5. Nonlocal chiral charges were first shown to be Noether charges in L. Dolan and A. s, Phys. Rev. D **22** (1980), 2018.
6. M. Luscher and K. Pohlmeyer, Nuclear Phys. B **137** (1978), 46.
7. Y.-S. Wu, Nucl. Phys. B **211** (1983), 160.
8. In a private communication, S. Michailou has stated that the classical initial value blem for the nonlinear sigma model can be solved using a Riemann-Hilbert analysis.
9. C. N. Yang, Phys. Rev. Lett. **38** (1979), 1377.
10. L. Dolan, Proceedings of XI Internat. Colloq. Group Theoretical Methods in Physics, nbul. Turkey, August 1982.

EPARTMENT OF PHYSICS, ROCKEFELLER UNIVERSITY, NEW YORK, NEW YORK 10021

Volume 21, 1985

Representations of Kac-Moody Algebras and Dual Resonance Models

I. B. Frenkel[1]

CONTENTS

Introduction. The theories of Kac-Moody algebras and dual resonance models were born at approximately the same time (1968). The second theory underwent enormous development until 1974 (see reviews [**25**, **26**]) followed by years of decline, while the first theory moved slowly until the work of Kac [**14**] in 1974 followed by accelerated progress. Now both theories have gained considerable interest in their respective fields, mathematics and physics. Despite the fact that these theories have no common motivations, goals or problems, their formal similarity goes remarkably far. In this paper we discuss primarily the mathematical

1980 *Mathematics Subject Classification*. Primary 17B65, 81E99.
[1]Supported in part by NSF Grants MCS-8108814(A01) and MCS 81-02534.

theory. For a review of the physical theory see the paper of J. Schwarz in this volume [**27**].

At an early stage of the history of dual resonance models, it was understood that many results could be naturally interpreted in terms of the representation space V of the commutation relations

$$[\mathbf{x}^\mu, \mathbf{p}^\nu] = ig^{\mu\nu}, \qquad [\alpha^\mu(m), \alpha^\nu(n)] = mg^{\mu\nu}\delta_{m,-n} \tag{1}$$

where $\mu, \nu \in \{0, 1, \ldots, d-1\}$, $m, n \in \mathbf{Z}$, $g^{\mu\nu} = \operatorname{diag}(-1, 1, \ldots, 1)$. Therefore, V is the linear span of the elements of type[2]

$$\prod_{i=1}^{N} \alpha^{\mu_i}(m_i)|p\rangle \tag{2}$$

where $m_i < 0$, $\mathbf{p}^\mu|p\rangle = p^\mu|p\rangle$, $\mathbf{x}^\mu|p\rangle = ig^{\mu\nu}(\partial/\partial p^\mu)|p\rangle$, $p = p^\mu e_\mu \in \mathbf{R}^{1,d-1}$ and $\{e_\mu\}_{\mu=0}^{d-1}$ is a basis. Later it was observed that (1) is just the canonical quantization of a string moving in d-dimensional space-time; hence, the name of the theory was changed to dual string models. In order to obtain the dual amplitudes physicists defined a "vertex operator"

$$X(p, z) = \exp\left(p \cdot \sum_{n=1}^{\infty} \frac{\alpha(-n)}{n} z^n\right)$$
$$\cdot \exp(p \cdot (\mathbf{p}\log z - i\mathbf{x}))\exp\left(-p \cdot \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} z^{-n}\right), \tag{3}$$

where $p \cdot \alpha(-n)$ denotes

$$p_\mu \alpha^\mu(-n) = g_{\mu\nu} p^\mu \alpha^\nu(-n), \text{ etc.}$$

Ten years later this operator was reborn in the representation theory of Kac-Moody algebras. For some time mathematicians tried to obtain as simple a representation of the infinite-dimensional Lie algebras as they have for classical finite-dimensional Lie algebras. The solution was first given in [**21**] for one algebra, and Garland observed some similarity with dual resonance models. Then in [**9**, **28**] the "vertex construction" was found for the whole class of affine Lie algebras and the similarity became a precise correspondence.

It happens that if one restricts p in (2) to be in the even integer lattice called the root lattice, then the space V is exactly the space of an irreducible representation of an affine Lie algebra $\hat{\mathfrak{g}}$. Furthermore, the operators $\alpha^\mu(m)$, $\mathbf{p}^\mu$, together with the modified vertex operators $X(p, z) \cdot \varepsilon_p$, $\|p\|^2 = 2$, provide the "basic" representation of the affine Lie

[2] $|p\rangle$ is a standard bracket notation used in physics. In mathematical literature this element is usually denoted by e^p.

algebra $\hat{\mathfrak{g}}$. The only modification is the operator ε_p of multiplication by ± 1 on vectors $|q\rangle$, which nevertheless is quite important. In this paper we will consider two further applications of these vertex representations. The first application is the study of all the representations of $\hat{\mathfrak{g}}$ with a dominant highest weight (standard representations) inside the basic representation V. This is possible thanks to the existence of subalgebras $\hat{\mathfrak{g}}_{[n]} \subset \hat{\mathfrak{g}}$, $n \in \mathbf{N}$, isomorphic to $\hat{\mathfrak{g}}$. We prove the conjecture about multiplicities of irreducible representations of $\hat{\mathfrak{g}}_{[n]}$ in V which was formulated in [**8**].

