the end; this gap in efforts is increasing in the (normalized) prize  $\delta\pi$  for all  $z > 0$ ;
- the sum of efforts by the two players  $m(z) + m(-z)$  decreases monotonically in the score gap  $2z > 0$  between the leader and the follower, hence it is highest when the score is tied at  $z = 0$  and equals (normalized)

$$2m(0) = \frac{e^{\delta\pi} - e^{-\delta\pi}}{4} + \frac{\delta\pi}{2};$$

- the value to a player from participating in the race  $\bar{V}(0)$  is given by (21), increasing in  $\pi$  with bounded limit  $\delta^{-1} \log 2$  as  $\pi$  grows unbounded, and hump-shaped in  $\mu, \sigma, c_0$ .

To complete our characterization of equilibrium outcomes, we now quantify the expected duration of the contest.

### 3.6 Expected Duration

Let  $\mu = \sigma = 1$  for simplicity of notation. Let  $T(z)$  denote the expected residual duration of the race given equilibrium strategies when the race score is at  $z$ . This solves (Karlin and Taylor) the ODE

$$\frac{1}{2}T''(z) + [m(z) - m(-z)]T'(z) + 1 = 0 \tag{22}$$

s.t.  $T(-1) = T(1) = 0$ . We are interested primatily in  $T(0)$ .

In the Appendix, we show that the solution to (22) is:

$$T(y) = 2 \frac{\int_{-1}^1 \phi(z) \int_{-1}^z \frac{ds}{\phi(s)} dz}{\int_{-1}^1 \phi(z) dz} \int_{-1}^y \phi(z) dz - 2 \int_{-1}^y \phi(z) \int_{-1}^z \frac{ds}{\phi(s)} dz$$

where

$$\phi(z) := \exp \left\{ 2 \int_{-1}^z [m(-x) - m(x)] dx \right\} = \left[ \frac{1 + \rho(z)}{1 + \rho(-1)} \right]^2 \left[ \frac{3 + \rho(-1)}{3 + \rho(z)} \right]^4.$$

Using our key change of variable, we arrive to an explicit expression:

$$T(0|x = 2e^{\delta\pi} - 1) = \frac{\int_{\frac{3-x}{1+x}}^1 \frac{1}{(3+r)^2} \int_r^x \frac{(3+s)^6}{(1+s)^4} ds dr - \int_1^x \frac{1}{(3+r)^2} \int_r^x \frac{(3+s)^6}{(1+s)^4} ds dr}{\left(1 + x - \frac{4}{1+x} + 4 \ln \left(\frac{1+x}{2}\right)\right)^2}$$

which integrates to

$$T(0|x = 2e^{\delta\pi} - 1) = \frac{B(2e^{-\delta\pi} - 1) + B(2e^{\delta\pi} - 1) - \frac{127}{2}}{(2e^{\delta\pi} - 2e^{-\delta\pi} + 4\delta\pi)^2} \quad (23)$$

where

$$B(x) = 5x + \frac{1}{6}x^2 + 40\frac{x-1}{3+x}\ln(x+1) + \frac{630x + 255x^2 + 391}{21x + 15x^2 + 3x^3 + 9}$$

A plot of  $T(0|x)$  for  $x = 2e^{\delta\pi} - 1 > 1$  (i.e. for  $\delta\pi > 0$ )

proves the following:

**Proposition 7 (The Expected Duration of the Contest)** *The expected duration of a contest with  $\mu = \sigma = 1$  given the normalized prize  $\delta\pi = \pi/c_0$  is  $T(0|2e^{\delta\pi} - 1)$  where the function  $T$ , defined in (23), is globally decreasing with  $T(0|1) = 1$ ,  $T(0|\infty) = 0$ . Therefore, the contest ends sooner the higher the prize.*

From an accounting viewpoint, this result is driven by two forces. First, as the prize rises, players put more effort and produce more output in the middle, but less at the ends of the game. Second, the difference in efforts between the leader and the follower increases globally with the prize, which accelerates the contest and reduces its duration.

## 4 Designing a Race: the Optimal Reward

Our main focus is on the optimal design of the race. Once we have characterized and computed the equilibrium strategy  $n$ , we can focus on the optimal design of the race. Let  $T$  denote the stopping time of the game when the finish line is  $\omega$  and the prize is  $\pi$ , namely

$$T = \inf_{t \geq 0} \{z_t = \omega \text{ or } z_t = -\omega\}.$$

Let total expected output starting from  $z_0$ :

$$\begin{aligned} Q_{\omega\pi}(z_0) &= E[x_{AT} + x_{BT}|z_0] \\ &= E\left[\int_0^T d(x_{At} + x_{Bt})|z_0\right] \\ &= E\left[\int_0^T (\mu(n_{At} + n_{Bt}) dt + \sigma dW_t)|z_0\right] \\ &= \mu E\left[\int_0^T (n_{At} + n_{Bt}) dt|z_0\right] \\ &= \frac{\mu}{\omega} E\left[\int_0^T \left(n\left(\frac{z_t}{\omega}\right) + n\left(-\frac{z_t}{\omega}\right)\right) dt|z_0\right] \end{aligned}$$

having used: the optimal best responses of the players to the normalized game designed by the principal, the effort scaling relationship (5) with  $n_1 = n$ , and the martingale property of the Wiener noise. Similarly, let

$$K_{\omega,\pi}(z_0) = \frac{c_0}{2} E \left[ \int_0^T (n_{At}^2 + n_{Bt}^2) dt | z_0 \right] = \frac{c_0}{2\omega^2} E \left[ \int_0^T \left( n^2 \left( \frac{z_t}{\omega} \right) + n^2 \left( -\frac{z_t}{\omega} \right) \right) dt | z_0 \right]$$

denote the expected total costs of effort during the game.

These values solve (resp.) the HJB equation

$$0 = n_\omega(z) + n_\omega(-z) + [n_\omega(z) - n_\omega(-z)] Q'_\omega(z) + \frac{\sigma^2}{2\mu} Q''_\omega(z) \quad (24)$$

s.t.

$$Q_\omega(-\omega) = Q_\omega(\omega) = 0$$

and

$$0 = \frac{c_0}{2} \{n_\omega^2(z) + n_\omega^2(-z)\} + [n_\omega(z) - n_\omega(-z)] K'_\omega(z) \mu + \frac{\sigma^2}{2} K''_\omega(z)$$

s.t.

$$K_\omega(-\omega) = K_\omega(\omega) = 0.$$

Clearly, net output

$$U_\omega = Q_\omega - K_\omega$$

solves

$$0 = -\frac{c_0}{2} \{n_\omega^2(z) + n_\omega^2(-z)\} + \{n_\omega(z) + n_\omega(-z) + [n_\omega(z) - n_\omega(-z)] U'_\omega(z)\} \mu + \frac{\sigma^2}{2} U''_\omega(z)$$

s.t.

$$U_\omega(-\omega) = U_\omega(\omega) = 0.$$

In the Appendix, we derive the solutions to the Principal's HJB (linear ordinary differential) equations in terms of the equilibrium symmetric effort strategy:

$$Q_\omega(z) = \frac{2\mu}{\sigma^2} \int_{-\omega}^z \left[ \int_j^0 \frac{n_\omega(s) + n_\omega(-s)}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_j^s [n_\omega(-x) - n_\omega(x)] dx \right\}} ds \right] dj$$

and

$$U_\omega(z) = \frac{2\mu}{\sigma^2} \int_{-\omega}^z \int_j^0 \frac{[n_\omega(s) + n_\omega(-s)] - \frac{c_0}{2\mu} [n_\omega^2(s) + n_\omega^2(-s)]}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_j^s [n_\omega(-x) - n_\omega(x)] dx \right\}} ds dj.$$

The problem of the principal can be taken to be any of the following:

$$\begin{aligned} & \max_{\pi, \omega} \{Q_\omega(0) - \pi\} \\ & \max_{\pi, \omega} \{U_\omega(0) - \pi\} \\ & \max_{\pi, \omega} U_\omega(0). \end{aligned}$$