As a second application of vertex representations, we give a construction of an arbitrary Kac-Moody algebra with real roots of equal length. This representation is not irreducible, but it yields some new information about hyperbolic algebras. In particular, we obtain the explicit form of operators corresponding to imaginary root vectors of zero norm. Again we should acknowledge the priority of physicists, who first introduced these operators [**4**]

$$A^i(n) = \int_C \alpha^i(z) X(nk, z)\, \frac{dz}{z} \tag{4}$$

where $n \in \mathbf{Z}$, k, $\alpha^i \in \mathbf{R}^{1,d-1}$, $\langle k, k\rangle = \langle k, \alpha^i\rangle = 0$, $\langle \alpha^i, \alpha^j\rangle = \delta_{ij}$. They noticed, in particular, that for fixed k these operators form a Heisenberg subalgebra

$$[A^i(m), A^j(n)] = m\delta_{ij}\delta_{m,-n}k(0). \tag{5}$$

Another important object in the dual resonance models is the Virasoro algebra spanned by the operators

$$L(n) = \frac{1}{2} \sum_{m=-\infty}^{\infty} :\alpha(-m) \cdot \alpha(n+m):, \tag{6}$$

where $: \,:$ denotes the normal ordering, i.e., $:\alpha(m) \cdot \alpha(n): = \alpha(m) \cdot \alpha(n)$ if $n \geqslant m$ and $= \alpha(n) \cdot \alpha(m)$ if $n < m$. These operators satisfy

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{d}{12}(m^3 - m)\delta_{m,-n}. \tag{7}$$

This algebra is known to mathematicians as a central extension of the algebra of vector fields on a circle. In dual resonance models the Virasoro algebra is used for the definition of "physical" subspaces of V:

$$V_c = \{v \in V: (L(n) - c\delta_{n,0})v = 0, n \geqslant 0\} \tag{8}$$

where c is a constant. In the representation theory the Virasoro algebra commutes with the Kac-Moody algebra $\mathfrak{g}$. The subspace V_c becomes a representation of $\mathfrak{g}$. It turns out that as such, V_c is irreducible whenever $V_c \neq 0$, if $\mathfrak{g} = \mathfrak{sl}(2)$, [**16**, **28**]. This is no longer true in general. In this paper we introduce a maximal commuting algebra $\hat{S}^W$ for arbitrary simple finite-dimensional $\mathfrak{g}$, which contains the Virasoro algebra. We obtain the decomposition

$$V = \sum_{\mu} V(\mu) \otimes \Omega(\mu) \tag{9}$$

where $V(\mu)$ is an irreducible finite-dimensional representation of $\mathfrak{g}$, and $\Omega(\mu)$ is the corresponding irreducible representation of $\hat{S}^W$. We also conjecture a resolution of $\Omega(\mu)$.

One of the highest achievements of the dual resonance models is the no-ghost theorem which asserts that for the critical dimension $d = 26$, the physical space V_1 contains no vectors of negative norm. We use this theorem in order to obtain an upper bound for the multiplicities of hyperbolic algebras of rank 26

$$\dim \mathfrak{g}_\alpha \leqslant \mu(-\langle \alpha, \alpha \rangle / 2), \tag{10}$$

where

$$\sum_{n=-1}^{\infty} \mu(n) q^n = \frac{1}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}}. \tag{11}$$

In fact, (10) is true for an algebra containing all the hyperbolic algebras of rank 26. This algebra, called the Monster Lie algebra, has been introduced in **[2]** with the hope that it will "explain" the biggest sporadic group F_1 of Fisher and Griess, known as Monster. One of the proofs of the no-ghost theorem uses the space $V_{1,k,p}$, spanned by the elements of type

$$\prod_{i=1}^{N} A^{m_i}(-n_i) | p \rangle \tag{12}$$

where $A^{m_i}(-n_i)$ is defined in (4), $\langle k, k \rangle = 0$ and satisfies the properties

$$\langle p, k \rangle = 1, \qquad \langle p, p \rangle = 2. \tag{13}$$

In the case when p is in the unique even unimodular lattice in $\mathbf{R}^{1,25}$, the space $V_{1,k} = \oplus_p V_{1,k,p}$, where p runs through all the elements satisfying (13), has the character

$$\operatorname{Ch} V_{1,k} = q^{-1} + \text{Const} + 196884q + \cdots. \tag{14}$$

Here, Const depends on k and its minimal value is 24. The number 196884 exceeds only by 1 the dimension of the minimal representation of F_1. It was conjectured in **[3]** that there is a "natural" representation of F_1 in a space with the character given by the right side of (14). Recently **[10]**, considerable progress was achieved in the understanding of the connection of dual resonance models and the natural representation of the Fisher-Griess Monster.

Predicting this connection, F. Dyson, in his talk at the von Humboldt Foundation Colloquium, said:

"Stranger things have happened in the history of physics than the unexpected appearance of sporadic groups." And he continued: "We have strong evidence that the creator of the universe loves symmetry, what lovelier symmetry could he find than the symmetry of the Monster?"

1. Kac-Moody algebras and standard representations.

1.1. We begin by recalling first the definition of Kac-Moody algebras. Let I be a finite subset in $\mathbf{Z}$ and $|I| = l$. An $l \times l$ integer matrix $A = (a_{ij})_{i,j \in I}$ is called a Cartan matrix if it satisfies the following properties:

(i) $a_{ii} = 2$, a_{ij} are nonpositive integers for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$, $i, j \in I$.

(ii) There exists a nondegenerate diagonal $l \times l$ matrix D such that the matrix DA is symmetric.

We denote by $\mathfrak{g}(A)$ a Lie algebra over $\mathbf{C}$ with $3l$ generators e_i, f_i, h_i, $i \in I$, and the following defining relations:

$$\begin{gathered} [h_i h_j] = 0, \quad [e_i, f_j] = -\delta_{ij} h_i, \\ [h_i, e_j] = a_{ij} e_j, \quad [h_i f_j] = -a_{ij} f_j, \\ (\operatorname{ad} e_i)^{1-a_{ij}} e_j = (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0, \qquad i \neq j, \end{gathered} \tag{1.1}$$

where $i, j \in I$. The algebra $\mathfrak{g}(A)$ is called a *Kac-Moody algebra* **[13, 24]**. In this paper we will suppose that A is symmetric.

We define the following notions: the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{\mathbf{R}} \otimes \mathbf{C}$, where $\mathfrak{h}_{\mathbf{R}} = \sum_{i \in I} \mathbf{R} h_i$; the bilinear symmetric form $\langle \, , \rangle$ on $\mathfrak{h}$, such that $\langle h_i, h_j \rangle = a_{ij}$, $i, j \in I$; the Weyl group W generated by reflections with respect to h_i in $\mathfrak{h}$; the root lattice $Q = \sum_{i \in I} \mathbf{Z} \alpha_i \subset \mathfrak{h}_{\mathbf{R}}$ (for symmetric A we identify $\alpha_i = h_i$), the weight lattice $P = \sum_{i \in I} \mathbf{Z} \omega_i \subset \mathfrak{h}_{\mathbf{R}}^*$, $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$; the system of real roots $\Delta_R = \bigcup_{i \in I} W \alpha_i \subset Q$. We will define several subalgebras of $\mathfrak{g}(A)$ as follows: $\mathfrak{n}^+$ generated by e_i, $i \in I$, $\mathfrak{n}$ generated by f_i, $i \in I$, $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$, $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$. There is a unique bilinear invariant symmetric form on $\mathfrak{g}(A)$ **[13, 24]**. We choose such a form, which extends the bilinear symmetric form $\langle \, , \rangle$ on $\mathfrak{h}$ and denote it by the same symbol.

Let ε: $Q \times Q \to \{\pm 1\}$ be a bilinear function satisfying the following conditions:

$$\begin{aligned} \varepsilon(\alpha, \beta) \varepsilon(\beta, \alpha) &= (-1)^{\langle \alpha, \beta \rangle}, \qquad \alpha, \beta \in Q, \\ \varepsilon(\alpha, \alpha) &= -1 \quad \text{for } \langle \alpha, \alpha \rangle = 2, \alpha \in Q. \end{aligned} \tag{1.2}$$

The existence of such a cocycle follows by extending its arbitrary definition on the basis elements $\varepsilon(\alpha_i, \alpha_j)$, $i < j$, $i, j \in I$.

Kac-Moody algebras admit a rather explicit description for a special choice of Cartan matrices. The first type of Kac-Moody algebras we are considering is distinguished by the condition that the Cartan matrix A is positive definite. The corresponding Lie algebras $\mathfrak{g} = \mathfrak{g}(A)$ are finite-dimensional simple Lie algebras with roots of equal length, i.e., of one of the types A_l, $l \geqslant 1$, D_l, $l \geqslant 4$, E_l, $l = 6, 7, 8$. These algebras have a simple realization in terms of a Chevalley basis:

$$\begin{gathered}
\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathbf{C} x_\alpha, \\
[h, x_\alpha] = \langle h, \alpha \rangle x_\alpha, \qquad h \in \mathfrak{h}, \\
[x_\alpha, x_\beta] = 0 \quad \text{if } \alpha + \beta \notin \Delta \cup 0, \\
[x_\alpha, x_\beta] = \varepsilon(\alpha, \beta) x_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \Delta, \\
[x_\alpha, x_{-\alpha}] = -\alpha.
\end{gathered} \tag{1.3}$$

The identification with the first definition is determined by setting $x_{\alpha_i} = e_i$, $x_{-\alpha_i} = f_i$, $i \in I$.

The second type of Kac-Moody algebras we consider is distinguished by the condition that the Cartan matrix is positive semidefinite. It can be defined as the unique extension of a positive-definite symmetric Cartan matrix A and will be denoted by $\hat{A}$. The corresponding Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}(\hat{A})$ is the affine Lie algebra (or loop algebra) with real roots of equal length, i.e., of one of the types $A_l^{(1)}$, $l \geqslant 1$, $D_l^{(1)}$, $l \geqslant 4$, $E_l^{(1)}$, $l = 6, 7, 8$. The Lie algebra $\hat{\mathfrak{g}}$ admits the following realization:

$$\begin{gathered}
\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c, \\
[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m \langle x, y \rangle \delta_{m,-n} c
\end{gathered} \tag{1.4}$$

where $x, y \in \mathfrak{g}$, $m, n \in \mathbf{Z}$, c is a central element in $\hat{\mathfrak{g}}$. We will identify $\mathfrak{g} = \mathfrak{g} \otimes 1$ and set $e_0 = x_{-\alpha} \otimes t$, $f_0 = x_\alpha \otimes t^{-1}$, $h_0 = c - \alpha$, where α is the maximal root in Δ. This provides one with the equivalence of the two definitions of $\hat{\mathfrak{g}}$. We denote by $\tilde{\mathfrak{g}}$ a semidirect product of $\hat{\mathfrak{g}}$ with $\mathbf{C}d$, where $d = t(d/dt)$ is a derivation of $\hat{\mathfrak{g}}$.