#### 4.1 Homogeneity of the Principal's Value in the Finish Line

Fix  $\omega$ . For  $z \in [-\omega, \omega]$ , we guess

$$Q_\omega(z) \equiv \omega Q_1\left(\frac{z}{\omega}\right)$$

where, by definition,  $Q_1(z/\omega)$  solves

$$0 = \left\{ n_1\left(\frac{z}{\omega}\right) + n_1\left(-\frac{z}{\omega}\right) + \left[ n_1\left(\frac{z}{\omega}\right) - n_1\left(-\frac{z}{\omega}\right) \right] Q'_1\left(\frac{z}{\omega}\right) \right\} \mu + \frac{\sigma^2}{2} Q''_1\left(\frac{z}{\omega}\right)$$

subject to  $Q_1(-1) = Q_1(1) = 0$ . Substituting  $Q'_\omega(z) = Q'_1(z/\omega)$ ,  $Q''_\omega(z) = Q''_1(z/\omega)/\omega$  and  $n_1\left(\frac{z}{\omega}\right) = \omega n_\omega(z)$  from (5) in the last ODE for  $z \in [-\omega, \omega]$  yields

$$\begin{aligned} 0 &= \left\{ \omega n_\omega(z) + \omega n_\omega(-z) + [\omega n_\omega(z) - \omega n_\omega(-z)] Q'_\omega(z) \right\} \mu + \frac{\sigma^2}{2} \omega Q''_\omega(z) \\ &= \left\{ n_\omega(z) + n_\omega(-z) + [n_\omega(z) - n_\omega(-z)] Q'_\omega(z) \right\} \mu + \frac{\sigma^2}{2} Q''_\omega(z) \end{aligned}$$

so  $Q_\omega(z)$  solves (24) for  $n(z) = n_\omega(z)$  and  $z \in [-\omega, \omega]$ . In addition

$$\begin{aligned} Q_\omega(-\omega) &= -\omega Q_1\left(-\frac{\omega}{\omega}\right) = -\omega Q_1(-1) = -\omega \cdot 0 = 0 \\ Q_\omega(\omega) &= \omega Q_1\left(\frac{\omega}{\omega}\right) = \omega Q_1(1) = \omega \cdot 0 = 0 \end{aligned}$$

so  $Q_\omega$  also solves the appropriate boundary conditions at  $z = \pm\omega$ . Therefore the guess is verified.

In particular

$$Q_\omega(0) = \omega Q_1(0).$$

Since  $Q_1(0) > 0$ , expected output at the beginning of the game is linear and unbounded in the finish line  $\omega$ .

For effort costs, we guess

$$K_1\left(\frac{z}{\omega}\right) = K_\omega(z)$$

plugging back and using again  $n_1\left(\frac{z}{\omega}\right) = \omega n_\omega(z)$

$$0 = \frac{c_0}{2\omega^2} \left\{ n_1\left(\frac{z}{\omega}\right) + n_1\left(-\frac{z}{\omega}\right) \right\} + \frac{1}{\omega} \left[ n_1\left(\frac{z}{\omega}\right) - n_1\left(-\frac{z}{\omega}\right) \right] \frac{1}{\omega} K_1'\left(\frac{z}{\omega}\right) \mu + \frac{\sigma^2}{2\omega^2} K_\omega''(z)$$

simplifying  $\omega^2$  we see that the guess is verified, and the boundary conditions hold. So

$$K_\omega(0) = K_1(0)$$

the expected effort costs are independent of the finish line: players spread themselves thinner on a wider game domain. Clearly, if the expected squared effort  $K_\omega$  is constant in  $\omega$  the expected effort  $Q_\omega$  must be increasing. So

$$U_\omega(0) = \omega Q_1(0) - K_1(0)$$

output net of effort costs is linear in the finish line. There exists no optimal finish line. Thus, we set  $\omega = 1$  and we restrict attention to the choice of the optimal prize  $\pi$ .

## 4.2 Solution of the Principal's Objective

To recap: we look for a solution  $m$  to the parameter-free ODE (12) subject to (7). We then plug this function into the Principal's objective at  $z = 0$  and we use (10) and the definition of  $\delta$  to write ex ante expected output gross and net of effort costs, as follows:

$$Q(0) = 2 \int_{-1}^0 \left[ \int_j^0 \frac{m(s) + m(-s)}{\exp \left\{ 2 \int_j^s [m(-x) - m(x)] dx \right\}} ds \right] dj \quad (25)$$

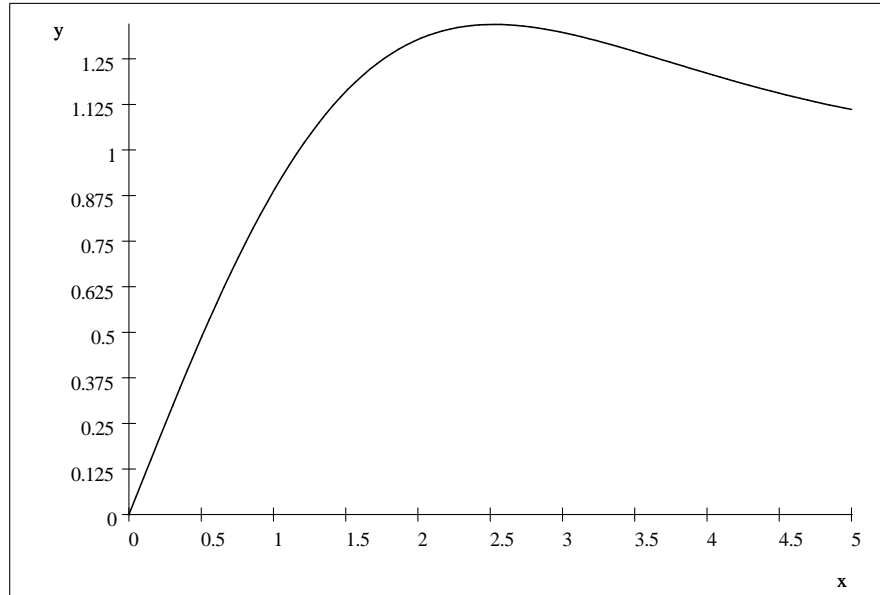
$$U(0) = \int_{-1}^0 \int_j^0 \frac{2 [m(s) + m(-s)] - \delta^{-1} [m^2(s) + m^2(-s)]}{\exp \left\{ 2 \int_j^s [m(-x) - m(x)] dx \right\}} ds dj. \quad (26)$$

Since the equilibrium strategy  $m$  only depends on parameters through  $\delta\pi$ , and the principal's value  $U$  depends only on  $m(s) = m(s|\delta\pi)$  and  $\delta$ , we have reduced the parameters of the problem from four  $(\mu, \sigma, c_0, \pi)$  to two:  $\delta = \frac{\mu^2}{\sigma^2 c_0}$  and  $\pi$ .

In the Appendix we show how to use the change of variable  $r = \rho(z)$  to integrate (25) and obtain an explicit solution for the expected effort and output in the race as a function of the composite parameter  $\pi\delta$ :

$$Q(0) = \frac{e^{3\pi\delta} - 6e^{\pi\delta}\pi\delta - e^{2\pi\delta} - e^{\pi\delta} + 6\pi\delta e^{2\pi\delta} + 1}{(e^{\pi\delta} - e^{-\pi\delta} + 2\pi\delta)(e^{\pi\delta} + e^{2\pi\delta})}. \quad (27)$$

For  $x = \pi\delta$  let  $W(x)$  be the function such that  $Q(0) = W(\pi\delta)$ . Here is a plot of  $W(x)$  for  $x > 0$



This value is maximized at (approx.)  $\delta\pi = 2.527$ . This result is surprising. The optimal prize is finite even if it has no opportunity cost. Contrast this result to a one-agent incentive problem, where the agent is motivated to exert effort by either a flow payment or a prize once he achieves a certain target. Clearly, as long as the Principal does not care either about the agent’s effort cost and the opportunity cost of the prize, the optimal payment is unbounded above.

Our earlier findings suggest the reason for this result. When the prize is very high, players engage in an initial “war” phase, where their efforts are very large near  $z = 0$ . The reason is that the prize justifies the high effort, and strategic complementarity reinforces that. Large efforts nearly cancel out and the score is moved by randomness. Once a player attains a lead, he can first reinforce it and then maintain it using the threat that, should the opponent try to catch up, the game would enter again the “war” phase. A player can always react in time because the score  $z$  evolves continuously. So both players maintain low efforts unless the score is almost a tie. This equilibrium pattern makes output rise modestly with the prize and is terrible for cost smoothing. The finite optimal prize makes for a more “interesting” race, fought all the way through.

If the principal cares about money and maximizes  $Q(0) - \pi$ , then clearly the profit-

maximizing prize is even smaller. In this case we look for the  $x = \delta\pi$  such that

$$\pi^*(\delta) = \arg \max_p \{W(\delta p) - p\}.$$

Plotting the FOC  $W'(p\delta)\delta = 1$  as a function of  $\delta$  reveals that the solution  $\pi^*(\delta)$  is increasing in  $\delta$ , with  $\pi^*(1) = 0$ . For any  $\delta < 1$  the expected profit is negative and the principal does not want to use the contest. For every  $\delta > 1$ , he does. For example, for  $\delta = 2$  we obtain  $\pi = 0.67$ .

To further illuminate the intuition behind the single-peaked contest performance as a function of the normalized prize, we also characterize the expected duration of the race as a function of parameters.

### 4.3 Comparison with Holmstrom Milgrom

Let  $\mu = \sigma = 1$ . Then  $m(z) = n(z)$  and the only parameter left is  $c_0 = 1/\delta$ . HM show that two independent contracts of time length 1 each offered to risk neutral workers implement the first best effort policy, and yield the principal a total profit of  $1/c_0$  (twice  $1/2c_0$ ). In our notation, the principal's total expected payoff is  $\delta$ . More generally, we can take  $T\delta$  to be the expected net payoff to the principal from two efficient HM contracts of length  $T$  each.

We investigate whether a contest can do better for the principal. To do so, we find the reward  $\pi_T$  that guarantees a race of expected duration  $T$ , and ask whether the principal's value  $Q(0) - \pi_T$  at that level of the reward exceeds  $\delta$  for some  $T > 0$ . For comparison with HM, we need to subtract the prize from the principal's expected payoff. If the principal did not care about money, in HM he would offer the agent an infinite amount.

Since an expected duration of 1 can be obtained only in the limit with a zero prize, which yields the principal no expected out, evidently the HM contracts do better whenever  $T \geq 1$ . But we can reformulate the question. Suppose we let the two HM contracts run only for a period of deterministic length  $T(2e^{\delta\pi} - 1) < 1$ , with net profits  $\delta T (2e^{\delta\pi} - 1)$ . By construction, these contracts last exactly as long as a contest of prize  $\delta\pi$  (in expectation.) The payoff to the principal from such a contest is  $Q(0|\delta\pi) - \pi$ . So we ask whether there exists  $\delta$  and  $\pi$  such that

$$Q(0|\delta\pi) - \pi > \delta \cdot T (2e^{\delta\pi} - 1).$$

Recall that  $W$  denotes the function such that  $Q(0|\delta\pi) = W(\delta\pi)$ . Then we ask whether there



exist  $\delta, \pi$  such that

$$W(x) - \frac{x}{\delta} > \delta T(2e^x - 1)$$

For example, for  $\delta = 2$  the optimal contest prize is  $\pi = .67$  and the resulting net payoff is approximately 0.41. The corresponding duration is

$$T(2e^{1.34} - 1) = 0.76103$$

and the principal gets twice that from a pair of HM contracts, which is better than 0.41,

For  $\delta = 10$ , the optimal contest prize is about 0.225 and the net payoff is approximately 1.1. The expected duration is

$$T(2e^{2.25} - 1) = 0.52510$$

and the principal obtains ten times that from two separate HM contracts, which is better than 1.1.

One limitation of the contest, relative to HM's incentive contracts, is agents' participation. Since the HM contract for risk neutral agents is efficient, it yields the agent the highest possible expected payoff, so agents will be more willing to accept an individual incentive contract than a contest.

## 5 Designing a Race: the Optimal Scoring Function (in progress)

Suppose now that the principal designs the rules of the game in terms of a "score"

$$s = f(z) \tag{28}$$

so that player  $B$  wins if  $s = \omega$ . We aim to solve for the SMPE of this game  $n^f$ . Then the principal solves the same problem as before, namely, he cares about actual output and effort costs, not about the score per se, so he maximizes either  $Q(0)$  or  $U(0)$ , where the strategy  $n^f$  replaces  $m$ . Since the principal can always set  $s = z$ , as in the previous case, and obtain an objective that grows linearly in  $\omega$ , to make the problem nontrivial we still set  $\omega = 1$ . For simplicity, we also omit the superscript  $f$  and let  $n : [-1, 1] \rightarrow R_+$  be the equilibrium strategy in score space.

We assume that  $f$  is a  $C^2$  function, so that by Ito's lemma

$$ds_t = f'(z_t) (n_{At} - n_{Bt}) \mu dt + \sigma f'(z_t) dW_t + \frac{\sigma^2 f''(z_t)}{2} dt.$$

**Conjecture 8** *The optimal scoring function  $f$  is increasing.*

If it was not, then player  $A$  would like to let output decline locally to give the score the best chance to rise, by setting effort to zero (effort cannot be negative, one cannot “intentionally score an own goal,” say.) Zero effort is detrimental to the principal.

If  $f$  is increasing and thus invertible, then

$$\phi(s) := f'(f^{-1}(s)) = f'(z) \quad (29)$$

so that

$$\phi'(s) = \frac{f''(f^{-1}(s))}{f'(f^{-1}(s))} = \frac{f''(z)}{\phi(s)}$$

and replacing

$$ds_t = \phi(s_t) \left\{ \left[ (n_{At} - n_{Bt}) \mu + \frac{\sigma^2}{2} \phi'(s_t) \right] dt + \sigma dW_t \right\}$$

The winner is the first to hit score  $s = 1$  or  $s = -1$ .

The HJB equation in a SMPE of this distorted game is

$$0 = \max_u -c_0 \frac{u^2}{2} + \mu [u - n(-s)] \phi(s) \bar{V}'(s) + \frac{\sigma^2 \phi'(s) \phi(s)}{2} \bar{V}'(s) + \frac{\sigma^2}{2} \phi^2(s) \bar{V}''(s).$$

The NFOC for optimal effort is

$$c_0 n(s) = \mu \phi(s) \bar{V}'(s)$$

substituting back

$$\begin{aligned} 0 &= -c_0 \frac{\mu \phi(s) \bar{V}'(s)}{2} + \mu \left[ \mu \phi(s) \bar{V}'(s) - \mu \phi(-s) \bar{V}'(-s) \right] \phi(s) \bar{V}'(s) + \frac{\sigma^2 \phi'(s) \phi(s)}{2} \bar{V}'(s) + \frac{\sigma^2 \phi^2(s)}{2} \bar{V}''(s) \\ &= \frac{\mu^2 \phi(s)}{c_0} \left[ \frac{1}{2} \phi(s) \bar{V}'(s) - \phi(-s) \bar{V}'(-s) \right] \bar{V}'(s) + \frac{\sigma^2 \phi'(s) \phi(s)}{2} \bar{V}'(s) + \frac{\sigma^2}{2} \phi^2(s) \bar{V}''(s) \\ 0 &= \frac{\mu^2}{c_0} \left[ \frac{1}{2} \bar{V}'(s) - \frac{\phi(-s)}{\phi(s)} \bar{V}'(-s) \right] \bar{V}'(s) + \frac{\sigma^2 \phi'(s)}{2 \phi(s)} \bar{V}'(s) + \frac{\sigma^2}{2} \bar{V}''(s) \end{aligned}$$

Rearranging and using the definition of  $\delta$

$$\begin{aligned} \bar{V}''(s) &= \frac{\mu^2}{c_0 \sigma^2} \left[ 2 \frac{\phi(-s)}{\phi(s)} \bar{V}'(-s) - \bar{V}'(s) \right] \bar{V}'(s) - \frac{\phi'(s)}{\phi(s)} \bar{V}'(s) \\ &= \delta \left[ 2 \frac{\phi(-s)}{\phi(s)} \bar{V}'(-s) - \bar{V}'(s) \right] \bar{V}'(s) - \frac{\phi'(s)}{\phi(s)} \bar{V}'(s). \end{aligned}$$

If  $s = z$ , we have  $\phi(s) = 1$  and  $\phi'(s) = 0$ , so this reduces to the familiar equation (2).