We will also consider a class of hyperbolic Kac-Moody algebras, which are determined by the condition that the signature of $\langle\,,\rangle$ on $\mathfrak{h}_{\mathbf{R}}$ has type $(1, l-1)$. In particular, if we define the overextended Cartan matrix $\widehat{A} = (a_{ij})_{i,j=-1}^{l}$ by the conditions $a_{0,-1} = -1$, $a_{0,i} = 0$, $i = 1, \ldots, l$, and $A = (a_{ij})_{i,j=0}^{l}$ is the extended Cartan matrix. Then the hyperbolic algebra $\widehat{\mathfrak{g}} = \mathfrak{g}(\widehat{A})$ admits the following construction [5]. Let V be the basic representation of $\tilde{\mathfrak{g}}$ with the grading starting from 1, i.e., the irreducible representation with the vector $v_0 \in V$ satisfying the conditions

$$\mathfrak{g} \otimes \mathbf{C}[t] v_0 = 0, \qquad c v_0 = d v_0 = v_0, \tag{1.5}$$

and let V^* be the contragradient representation. We define a map ϕ: $V^* \times V \to \tilde{\mathfrak{g}}$ by

$$\phi(v^*, v) = -\sum_{j \in J} \langle v^*, x_j v \rangle x_j \tag{1.6}$$

where $v^* \in V^*$, $v \in V$, $\{x_j : j \in J\}$ is an orthonormal basis for $\tilde{\mathfrak{g}}$ with respect to the form. We define a $\mathbf{Z}$-graded Lie algebra $\mathfrak{a} = \Sigma_{n \in \mathbf{Z}} \mathfrak{a}$ where $\mathfrak{a}_0 = \tilde{\mathfrak{g}}$, $\mathfrak{a}_{-1} = V$, $\mathfrak{a}_1 = V^*$, and where $\mathfrak{a}^+ = \Sigma_{n \geqslant 1} \mathfrak{a}_n$ and $\mathfrak{a}^- = \Sigma_{n \geqslant 1} \mathfrak{a}_{-n}$ are the free Lie algebras generated by $\mathfrak{a}_1$ and $\mathfrak{a}_{-1}$, respectively. The Lie brackets between $\mathfrak{a}$, and $\mathfrak{a}_{-1}$ are defined by $[v^*, v] = \phi(v^*, v)$ and between $\mathfrak{a}_k$ and $\mathfrak{a}_{-l}$, $k, l \geqslant 1$, are defined inductively. Let $\mathcal{J} = \mathcal{J}^+ + \mathcal{J}^-$, $\mathcal{J}^\pm = \Sigma_{k \geqslant 1} \mathcal{J}_k^\pm$, $\mathcal{J}_k^\pm = \{x \in \mathfrak{a}_{\pm k}: (\operatorname{ad} \mathfrak{a}_{\pm 1})^{k-1} x = 0\}$. Then we have

THEOREM 1.1 [5]. *$\mathcal{J}$ is an ideal in $\mathfrak{a}$ and $\mathfrak{a}/\mathcal{J} \approx \mathfrak{g}(\widehat{A})$.*
In particular one has for $\alpha \in \Delta$, such that $\langle \alpha, \omega_{-1} \rangle = 1$, that

$$\dim \widehat{\mathfrak{g}}_\alpha = p^{(l)}(-\langle \alpha, \alpha \rangle / 2 + 1) \tag{1.7}$$

where

$$\sum_{n=0}^{\infty} p^{(l)}(n) q^n = \varphi(q)^{-l} \tag{1.8}$$

and we denote $\varphi(q) = \prod_{m=1}^{\infty}(1 - q^m)$.

1.2. We recall Kac's results about standard representations [14]. Let us denote $Q^+ = \{\alpha = \Sigma_{i \in I} n_i \alpha_i,\ n_i \in \mathbf{Z}_+,\ i \in I\}$, $P^{++} = \{\lambda = \Sigma_{i \in I} h_i \omega_i,\ n_i \in \mathbf{Z}_+,\ i \in I\}$. An element $\lambda \in P^{++}$ is called a dominant highest weight. The standard representation on irreducible representation with dominant highest weight λ of Kac-Moody algebra $\mathfrak{g}$ is by definition the unique irreducible representation $V(\lambda)$ satisfying the following property: there exists a vector $v_0 \in V(\lambda)$ such that

$$\mathfrak{n}^+ v_0 = 0, \qquad h v_0 = \langle \lambda, h \rangle v_0, \quad h \in \mathfrak{h}. \tag{1.9}$$

Let $\mathbf{C}(P)$ denote the algebra of all formal sums of elements of the group algebra $\mathbf{C}[P]$ with support in a finite union of the sets $\nu + P^+$. Let V be a representation of $\mathfrak{g}$. We call the character of V the formal sum

$$\operatorname{ch} V = \sum_{\mu \in P} \dim V_\mu \cdot e^\mu \tag{1.10}$$

where $V_\mu = \{v \in V: hv = \langle \mu, h \rangle v \text{ for all } h \in \mathfrak{h}\}$. We will consider the category of representations V with $\operatorname{ch} V \in \mathbf{C}(P)$.

PROPOSITION 1.2 (KAC). (i) *For $\lambda \in P^{++}$ one has*

$$\operatorname{ch} V(\lambda) = \frac{\Sigma_{w \in W} \det(w) e^{w(\lambda+\rho)}}{e^\rho \prod_{\alpha \in \Delta_+}(1 - e^{-\alpha})}, \tag{1.11}$$

or equivalently for $\mu \in P^{++}$, $\mu \neq \lambda$,

$$\sum_{w \in W} \det(w) \dim V(\lambda)_{\mu+\rho-w\rho} = 0. \tag{1.12}$$

(ii) *In particular, for* $\lambda = 0$

$$\sum_{w \in W} \det(w) e^{w\rho} = e^{\rho} \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}), \tag{1.13}$$

where $\rho = \sum_{i \in I} \omega_i$.