Once again, normalize the value function

$$V(z) = \delta \bar{V}(z)$$

so that the value of the game to the player solves

$$\begin{aligned} \frac{V''(s)}{\delta} &= \delta \left[ 2 \frac{\phi(-s)}{\phi(s)} \frac{V'(-s)}{\delta} - \frac{V'(s)}{\delta} \right] \frac{V'(s)}{\delta} - \frac{\phi'(s)}{\phi(s)} \frac{V'(s)}{\delta} \\ V''(s) &= \left[ 2 \frac{\phi(-s)}{\phi(s)} V'(-s) - V'(s) \right] V'(s) - \frac{\phi'(s)}{\phi(s)} V'(s) \end{aligned}$$

subject to

$$V(-1) = 0 \text{ and } V(1) = \delta\pi.$$

In terms of strategies, let

$$m(s) = \phi(s)V'(s)$$

which clearly depends only on the scoring function through  $\phi$  and on  $\delta\pi$  through  $V$ . As before, we will derive an ODE for  $m$ . As in the original, undistorted game, from the solution  $m(s)$  we can then recover the equilibrium strategy of the original game

$$n(s) = \frac{\mu}{c_0} \phi(s) \bar{V}'(s) = \frac{\mu}{c_0} \phi(s) \frac{V'(s)}{\delta} = \frac{\mu}{c_0 \delta} \phi(s) V'(s) = \frac{\mu}{c_0 \delta} m(s).$$

Plugging back this effort strategy into the Principal's objective, we see that the latter depends only on the composite parameter  $\delta$  and on the prize  $\pi$ , for every scoring function  $\phi$ .

Differentiating  $m$

$$m'(s) = \phi'(s)V'(s) + \phi(s)V''(s)$$

we get

$$\begin{aligned} V'(s) &= \frac{m(s)}{\phi(s)} \\ V''(s) &= \frac{m'(s)}{\phi(s)} - \frac{\phi'(s)}{\phi(s)} V'(s) = \frac{m'(s)}{\phi(s)} - \frac{\phi'(s)}{\phi(s)} \frac{m(s)}{\phi(s)} \end{aligned}$$

plugging back into the HJB equation

$$\frac{m'(s)}{\phi(s)} - \frac{\phi'(s)}{\phi(s)} \frac{m(s)}{\phi(s)} = \left[ 2 \frac{\phi(-s)}{\phi(s)} \frac{m(-s)}{\phi(-s)} - \frac{m(s)}{\phi(s)} - \frac{\phi'(s)}{\phi(s)} \right] \frac{m(s)}{\phi(s)}.$$

After simplifying, we obtain a DDE in effort  $m$ ,

$$m'(s)\phi(s) = 2m(s) \left[ m(-s) - \frac{1}{2}m(s) \right] \quad (30)$$

which is almost identical to (6), in fact identical except for the  $\phi(s)$  on the LHS.

To turn (30) into an Ordinary DE, we use its two equivalent forms.

$$\begin{aligned} m(-s) &= \frac{1}{2}m(s) + \frac{m'(s)\phi(s)}{2m(s)} \\ m'(-s) &= 2\frac{m(-s)}{\phi(-s)} \left[ m(s) - \frac{1}{2}m(-s) \right] \end{aligned}$$

We proceed as before: differentiating once more (30)

$$m''(s)\phi(s) + m'(s)\phi'(s) = 2 \{ m'(s)[m(-s) - m(s)] - m(s)m'(-s) \}.$$

Substituting from the DDE we get

$$\begin{aligned} \phi(s)m''(s) + m'(s)\phi'(s) &= 2m'(s) \left[ m(-s) - \frac{1}{2}m(s) - \frac{1}{2}m(s) \right] - 2m(s) \cdot 2\frac{m(-s)}{\phi(-s)} \left[ m(s) - \frac{1}{2}m(-s) \right] \\ &= 2m'(s) \left[ \frac{m'(s)\phi(s)}{2m(s)} - \frac{1}{2}m(s) \right] - 2m(s) \cdot 2\frac{\frac{1}{2}m(s) + \frac{m'(s)\phi(s)}{2m(s)}}{\phi(-s)} \left\{ m(s) - \frac{1}{2} \left[ \frac{1}{2}m(s) + \frac{m'(s)\phi(s)}{2m(s)} \right] \right\} \\ &= \frac{[m'(s)]^2 \phi(s)}{m(s)} - m'(s)m(s) - m(s) \frac{m(s) + \frac{m'(s)\phi(s)}{m(s)}}{\phi(-s)} \left\{ 2m(s) - \frac{1}{2}m(s) - \frac{m'(s)\phi(s)}{2m(s)} \right\} \\ &= \frac{[m'(s)]^2 \phi(s)}{m(s)} - m'(s)m(s) - \frac{m(s)}{\phi(-s)} \left\{ m(s) + \frac{m'(s)\phi(s)}{m(s)} \right\} \left\{ \frac{3}{2}m(s) - \frac{m'(s)\phi(s)}{2m(s)} \right\} \\ &= \frac{[m'(s)]^2 \phi(s)}{m(s)} - m'(s)m(s) - \frac{m(s)}{\phi(-s)} \left\{ \frac{3}{2}m^2(s) + \frac{3}{2}m'(s)\phi(s) - \frac{m'(s)\phi(s)}{2} - \frac{1}{2} \left[ \frac{m'(s)\phi(s)}{m(s)} \right]^2 \right\} \\ &= \frac{[m'(s)]^2 \phi(s)}{m(s)} - m'(s)m(s) - \frac{m(s)}{\phi(-s)} \left\{ \frac{3}{2}m^2(s) + m'(s)\phi(s) - \frac{1}{2} \left[ \frac{m'(s)\phi(s)}{m(s)} \right]^2 \right\} \\ &= \frac{[m'(s)]^2 \phi(s)}{m(s)} - m'(s)m(s) - \frac{3}{2} \frac{m^3(s)}{\phi(-s)} - m(s)m'(s) \frac{\phi(s)}{\phi(-s)} + \frac{1}{2} \frac{[m'(s)\phi(s)]^2}{m(s)\phi(-s)} \end{aligned}$$

so finally

$$\phi(s)m''(s) = -m'(s)\phi'(s) + \frac{[m'(s)]^2 \phi(s)}{m(s)} \left[ 1 + \frac{\phi(s)}{2\phi(-s)} \right] - m'(s)m(s) \left[ 1 + \frac{\phi(s)}{\phi(-s)} \right] - \frac{3}{2} \frac{m^3(s)}{\phi(-s)} \quad (31)$$

subject to

$$\int_{-1}^1 V'(s)ds = \int_{-1}^1 \frac{m(s)}{\phi(s)} ds = \delta\pi. \quad (32)$$

Given all parameters, as summarized by  $\delta$ , a scoring function  $f$  and the implied marginal score function  $\phi(s) := f'(f^{-1}(s))$ , the SMPE of the contest is the function  $m$  that solves (31) subject to (32). Given this  $m$ , the principal obtains payoffs  $Q$  or  $U$  as defined in (25) or (26). The principal chooses the optimal prize  $\pi$  and scoring function  $f$  to maximize his payoff. With a nonlinear scoring function  $f$ , the solution to (25) is no longer (27), but has to be found anew for given  $f$ . We are currently attempting to find the relevant change of variable corresponding to  $r = \rho(z)$  in the natural score case.

We can immediately replicate some of the earlier results, using the assumption that  $\phi > 0$  everywhere (scoring is strictly increasing).

First,  $m'(s) = 0$  implies

$$m''(s)\phi(s) = -\frac{3m^3(s)}{2\phi(-s)} \leq 0.$$

If  $m(s) = 0$  then the stationary point  $s$  must be a local minimum, because effort cannot become negative. But then we require  $m'''(s) > 0$ . Using (31), we see that  $m(s) = m'(s) = m''(s) = 0$  implies  $m'''(s) = 0$ , and so on by induction, so that  $m(s)$  is identically zero everywhere, which cannot be possibly optimal for the principal. Therefore, any stationary point must be a local maximum and equilibrium effort remains either globally increasing or hump-shaped.

Second, at the outset of the game

$$m'(0)\phi(0) = m^2(0) > 0$$

so effort is increasing at first, otherwise it would be again identically zero on the whole score domain.

Third, the leader tries harder than the follower.

Define  $q(z) = m(z) - m(-z)$ , with  $q(0) = 0$ . We claim that  $q(z) \geq 0$  as  $z \geq 0$ . Using (30)

$$\begin{aligned} q'(z) &= m'(z) + m'(-z) \\ &= \frac{m(z)}{\phi(z)}[2m(-z) - m(z)] + \phi(-z)[2m(z) - m(-z)] \\ &= 2m(z)m(-z) \left[ \frac{1}{\phi(z)} + \frac{1}{\phi(-z)} \right] - \frac{m^2(z)}{\phi(z)} - \frac{m^2(-z)}{\phi(-z)}. \end{aligned}$$

So

$$q'(0) = \frac{2m^2(0)}{\phi(0)} > 0.$$

Let  $z'$  be the least  $z' \in (0, 1)$  with  $q(z') = 0$ . Then  $m(z') = m(-z')$ , and

$$q'(z') = \frac{2m^2(z')}{\phi(z')} > 0.$$

But then there is a smaller zero  $z'' \in (0, z')$  of  $q(z)$ , contradicting the choice of  $z'$ .

Now we try the same transformation as in the standard case. Let

$$\rho(s) = \frac{m'(s)\phi(s)}{m^2(s)}$$

taking a derivative and replacing from the ODE for  $m$  and simplifying

$$\begin{aligned} -\frac{2\rho'(s)}{m(s)} &= \left[ \frac{2}{\phi(s)} - \frac{1}{\phi(-s)} \right] \rho^2(s) + 2 \left[ \frac{1}{\phi(s)} + \frac{1}{\phi(-s)} \right] \rho(s) + \frac{3}{\phi(s)} \\ \rho'(s) &= -\frac{m(s)}{\phi(s)} \left[ \rho^2(s) + \rho(s) + \frac{3}{2} \right] + \frac{1}{\phi(-s)} \rho(s) \left[ 1 - \frac{\rho(s)}{2} \right] \end{aligned}$$

First, at  $s = 0$  we get immediately  $\rho'(0) < 0$  because  $\rho(0) = 2 - 1 = 1$ .

Second,  $\rho'(s) < 0$  for  $\rho(s) < 2$ , so as  $s$  rises above 0  $\rho$  keeps declining.  $\rho(s) < 0$  for all  $s \geq 0$ .

To establish  $\rho' < 0$  a sufficient condition is

$$\frac{2}{\phi(s)} > \frac{1}{\phi(-s)} > \frac{1}{2\phi(s)}$$

Next

$$\rho' = -mH(\rho, s)$$

implies

$$\begin{aligned} \rho'' &= m' \frac{\rho'}{m} - m \left[ -\frac{\phi'(s)}{\phi^2(s)} \left[ \rho^2(s) + \rho(s) + \frac{3}{2} \right] + \frac{\rho'(s)}{\phi(s)} [2\rho(s) + 1] \right] \\ &\quad - m \left[ -\frac{\phi'(-s)}{\phi^2(-s)} \rho(s) \left[ 1 - \frac{\rho(s)}{2} \right] + \frac{\rho'(s)}{\phi(-s)} [1 - \rho(s)] \right] \end{aligned}$$

so if  $\rho'(s) = 0$ , which requires  $\rho(s) > 2$ ,

$$\rho'' = m \left\{ \frac{\phi'(s)}{\phi^2(s)} \left[ \rho^2(s) + \rho(s) + \frac{3}{2} \right] + \frac{\phi'(-s)}{\phi^2(-s)} \rho(s) \left[ 1 - \frac{\rho(s)}{2} \right] \right\}$$

using

$$\frac{1}{\phi(s)} \left[ \rho^2(s) + \rho(s) + \frac{3}{2} \right] = -\frac{1}{\phi(-s)} \rho(s) \left[ 1 - \frac{\rho(s)}{2} \right]$$

$$\begin{aligned}
\rho'' &= m \left\{ -\frac{\phi'(s)}{\phi(s)} \frac{1}{\phi(-s)} \rho(s) \left[ 1 - \frac{\rho(s)}{2} \right] + \frac{\phi'(-s)}{\phi^2(-s)} \rho(s) \left[ 1 - \frac{\rho(s)}{2} \right] \right\} \\
&= m \left[ 1 - \frac{\rho(s)}{2} \right] \frac{\rho(s)}{\phi(-s)} \left\{ \frac{\phi'(-s)}{\phi(-s)} - \frac{\phi'(s)}{\phi(s)} \right\}
\end{aligned}$$

this can be negative only if for some  $s < 0$

$$\frac{\phi'(-s)}{\phi(-s)} > \frac{\phi'(s)}{\phi(s)}$$

or for  $s > 0$

$$\frac{\phi'(s)}{\phi(s)} > \frac{\phi'(-s)}{\phi(-s)}.$$

But  $\phi(s) = f'(f^{-1}(s))$ , thus

$$\frac{\phi'(s)}{\phi(s)} = -\frac{d}{ds} \left( \frac{1}{f'(f^{-1}(s))} \right)$$

If  $f$  is concave then  $f'$  keeps falling as  $s$  rises above 0, so  $\frac{\phi'(s)}{\phi(s)}$  falls from  $\frac{\phi'(0)}{\phi(0)}$ , so this impossible. We conclude that for any concave scoring function  $\rho'(s) < 0$  everywhere, and we can change variables from  $s$  to

$$r = \rho(s) = \frac{m'(s)\phi(s)}{m^2(s)}.$$

Next let effort as a function of the new variable

$$\hat{m}(r) = m(\rho^{-1}(r))$$

$$\begin{aligned}
\hat{m}'(r) &= \frac{m'(\rho^{-1}(r))}{\rho'(\rho^{-1}(r))} \\
&= \frac{m'(\rho^{-1}(r))}{\hat{m}(r)} \frac{1}{\frac{1}{\phi(\rho^{-1}(r))} \left( r^2 + r + \frac{3}{2} \right) + \frac{1}{\phi(-\rho^{-1}(r))} r \left( 1 - \frac{r}{2} \right)} \\
&= \frac{r [\hat{m}(r)]^2}{\hat{m}(r)\phi(\rho^{-1}(r))} \frac{1}{\frac{1}{\phi(\rho^{-1}(r))} \left( r^2 + r + \frac{3}{2} \right) + \frac{1}{\phi(-\rho^{-1}(r))} r \left( 1 - \frac{r}{2} \right)}
\end{aligned}$$

so finally

$$\begin{aligned}
\hat{m}'(r) &= \frac{2r\hat{m}(r)}{(2r^2 + 2r + 3) + \frac{\phi(\rho^{-1}(r))}{\phi(-\rho^{-1}(r))} r (2 - r)} \\
&= \frac{2r\hat{m}(r)}{r^2 \left( 2 - \frac{\phi(\rho^{-1}(r))}{\phi(-\rho^{-1}(r))} \right) + 2r \left( 1 + \frac{\phi(\rho^{-1}(r))}{\phi(-\rho^{-1}(r))} \right) + 3}
\end{aligned}$$

This is an ODE in  $\hat{m}(r)$  which depends on the unknown function  $\rho$ .

**Conjecture 9** *The optimal scoring function  $f$  is concave.*

This conjecture says that it is optimal to penalize the leader, as the marginal net output  $z$  that he produces translates into lower and lower marginal score.

In the natural score scale  $s = z$  we have  $\phi(s) = 1$ ,  $\phi'(s) = 0$  and this boundary value problem reduces to (12).

With an affine scoring function  $f(z) = a + bz$  we have  $\phi(s) = f'(z) = b$  and  $\phi'(s) = 0$ .  
Let

$$u(s) = \frac{m(s)}{b}$$

then replacing into (31)-(32)

$$\begin{aligned} bu''(s) &= \frac{[bu'(s)]^2 b}{bu(s)} \left[1 + \frac{b}{2b}\right] - bu'(s)bu(s) \left[1 + \frac{b}{b}\right] - \frac{3b^3 u^3(s)}{2b} \\ \frac{u''(s)}{b} &= \frac{3[u'(s)]^2}{2u(s)} - 2u'(s)u(s) - \frac{3}{2}u^3(s) \end{aligned}$$

s.t.

$$\int_{-1}^1 u(s)ds = \delta\pi.$$

So the score location  $a$  is obviously irrelevant, but the score scale  $b$  is relevant.

To recap: for any prize  $\pi$  and scoring function  $f$ , which gives rise to  $\phi$  through (29), we solve the boundary value problem (31)-(32). The resulting solution  $m$  depends on parameters only through the composite parameter  $\delta\pi$ . This  $m$  is a scaled equilibrium strategy, that can be used directly in the objective function of the principal, gross output (25) or net output (26), without even bothering with recovering the true equilibrium strategy  $n$ .

Clearly, the problem of finding the optimal  $f$  is non-recursive.

For any given scoring function  $f$  we can solve (31)-(32) equation numerically, by computing the numerical equivalent of  $\phi$  and  $\phi'$ , and using as “pivot”

$$m'(0) = \frac{m^2(0)}{\phi(0)}.$$

We guess  $m(0)$ , compute  $m'(0)$ , then  $m''(0)$  from the ODE above, and extend the solution to  $[-1, 1]$ . Then we check whether the boundary condition (32) is satisfied, otherwise we iterate. A brute force algorithms considers and evaluated for the principal all increasing functions onto  $[-1, 1]$ . If  $f$  is discretized into a vector of size  $N$ , there is a finite set of such functions to be checked.