When $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, Proposition 1.2 becomes the classical result of H. Weyl. Later on we will use the following corollary of (ii), in this case taking the value at $t(\lambda + \rho)$ and setting $q = e^{-t}$,

$$q^{\|\lambda\|^2/2} \prod_{\alpha \in \Delta_+} \left(1 - q^{\langle \alpha, \lambda+\rho \rangle}\right) = \sum_{w \in W} \det(w) q^{\|w(\lambda+\rho)-\rho\|^2/2}. \tag{1.14}$$

When the Kac-Moody algebra is an affine Lie algebra $\hat{\mathfrak{g}}$ (ii) is known as the Macdonald identity. Again taking the value at $t(\lambda + \hat{\rho})$ and setting $q = e^{-t}$, we get

$$q^{\|\lambda\|^2/2} \prod_{\alpha \in \hat{\Delta}_+} \left(1 - q^{\langle \alpha, \lambda+\hat{\rho} \rangle}\right) = \sum_{w \in \hat{W}} \det(w) q^{\|w(\lambda+\hat{\rho})-\hat{\rho}\|^2/2}. \tag{1.15}$$

In subsection 3.1 we will show how to "divide" the numerator of (1.11) by the denominator in the affine case. For the representations of level 1 the character of $V(\lambda)$ has an especially simple and beautiful form [**6, 15**]:

PROPOSITION 1.3. *Let* $\hat{\omega} = d + \omega$ *be a level* 1 *weight of* $\hat{\mathfrak{g}}$. *Then*

$$\operatorname{ch} V(\hat{\omega}) = \varphi(q)^{-l} \sum_{\gamma \in \omega + Q} e^{\gamma} \cdot q^{\langle \gamma, \gamma \rangle/2}, \tag{1.16}$$

where $q = e^{-c}$.

2. Vertex representations of Kac-Moody algebras.

2.1. Kac-Moody algebras admit a particular representation, which plays an important role in the theory. We will call it the vertex representation and we will use it for several applications later on. Now we will give a construction of the vertex representation.

Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with the nondegenerate invariant bilinear symmetric form $\langle\, ,\rangle$. We call the Lie algebra $\hat{\mathfrak{g}}(A) = \mathfrak{g}(A) \otimes \mathbf{C}[T, T^{-1}] \oplus \mathbf{C}C$, with the Lie bracket given by (1.4), the affinization of Kac-Moody algebra $\mathfrak{g}(A)$. In particular, the subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbf{C}[T, T^{-1}] \oplus \mathbf{C}C$ is called the Heisenberg subalgebra of $\hat{\mathfrak{g}}(A)$. We define the action of Q on $\hat{\mathfrak{h}}$ by the formula

$$\alpha \cdot h \otimes T^m = h \otimes T^m - \langle \alpha, h \rangle \delta_{0,m} C, \qquad \alpha \cdot C = C \tag{2.1}$$

where $\alpha \in Q$, $h \in \mathfrak{h}$. The pair $(\hat{\mathfrak{h}}, Q)$ is called the Heisenberg system of $\hat{\mathfrak{g}}(A)$. In this paper we will always use the notation $h(n)$ for $h \otimes T^n$.

Let Γ be a lattice in $\mathfrak{h}_{\mathbf{R}}$ such that $Q \subset \Gamma \subset P$, and let ε: $Q \times \Gamma \to \mathbf{C}_1^* = \{z \in \mathbf{C}\colon |z| = 1\}$ be a bilinear cocycle, whose restriction to $Q \times Q$ satisfies (1.2). We will fix a polarization of $\hat{\mathfrak{h}} = \mathfrak{h}^+ + (\mathfrak{h} + \mathbf{C}C) + \mathfrak{h}^-$, where $\mathfrak{h}^{\pm} = \mathfrak{h} \otimes T^{\pm}\mathbf{C}[T^{\pm 1}]$. Let $S(\mathfrak{h}^-)$ denote a symmetric algebra of $\mathfrak{h}^-$ and $\mathbf{C}[\Gamma]$ be a group algebra of Γ. We construct an irreducible projective representation with the cocycle ε of the Heisenberg system $(\hat{\mathfrak{h}}, Q)$ in the space $V = S(\mathfrak{h}^-) \otimes \mathbf{C}[\Gamma]$ in a standard way [**9**]:

$$\begin{aligned}
h(n) \cdot v \otimes e^{\alpha} &= n\left(\partial_{h(-n)} v\right) \otimes e^{\alpha}, \qquad n > 0,\\
h(-n) \cdot v \otimes e^{\alpha} &= \left(h(-n) v\right) \otimes e^{\alpha}, \qquad n > 0,\\
h(0) \cdot v \otimes e^{\alpha} &= v \otimes \left(\partial_h e^{\alpha}\right) = \langle h, \alpha \rangle v \otimes e^{\alpha},\\
\beta \cdot v \otimes e^{\alpha} &= v \otimes e^{\alpha+\beta}, \qquad C \cdot v \otimes e^{\alpha} = v \otimes e^{\alpha}
\end{aligned} \tag{2.2}$$

where $\alpha \in \Gamma$, $\beta \in Q$, $v \in S(\mathfrak{h}^-)$.

We define a class of *vertex operators* which was first introduced in dual resonance models:

$$\begin{aligned}
X(\alpha, z) = {}& \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \alpha(-n)\right)\\
& \cdot \exp(\log z\alpha(0) + \alpha) \exp\left(-\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \alpha(n)\right)
\end{aligned} \tag{2.3}$$

where $\alpha \in Q$, $z \in \mathbf{C}\backslash 0$. Generally speaking $X(\alpha, z)$ maps V into V', the space of formal series of elements from V. However, the homogeneous components $X_n(\alpha)$, $n \in \mathbf{Z}$, defined by the decomposition $X(\alpha, z) = \sum_{n \in \mathbf{Z}} X_n(\alpha) z^{-n}$ are well-defined operators on V. We will often denote $X_n(\alpha)$ by $\int_C X(\alpha, z) z^n \, dz/z$, where C is a circle containing the origin. This notation has a precise meaning in the following sense. Let $v \in V$, then

$$X_n(\alpha) v = \int_C X(\alpha, z) v \cdot z^n \frac{dz}{z}.$$

We will always operate with vertex operators having in mind this fact.