A restricted class of scoring functions is

$$s = z^\alpha$$

for some  $\alpha$  odd, so this power function is monotone and thus invertible. Then

$$\begin{aligned} f^{-1}(s) &= s^{\frac{1}{\alpha}} \\ \phi(s) &= \alpha s^{\frac{\alpha-1}{\alpha}} \\ \phi'(s) &= (\alpha - 1) s^{-\frac{1}{\alpha}} \end{aligned}$$

We use these expressions and search for the optimal  $\alpha$ . In particular, we can ask is the optimal  $\alpha$  positive or negative? I.e., is the leader encouraged or discouraged? Here we only need to look over a relatively small number of possible values of  $\alpha$ .

## 6 Conclusions

We have fully characterized the equilibrium strategies and the total expected performance of a stochastic dynamic contest between two players who compete for a prize based on unobservable effort but observable output. For tractability, we have restricted our analysis to undiscounted payoffs and quadratic effort costs. Discounting may introduce additional strategic considerations, but is likely to reinforce the result, as a laggard sees the prize farther and farther away not only in terms of effort costs but also in terms of time, thus cost of delay, necessary to win it. Our results are clearly robust to perturbations of the basic model, including to the cost function. We cannot think of any obvious reason why a different convexity of the cost function may overturn the results, but we plan to explore this issue in detail.

Our main objective is to exploit the expressions that we obtained for the total performance and the strategies of the players in the equilibrium of the contest to characterize the design of an optimal race. This has obvious applications to the design of workplace tournaments, R&D tournaments, and professional sports leagues. The space of mechanism is large, encompassing duration, prize, handicaps etc. The interesting trade-off arises from the need to promise, on the one hand, to support the leader ex post and to make it easy for him to maintain a lead, in order to elicit high effort from both players ex ante, and, on the

other hand, insurance to the laggard that he will always have a chance, to prevent him from giving up too early, because this would allow the leader to relax too and would make for an uninteresting and quickly resolved race.

# A Appendix

## A.1 A Useful Functional of the Equilibrium Effort Strategy

**Lemma 10** *Given the SMPE effort strategy  $m$  and the change of variable  $r = \rho(z)$ ,*

$$\exp \left\{ 2 \int_j^s [m(-x) - m(x)] dx \right\} = \left[ \frac{1 + \rho(s)}{1 + \rho(j)} \right]^2 \left[ \frac{3 + \rho(j)}{3 + \rho(s)} \right]^4.$$

**Proof.** Changing variable from  $y$  to  $r = \rho(y)$  and using (20) and (33)

$$m(-y) - m(y) = C \frac{1+r}{(3+r)^3} \left( -1 + \frac{1+r}{2} \right) = -\frac{C}{2} \frac{1-r^2}{(3+r)^3}$$

$$[m(-y) - m(y)] dy = \frac{1-r^2}{(3+r)^3} \left( \frac{3+r}{1+r} \right)^2 dr = \frac{1-r}{(3+r)(1+r)} dr$$

$$\begin{aligned} 2 \int_j^y [m(-x) - m(x)] dx &= 2 \int_{\rho(j)}^{\rho(y)} \frac{1-r}{(3+r)(1+r)} dr \\ &= 2 \int_{\rho(j)}^{\rho(y)} \left( \frac{1}{1+r} - \frac{2}{3+r} \right) dr \\ &= 2 \log \left( \frac{1 + \rho(y)}{1 + \rho(j)} \right) - 4 \log \frac{3 + \rho(y)}{3 + \rho(j)} \\ &= \log \left[ \frac{1 + \rho(y)}{1 + \rho(j)} \right]^2 \left[ \frac{3 + \rho(j)}{3 + \rho(y)} \right]^4 \end{aligned}$$

Taking antilogs we obtain the claim.

## A.2 Solving for the Expected Duration of the Contest

We want to solve the ODE (22). Let

$$h(z) = T'(z)$$

so that

$$h'(z) = -2 [m(z) - m(-z)] h(z) - 2$$

The solution of the associated homogeneous equation by variation of the arbitrary constant is

$$h(z) = K_0 e^{-2 \int_{-1}^z [m(s) - m(-s)] ds}$$

Differentiating

$$h'(z) = K_0' e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} - K_0 2 [m(z) - m(-z)] e^{-2 \int_{-1}^z [m(s) - m(-s)] ds}$$

while in the original ODE

$$h'(z) = -2[m(z) - m(-z)] K_0 e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} - 2$$

Equating terms

$$\begin{aligned} K_0' e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} &= -2 \\ K_0' &= -2 e^{2 \int_{-1}^z [m(s) - m(-s)] ds} \\ K_0 &= K_1 - 2 \int_{-1}^z e^{2 \int_{-1}^s [m(x) - m(-x)] dx} ds \end{aligned}$$

So finally

$$T'(z) = K_1 e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} - 2 e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} \int_{-1}^z e^{2 \int_{-1}^s [m(x) - m(-x)] dx} ds$$

and integrating

$$T(y) = K_2 + K_1 \int_{-1}^y e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} dz - 2 \int_{-1}^y e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} \int_{-1}^z e^{2 \int_{-1}^s [m(x) - m(-x)] dx} ds dz$$

Notice that the boundary condition  $T(-1) = 0$  implies  $K_2 = 0$ , while  $T(1) = 0$  implies

$$K_1 = \frac{2 \int_{-1}^1 e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} \int_{-1}^z e^{2 \int_{-1}^s [m(x) - m(-x)] dx} ds dz}{\int_{-1}^1 e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} dz}$$

so finally the solution is

$$\begin{aligned} T(y) &= \frac{2 \int_{-1}^1 e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} \int_{-1}^z e^{2 \int_{-1}^s [m(x) - m(-x)] dx} ds dz}{\int_{-1}^1 e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} dz} \int_{-1}^y e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} dz \\ &\quad - 2 \int_{-1}^y e^{-2 \int_{-1}^z [m(s) - m(-s)] ds} \int_{-1}^z e^{2 \int_{-1}^s [m(x) - m(-x)] dx} ds dz \end{aligned}$$

which is (22) in the text, using the definition of  $\phi(z)$ .

Next, by Lemma 10

$$\phi(z) = \frac{(2 + 2e^{\delta\pi})^4 [1 + \rho(z)]^2}{(2e^{\delta\pi})^2 [3 + \rho(z)]^4} = 4e^{-2\delta\pi} (1 + e^{\delta\pi})^4 \frac{[1 + \rho(z)]^2}{[3 + \rho(z)]^4}$$

We now repeatedly change variable from  $z$  to  $r = \rho(z)$  using

$$\rho(-1) = 2e^{\delta\pi} - 1; \rho(0) = 1; \rho(1) = 2e^{-\delta\pi} - 1.$$

$$dz = \Omega \left( \frac{3+r}{1+r} \right)^2 dr$$

where

$$\Omega = -\frac{2}{C} = \frac{1}{2(e^{-\delta\pi} - e^{\delta\pi} - 2\delta\pi)}.$$

First,

$$\begin{aligned}
\phi(z) \int_{-1}^z \frac{ds}{\phi(s)} &= \frac{[1 + \rho(z)]^2}{[3 + \rho(z)]^4} \int_{-1}^z \frac{[3 + \rho(s)]^4}{[1 + \rho(s)]^2} ds \\
&= \Omega \frac{[1 + \rho(z)]^2}{[3 + \rho(z)]^4} \int_{\rho(-1)}^{\rho(z)} \frac{(3+r)^4}{(1+r)^2} \left(\frac{3+r}{1+r}\right)^2 dr \\
&= \Omega \frac{[1 + \rho(z)]^2}{[3 + \rho(z)]^4} \int_{\rho(-1)}^{\rho(z)} \frac{(3+r)^6}{(1+r)^4} dr
\end{aligned}$$