Let ε_α: $\mathbf{C}[\Gamma] \to \mathbf{C}[\Gamma]$, $\alpha \in Q$, be given by $\varepsilon_\alpha \cdot e^{\beta} = \varepsilon(\alpha, \beta) e^{\beta}$. We introduce the operators $X^{\varepsilon}(\alpha, z) = X(\alpha, z)\varepsilon_\alpha$. One can find directly the commutation relations between the homogeneous components of vertex operators; in particular, one has [**9**]

$$\begin{aligned}
\left[h(m), X_n^{\varepsilon}(\alpha)\right] &= \langle h, \alpha \rangle X_{n+m}^{\varepsilon}(\alpha),\\
\left[X_n^{\varepsilon}(\alpha), X_m^{\varepsilon}(\beta)\right] &= 0, \qquad \langle \alpha, \beta \rangle \geqslant 0,\\
\left[X_n^{\varepsilon}(\alpha), X_m^{\varepsilon}(\beta)\right] &= \varepsilon(\alpha, \beta) X_{n+m}^{\varepsilon}(\alpha + \beta), \qquad \langle \alpha, \beta \rangle = -1,\\
\left[X_n^{\varepsilon}(\alpha), X_m^{\varepsilon}(-\alpha)\right] &= -\left(\alpha(n+m) + n\delta_{m,-m}\right)
\end{aligned} \tag{2.4}$$

where $\alpha, \beta \in \Delta$, $n, m \in \mathbf{Z}$. The above calculations are based on the following formula for $|z|>|z_0|$:

$$X^\varepsilon(\alpha, z)X^\varepsilon(\beta, z_0) = \varepsilon(\alpha, \beta)(z - z_0)^{\langle\alpha,\beta\rangle} \cdot (zz_0)^{-\langle\alpha,\beta\rangle/2} \colon X^\varepsilon(\alpha, z)X^\varepsilon(\beta, z_0) \colon \quad (2.5)$$

where : : denotes the normal ordering, and the operator analogue of the Cauchy residue formula. Setting $h_i = h_i(0)$, $e_i = X_0^\varepsilon(\alpha_i)$, $f_i = X_0^\varepsilon(-\alpha_i)$, one can deduce from (2.4) the commutation relations (1.1).

THEOREM 2.1. *Let $\mathfrak{g}$ be a nonaffine Kac-Moody algebra. Then its representation π in V is defined by*

$$\pi(e_i) = X_0^\varepsilon(\alpha_i),$$
$$\pi(f_i) = X_0^\varepsilon(-\alpha_i), \qquad i \in I,$$
$$\pi(h_i) = h_i(0).$$

In particular, the operators $X_0^\varepsilon(\alpha)$, $\alpha \in \Delta_R$, define the representation of all real root elements.

PROOF. The relations (1.1) follow immediately from the commutation relations of vertex operators (2.4). Then the simplicity of $\mathfrak{g}$ [11] implies the result.

It is interesting to note that although the vertex representation has appeared first in the affine case, affine Lie algebras are excluded from the statement of Theorem 2.1. We will formulate this result separately.

THEOREM 2.2 [9]. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Then the representation π of the corresponding affine Lie algebra $\hat{\mathfrak{g}}$ is defined by*

$$\pi(x_\alpha(n)) = X_n^\varepsilon(\alpha), \qquad \alpha \in \Delta, n \in \mathbf{Z},$$
$$\pi(h(n)) = h(n), \qquad \pi(c) = \mathrm{Id}.$$

The representation V_Γ decomposes into the sum of fundamental representations of $\hat{\mathfrak{g}}$ corresponding to the orbits of Q in Γ.

We define a Hermitian form (,) in V by the conditions

$$(1 \otimes e^\alpha, 1 \otimes e^\beta) = \delta_{\alpha,\beta}, \qquad h(n)^* = h(-n), \quad h \in \mathfrak{h}_{\mathbf{R}}. \quad (2.6)$$

The form (,) is positive definite on V if and only if the form $\langle , \rangle$ is positive definite on $\mathfrak{h}_{\mathbf{R}}$. One can check the following property:

$$X_n^\varepsilon(\alpha)^* = X_{-n}^\varepsilon(-\alpha). \quad (2.7)$$

We introduce now an important algebra acting on V which is called the Virasoro algebra. This algebra is, in fact, a central extension of the algebra of vector fields on the unit circle and can be defined as follows:

$$\mathfrak{v} = \sum_{n \in \mathbf{Z}} \mathbf{C}L(n) \oplus \mathbf{C}c, \quad (2.8)$$

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m,-n}c \quad (2.9)$$

where c is a central element of $\mathfrak{v}$. Let h_i, $i = 1, \ldots, l$, be an orthonormal basis of $\mathfrak{h}$. One obtains a representation of the Virasoro algebra $\mathfrak{v}$ on V by setting

$$L(n) = \frac{1}{2}\sum_{i=1}^{l}\sum_{m \in \mathbf{Z}} \colon h_i(h-m)h_i(m)\colon. \quad (2.10)$$

In this case $c = l \cdot \mathrm{Id}$. One can verify the following commutation relations [9]:

$$[L(m), h(n)] = -nh(m+n), \qquad h \in \mathfrak{h}, \quad (2.11)$$
$$[L(m), X_n^\varepsilon(\alpha)] = -nX_{m+n}^\varepsilon(\alpha), \qquad \alpha \in \Delta_R. \quad (2.12)$$

The Virasoro algebra $\mathfrak{v}$ commutes with $\mathfrak{g}$ and therefore allows one to significantly "decrease" the reducible representation V. We define for $k \in \mathbf{C}$

$$V_k = \{v \in V\colon (L(n) - k\delta_{0,n})v = 0, n \in \mathbf{Z}\}. \quad (2.13)$$

Then for each $k \in \mathbf{C}$ such that $V_k \neq \phi$ we obtain a representation of $\mathfrak{g}$. There representations are irreducible only in the simplest case of $\mathfrak{g} = \mathfrak{sl}(2)$ [16, 28].