Next,

$$\begin{aligned}
\int_{-1}^y \phi(z) \int_{-1}^z \frac{ds}{\phi(s)} dz &= \Omega \int_{-1}^y \frac{[1 + \rho(z)]^2}{[3 + \rho(z)]^4} \int_{\rho(-1)}^{\rho(z)} \frac{(3+s)^6}{(1+s)^4} ds dz \\
&= \Omega^2 \int_{\rho(-1)}^{\rho(y)} \frac{(1+r)^2}{(3+r)^4} \int_{\rho(-1)}^r \frac{(3+s)^6}{(1+s)^4} ds \left(\frac{3+r}{1+r}\right)^2 dr \\
&= \Omega^2 \int_{\rho(y)}^{\rho(-1)} \frac{1}{(3+r)^2} \int_r^{\rho(-1)} \frac{(3+s)^6}{(1+s)^4} ds dr
\end{aligned}$$

Next,

$$\begin{aligned}
\int_{-1}^y \phi(z) dz &= 4e^{-2\delta\pi} (1 + e^{\delta\pi})^4 \int_{-1}^y \frac{[1 + \rho(z)]^2}{[3 + \rho(z)]^4} dz \\
&= 4\Omega e^{-2\delta\pi} (1 + e^{\delta\pi})^4 \int_{\rho(-1)}^{\rho(y)} \frac{(1+r)^2}{(3+r)^4} \left(\frac{3+r}{1+r}\right)^2 dr \\
&= 4\Omega e^{-2\delta\pi} (1 + e^{\delta\pi})^4 \int_{\rho(-1)}^{\rho(y)} \frac{1}{(3+r)^2} dr \\
&= 4\Omega e^{-2\delta\pi} (1 + e^{\delta\pi})^4 \left[ \frac{1}{3 + \rho(-1)} - \frac{1}{3 + \rho(y)} \right] \\
&= 4\Omega e^{-2\delta\pi} (1 + e^{\delta\pi})^4 \frac{\rho(y) - \rho(-1)}{2(1 + e^{\delta\pi}) [3 + \rho(y)]} \\
&= 2\Omega e^{-2\delta\pi} (1 + e^{\delta\pi})^3 \frac{1 + \rho(y) - 2e^{\delta\pi}}{3 + \rho(y)}.
\end{aligned}$$

Therefore

$$\frac{\int_{-1}^0 \phi(z) dz}{\int_{-1}^1 \phi(z) dz} = \frac{\frac{1 + \rho(0) - 2e^{\delta\pi}}{3 + \rho(0)}}{\frac{1 + \rho(1) - 2e^{\delta\pi}}{3 + \rho(1)}} = \frac{\frac{2 - 2e^{\delta\pi}}{4}}{\frac{2e^{-\delta\pi} - 2e^{\delta\pi}}{2 + 2e^{-\delta\pi}}} = \frac{(e^{\delta\pi} - 1)(1 + e^{-\delta\pi})}{2(e^{\delta\pi} - e^{-\delta\pi})} = \frac{1}{2}$$

Putting it all together, the expected duration of the race is:

$$\begin{aligned}
T(0) &= 2 \int_{-1}^1 \int_{-1}^z \frac{ds}{\phi(s)} \phi(z) dz \cdot \frac{\int_{-1}^0 \phi(z) dz}{\int_{-1}^1 \phi(z) dz} - 2 \int_{-1}^0 \int_{-1}^z \frac{ds}{\phi(s)} \phi(z) dz \\
&= \Omega^2 \int_{\rho(1)}^{\rho(-1)} \frac{1}{(3+r)^2} \int_r^{\rho(-1)} \frac{(3+s)^6}{(1+s)^4} ds dr - 2\Omega^2 \int_{\rho(0)}^{\rho(-1)} \frac{1}{(3+r)^2} \int_r^{\rho(-1)} \frac{(3+s)^6}{(1+s)^4} ds dr \\
&= \Omega^2 \int_{2e^{-\delta\pi}-1}^{2e^{\delta\pi}-1} \frac{1}{(3+r)^2} \int_r^{2e^{\delta\pi}-1} \frac{(3+s)^6}{(1+s)^4} ds dr - 2\Omega^2 \int_1^{2e^{\delta\pi}-1} \frac{1}{(3+r)^2} \int_r^{2e^{\delta\pi}-1} \frac{(3+s)^6}{(1+s)^4} ds dr \\
&= \Omega^2 \left[ \int_{2e^{-\delta\pi}-1}^1 \frac{1}{(3+r)^2} \int_r^{2e^{\delta\pi}-1} \frac{(3+s)^6}{(1+s)^4} ds dr - \int_1^{2e^{\delta\pi}-1} \frac{1}{(3+r)^2} \int_r^{2e^{\delta\pi}-1} \frac{(3+s)^6}{(1+s)^4} ds dr \right]
\end{aligned}$$

Let  $x = 2e^{\delta\pi} - 1$ . Then the expected duration of a contest with normalized prize  $\delta\pi$  such that  $x = 2e^{\delta\pi} - 1 > 1$  is the integral expression in the text, that we now integrate. Let

$$D(s) = \int \frac{(3+s)^6}{(1+s)^4} ds = 73s + 7s^2 + \frac{1}{3}s^3 + 160 \ln(s+1) + \frac{-1728s - 720s^2 - 1072}{9s + 9s^2 + 3s^3 + 3}$$

$$B(r) = \int \frac{D(r)}{(3+r)^2} dr = 5r + \frac{1}{6}r^2 + 40 \ln(r+1) - \frac{160}{r+3} \ln(r+1) + \frac{630r + 255r^2 + 391}{21r + 15r^2 + 3r^3 + 9}$$

So overall for  $x = 2e^{\delta\pi} - 1$

$$\begin{aligned}
T(0|x) &= \left(1 + x - \frac{4}{1+x} + 4 \ln\left(\frac{1+x}{2}\right)\right)^{-2} \left[ \int_{\frac{3-x}{1+x}}^1 \frac{D(x) - D(r)}{(3+r)^2} dr - \int_1^x \frac{D(x) - D(r)}{(3+r)^2} dr \right] \\
&= \frac{D(x) \int_{\frac{3-x}{1+x}}^1 \frac{dr}{(3+r)^2} - B(1) + B\left(\frac{3-x}{1+x}\right) - D(x) \int_1^x \frac{dr}{(3+r)^2} + B(x) - B(1)}{\left(1 + x - \frac{4}{1+x} + 4 \ln\left(\frac{1+x}{2}\right)\right)^2} \\
&= \frac{D(x) \left(-\frac{1}{4} + \frac{1}{3+\frac{3-x}{1+x}} + \frac{1}{3+x} - \frac{1}{4}\right) + B\left(\frac{3-x}{1+x}\right) + B(x) - 2B(1)}{\left(1 + x - \frac{4}{1+x} + 4 \ln\left(\frac{1+x}{2}\right)\right)^2} \\
&= \frac{B\left(\frac{3-x}{1+x}\right) + B(x) - 63.5}{\left(1 + x - \frac{4}{1+x} + 4 \ln\left(\frac{1+x}{2}\right)\right)^2}
\end{aligned}$$

as claimed in the text.

### A.3 Solving for the Principal's Value.

By symmetry, the Principal's objective function  $Q$  is even and its derivative  $h = Q'$  is odd. Using this notation, to find  $Q$  we solve for  $h$  from the first order ODE

$$0 = n(z) + n(-z) + [n(z) - n(-z)] h(z) + \frac{\sigma^2}{2\mu} h'(z)$$

subject to

$$\int_{-\omega}^{\omega} h(z) dz = 0.$$

The homogenous equation is

$$h'(z) = \frac{2\mu}{\sigma^2} \{[n(-z) - n(z)] h(z)\}$$

so

$$h(z) = C_1 \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\}$$

Vary  $C_1(z)$  so

$$h'(z) = \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\} C_1'(z) + \frac{2\mu}{\sigma^2} \{[n(-z) - n(z)] h(z)\}$$

$$\frac{2\mu}{\sigma^2} \{-n(z) - n(-z)\} = \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\} C_1'(z)$$

$$C_1(z) = \int^z \frac{\frac{2\mu}{\sigma^2} \{-n(s) - n(-s)\}}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^s [n(-x) - n(x)] dx \right\}} ds + C_0$$

and the nonhomogenous equation has solution

$$h(z) = \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\} \left( C_0 - \frac{2\mu}{\sigma^2} \int_{-\omega}^z \frac{n(s) + n(-s)}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^s [n(-x) - n(x)] dx \right\}} ds \right)$$

To pin down  $C_0$ , we use symmetry of  $Q$  which implies  $Q'(0) = 0$ , so  $h(0) = 0$  and

$$C_0 = \frac{2\mu}{\sigma^2} \int_{-\omega}^0 \frac{n(s) + n(-s)}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^s [n(-x) - n(x)] dx \right\}} ds$$

so finally

$$\begin{aligned} h(z) &= \frac{2\mu}{\sigma^2} \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\} \int_z^0 \frac{n(s) + n(-s)}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^s [n(-x) - n(x)] dx \right\}} ds \\ &= \frac{2\mu}{\sigma^2} \int_z^0 \frac{n(s) + n(-s)}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_z^s [n(-x) - n(x)] dx \right\}} ds. \end{aligned}$$

We can verify that this function indeed solves the ODE for  $h$  subject to  $h(0) = 0$ .