Let $\hat{\mathfrak{g}}$ be an affinization of a Kac-Moody algebra $\mathfrak{g}$. We can extend it to a semidirect product with the Virasoro algebra $\mathfrak{v}$ by using the following commutation relations:

$$[L(m), x \otimes t^n] = -t^{m+1}\frac{d}{dt}x \otimes t^n = -nx \otimes t^{n+m}. \quad (2.14)$$

Then (2.11), (2.12) imply that the vertex representation of $\hat{\mathfrak{g}}$ can be extended to the semidirect product if one defines $L(n)$ by (2.10).

2.2. Real root vectors exhaust the root system of Kac-Moody algebra only in the finite-dimensional case. In general, we have in addition infinitely many imaginary root vectors, and it is not known yet which operators represent a general vector of this type. We will find now the form of operators corresponding to the light-cone (isotropic) imaginary root vectors for a hyperbolic Lie algebra. Physicists can recognize "photon" vertex operators introduced in dual resonance models [4].

PROPOSITION 2.3. *Let $\mathfrak{g}$ be a hyperbolic algebra of rank l, and let $\gamma \in \Delta$ be an isotropic imaginary root vector. Then any $x \in \mathfrak{g}_\gamma$ has the following representation on V:*

$$\pi(x) = \int_C \alpha(z)X^\varepsilon(\gamma, z)\frac{dz}{z} \quad (2.14)$$

where $\alpha \in \mathfrak{h}$ is orthogonal to γ.

PROOF. First we will check the identity

$$\left[\int_C X^\varepsilon(\alpha, z)\frac{dz}{z}, \int_C X^\varepsilon(\beta, z)\frac{dz}{z}\right] = \varepsilon(\alpha,\beta)\int_C \alpha(z)X^\varepsilon(\alpha+\beta, z)\frac{dz}{z} \tag{2.15}$$

where $\alpha, \beta \in \Delta_R$, $\alpha + \beta = \gamma$. One has

$$\begin{aligned}\left[\int_C X^\varepsilon(\alpha, z)\frac{dz}{z}, \int_{C_0} X^\varepsilon(\beta, z_0)\frac{dz_0}{z_0}\right] &= \int_{C_0}\left(\int_{C_R} X^\varepsilon(\alpha, z)X^\varepsilon(\beta, z_0)\frac{dz}{z} - \int_{C_r} X^\varepsilon(\beta, z_0)X^\varepsilon(\alpha, z)\frac{dz}{z}\right)\frac{dz_0}{z_0}\\ &= \int_{C_0}\left(\int_{C_R\setminus C_r}\varepsilon(\alpha,\beta)\frac{zz_0}{(z-z_0)^2}:X^\varepsilon(\alpha, z)X^\varepsilon(\beta, z_0):\frac{dz}{z}\right)\frac{dz_0}{z_0}\\ &= \varepsilon(\alpha,\beta)\int_{C_0}\alpha(z_0)X^\varepsilon(\alpha+\beta, z_0)\frac{dz_0}{z_0}.\end{aligned}$$

This implies that $\mathfrak{g}_\gamma$ consists of elements of the form (2.14), where $\alpha \in \mathfrak{h}$ is orthogonal to γ. Theorem 1.1 implies that $\dim \mathfrak{g}_\gamma = l - 2$, and we note that $\int_C \gamma(z)X^\varepsilon(\gamma, z)\,dz/z = 0$. Therefore, every operator of this form belongs to $\mathfrak{g}_\gamma$.

Let us consider an affine Lie algebra as a subalgebra of hyperbolic algebra acting on the vertex representation. Then one has

PROPOSITION 2.4. *The operators*

$$\int_C X^\varepsilon(\alpha + nc, z)\frac{dz}{z}, \qquad \int_C h(z)X^\varepsilon(nc, z)\frac{dz}{z},$$
$$\alpha \in \Delta,\ n \in \mathbf{Z},\ h \in \mathfrak{h},\ \textit{and}\ h(0),$$

where $h \in \mathfrak{h}$ is orthogonal to c ($\langle c, c\rangle = 0$), define a representation of the affine Lie algebra $\hat{\mathfrak{g}}$ on V. In particular,

(i)

$$\left[\int_C h(z)X^\varepsilon(mc, z)\frac{dz}{z}, \int_C g(z)X^\varepsilon(nc, z)\frac{dz}{z}\right] = m\langle h, g\rangle\delta_{m,-n}c(0),$$

(ii)

$$\left[\int_C h(z)X^\varepsilon(mc, z)\frac{dz}{z}, \int_C X^\varepsilon(\alpha + nc, z)\frac{dz}{z}\right] = \varepsilon(mc, \alpha + nc)\langle h, \alpha\rangle\int_C X^\varepsilon(\alpha + (m+n)c, z)\frac{dz}{z},$$