From  $h$  and Lemma 10 we can recover the desired Principal's value:

$$\begin{aligned} Q(z) &= Q(-\omega) + Q(z) = \int_{-\omega}^z Q'(j) dj = \int_{-\omega}^z h(j) dj \\ &= \frac{2\mu}{\sigma^2} \int_{-\omega}^z \int_j^0 \frac{n(s) + n(-s)}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_j^s [n(-x) - n(x)] dx \right\}} ds dj. \end{aligned}$$

Next,  $U$  is also even, and  $h = U'$ , also symmetric. We solve the first order ODE

$$0 = -\frac{c_0}{2\mu} [n^2(z) + n^2(-z)] + n(z) + n(-z) + [n(z) - n(-z)]h(z) + \frac{\sigma^2}{2\mu}h'(z)$$

subject to

$$\int_{-\omega}^{\omega} h(z)dz = 0.$$

The homogenous equation is

$$h'(z) = \frac{2\mu}{\sigma^2} \{[n(-z) - n(z)]h(z)\}$$

so

$$h(z) = C_1 \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\}$$

Vary  $C_1(z)$  so

$$h'(z) = \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\} C_1'(z) + \frac{2\mu}{\sigma^2} \{[n(-z) - n(z)]h(z)\}$$

$$\frac{2\mu}{\sigma^2} \{-n(z) - n(-z)\} + \frac{c_0}{\sigma^2} [n^2(z) + n^2(-z)] = \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\} C_1'(z)$$

$$C_1(z) = \int_{-\omega}^z \frac{\frac{c_0}{\sigma^2} [n^2(s) + n^2(-s)] - \frac{2\mu}{\sigma^2} [n(s) + n(-s)]}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^s [n(-x) - n(x)] dx \right\}} ds + C_0$$

and the nonhomogenous equation has solution

$$h(z) = \exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^z [n(-x) - n(x)] dx \right\} \left( C_0 + \int_{-\omega}^z \frac{\frac{c_0}{\sigma^2} [n^2(s) + n^2(-s)] - \frac{2\mu}{\sigma^2} [n(s) + n(-s)]}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^s [n(-x) - n(x)] dx \right\}} ds \right)$$

To pin down  $C_0$ , we use symmetry of  $U$  which implies  $U'(0) = 0$ , so  $h(0) = 0$  and

$$C_0 = \int_{-\omega}^0 \frac{\frac{2\mu}{\sigma^2} [n(s) + n(-s)] - \frac{c_0}{\sigma^2} [n^2(s) + n^2(-s)]}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_{-\omega}^s [n(-x) - n(x)] dx \right\}} ds$$

so finally

$$h(z) = \int_z^0 \frac{\frac{2\mu}{\sigma^2} [n(s) + n(-s)] - \frac{c_0}{\sigma^2} [n^2(s) + n^2(-s)]}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_z^s [n(-x) - n(x)] dx \right\}} ds.$$

We can verify that this function indeed solves the ODE for  $h$  subject to  $h(0) = 0$ .

From  $h$  we can recover the desired Principal's value:

$$\begin{aligned} U(z) &= U(-\omega) + U(z) = \int_{-\omega}^z U'(j)dj = \int_{-\omega}^z h(j)dj \\ &= \int_{-\omega}^z \int_j^0 \frac{\frac{2\mu}{\sigma^2} [n(s) + n(-s)] - \frac{c_0}{\sigma^2} [n^2(s) + n^2(-s)]}{\exp \left\{ \frac{2\mu}{\sigma^2} \int_j^s [n(-x) - n(x)] dx \right\}} dsdj. \end{aligned}$$



The appropriate transformations give us (25) and (26). To integrate (25), and then similarly for (26), we change variable again to  $r = \rho(z)$ .

First, using (20) and the definition of  $\rho(z)$

$$m(-z) = m(z) \frac{m(-z)}{m(z)} = m(z) \frac{1 + \rho(z)}{2} \quad (33)$$

so for  $r = \rho(z)$

$$m(z) + m(-z) = C \frac{1+r}{(3+r)^3} \left(1 + \frac{1+r}{2}\right) = \frac{C}{2} \frac{1+r}{(3+r)^2}.$$

Using the change of variable (15) we obtain

$$[m(z) + m(-z)] dz = -\frac{1+r}{(3+r)^2} \left(\frac{3+r}{1+r}\right)^2 dr = -\frac{dr}{1+r}$$

Using Lemma 10 and putting all together,

$$\frac{[m(y) + m(-y)] dy}{\exp \left\{ 2 \int_j^s [m(-x) - m(x)] dx \right\}} = -\frac{dr}{1+r} \left[ \frac{3+r}{3+\rho(j)} \right]^4 \left[ \frac{1+\rho(j)}{1+r} \right]^2 = -\frac{[1+\rho(j)]^2 (3+r)^4}{[3+\rho(j)]^4 (1+r)^2} dr.$$

Using  $\rho(0) = 1$  and integrating for fixed  $j$ :

$$\int_j^0 \frac{m(y) + m(-y)}{\exp \left\{ 2 \int_j^y [m(-x) - m(x)] dx \right\}} dy = -\frac{[1+\rho(j)]^2}{[3+\rho(j)]^4} \int_{\rho(j)}^1 \frac{(3+r)^4}{(1+r)^2} dr$$

$$\int_1^{\rho(j)} \frac{(3+r)^4}{(1+r)^3} dr = 9\rho(j) - 24 \ln 2 + \frac{1}{2} [\rho(j)]^2 + 24 \ln(\rho(j) + 1) - \frac{32\rho(j) + 40}{[1+\rho(j)]^2} + \frac{17}{2}$$

Finally

$$\begin{aligned} Q(z) &= 2 \int_{-1}^z \left\{ 9\rho(j) - 24 \ln 2 + \frac{1}{2} [\rho(j)]^2 + 24 \ln(\rho(j) + 1) - \frac{32\rho(j) + 40}{[1+\rho(j)]^2} + \frac{17}{2} \right\} dj \\ &= 2 \int_{\rho(-1)}^{\rho(z)} \left\{ 9r - 24 \ln 2 + \frac{1}{2} r^2 + 24 \ln(r + 1) - \frac{32r + 40}{[1+r]^2} + \frac{17}{2} \right\} \left[ -\frac{2}{C} \left( \frac{3+r}{1+r} \right)^2 \right] dr \\ &= \frac{4}{C} \int_{\rho(z)}^{2e^{\delta\pi}-1} \frac{9r - 24 \ln 2 + \frac{1}{2} r^2 + 24 \ln(r + 1) - \frac{32r+40}{(1+r)^2} + \frac{17}{2}}{(3+r)^2} dr \end{aligned}$$

using  $C = 32m(0) = 32 \left( \frac{e^{\delta\pi} - e^{-\delta\pi}}{8} + \frac{\delta\pi}{4} \right) = 4(e^{\delta\pi} - e^{-\delta\pi}) + 8\delta\pi$  and  $\rho(0) = 1$ , integration yields

$$\begin{aligned} Q(0) &= \frac{4}{C} \int_1^{2e^{\delta\pi}-1} \frac{9r - 24 \ln 2 + \frac{1}{2} r^2 + 24 \ln(r + 1) - \frac{32r+40}{(1+r)^2} + \frac{17}{2}}{(3+r)^2} dr \\ &= \frac{4}{C(e^{\pi\delta} + e^{2\pi\delta})} (e^{3\pi\delta} - 6e^{\pi\delta}\pi\delta - e^{2\pi\delta} - e^{\pi\delta} + 6\pi\delta e^{2\pi\delta} + 1) \\ &= \frac{e^{3\pi\delta} - 6e^{\pi\delta}\pi\delta - e^{2\pi\delta} - e^{\pi\delta} + 6\pi\delta e^{2\pi\delta} + 1}{[(e^{\pi\delta} - e^{-\pi\delta}) + 2\pi\delta](e^{\pi\delta} + e^{2\pi\delta})} \end{aligned}$$

as in the text.