(iii)

$$\left[\int_C X^\varepsilon(\alpha + mc, z)\frac{dz}{z}, \int_C X(\beta + nc, z)\frac{dz}{z}\right] = \varepsilon(\alpha + mc, \beta + mc)\cdot\begin{cases}0, & \langle\alpha,\beta\rangle \geqslant 0,\\ \int_C X^\varepsilon(\alpha+\beta+(m+n)c)\frac{dz}{z}, & \langle\alpha,\beta\rangle = -1,\\ \int_C \alpha(z)X^\varepsilon((m+n)c, z)\frac{dz}{z} + m\delta_{m,-n}c(0), & \alpha = -\beta.\end{cases}$$

PROOF. The proof follows from Proposition 2.3 and Theorem 2.1; also, it is not difficult to check the commutation relations directly, e.g. (i)

$$h(z)g(z_0) = :h(z)g(z_0): + \frac{zz_0}{(z-z_0)^2}\langle h, g\rangle,$$

$$\begin{aligned}\left[\int_C h(z)X^\varepsilon(mc, z)\frac{dz}{z}, \int_C g(z_0)X^\varepsilon(nc, z_0)\frac{dz_0}{z_0}\right] &= \int_{C_0}\left(\int_{C_R} h(z)g(z_0)X^\varepsilon(mc, z)X(nc, z_0)\frac{dz}{z}\right.\\ &\qquad\left. - \int_{C_r} g(z_0)h(z)X^\varepsilon(mc, z)X^\varepsilon(nc, z_0)\frac{dz_0}{z}\right)\frac{dz_0}{z_0}\\ &= \int_{C_0}\left(\int_{C_R\setminus C_r} X^\varepsilon(mc, z)X^\varepsilon(nc, z_0)\langle h, g\rangle\frac{zz_0}{(z-z_0)^2}\right)\frac{dz_0}{z_0}\\ &= \langle h, g\rangle\int_{C_0}\left(z\frac{d}{dz}X^\varepsilon(mc, z)\right)_{z=z_0}X^\varepsilon(nc, z_0)\frac{dz_0}{z_0}\\ &= m\langle h, g\rangle\int_{C_0} c(z_0)X^\varepsilon((m+n)c, z_0)\frac{dz_0}{z_0}\\ &= m\langle h, g\rangle\delta_{m,-n}c(0)\end{aligned}$$

because

$$\int_C c(z)X^\varepsilon(kc, z)\frac{dz}{z} = \delta_{k,0}c(0). \quad \text{Q.E.D.}$$

If we consider the dimension of the space of operators of the form $\int_C h(z)X^\varepsilon(nc, z)\,dz/z$, then we can notice it is one more than $\dim \mathfrak{g}_c$.

One can ask a question: What is the meaning of this additional one-dimensional space? It turns out that this is a derivation of the affine Lie algebra $\hat{\mathfrak{g}}$.

PROPOSITION 2.5. *Under the conditions of Proposition* 2.4, *the operators* $\int_C d(z)X^\varepsilon(nc, z)\,dz/z$ *where* $\langle d, c\rangle = -1$, $\langle d, \alpha\rangle = 0$, $\alpha \in \Delta$, *define the extension of* $\hat{\mathfrak{g}}$ *by the Virasoro algebra. In particular,*

(i)

$$\left[\int_C d(z)X^\varepsilon(mc, z)\,\frac{dz}{z}, \int_C X^\varepsilon(\alpha + nc, z)\,\frac{dz}{z}\right] = -n\varepsilon(mc, \alpha + nc)\int_C X^\varepsilon(\alpha + (m+n)c, z)\,\frac{dz}{z},$$

(ii)

$$\left[\int_C d(z)X^\varepsilon(mc, z)\,\frac{dz}{z}, \int h(z)X^\varepsilon(nc, z)\,\frac{dz}{z}\right] = -n\int_C X^\varepsilon((m+n)c, z)\,\frac{dz}{z},$$

(iii)

$$\left[\int_C d(z)X^\varepsilon(mc, z)\,\frac{dz}{z}, \int_C d(z)X^\varepsilon(nc, z)\,\frac{dz}{z}\right] = (m-n)\int_C (z)X^\varepsilon((m+n)c, z)\,\frac{dz}{z} + 2m^3\delta_{m,-n}.$$

3. Standard representations of finite-dimensional and affine Lie algebras in the vertex.

3.1. In this section we will consider the vertex representation $V = V_P$ of finite-dimensional Lie algeba $\mathfrak{g}$. One can see that any irreducible finite-dimensional representation $V(\lambda)$ with the highest weight λ occurs in V, e.g. such subrepresentation of $\mathfrak{g}$ is generated by the highest weight vector $1 \otimes e^\lambda \in V$. Let $\Omega(\lambda)$ be the vector space of all the highest weight vectors of irreducible representations isomorphic to $V(\lambda)$ in V. One has a natural isomorphism

$$V \approx \sum_{\lambda \in P^{++}} V(\lambda) \otimes \Omega(\lambda). \tag{3.1}$$

There is a natural duality between vector spaces $V(\lambda)$ and $\Omega(\lambda)$ which we will make more apparent in the next subsection. In particular, we will define a Lie algebra acting irreducibly on $\Omega(\lambda)$. Now we will determine the multiplicities of $V(\lambda)$ in V, i.e., we will find the character of $\Omega(\lambda)$.

THEOREM 3.1 [17]. *Let* $\mathfrak{g}$ *be a simple Lie algebra. For any dominant* λ *the multiplicities of* $V(\lambda)$ *in the vertex representation* V *are given by the generating function*

$$\operatorname{ch}\Omega(\lambda) = q^{\langle\lambda,\lambda\rangle/2}\frac{\prod_{\alpha\in\Delta_+}(1 - q^{\langle\alpha,\lambda+\rho\rangle})}{\varphi(q)^l}. \tag{3.2}$$

PROOF. We give here a proof different from the one given in [17], which has a simple generalization to the case when $\mathfrak{g}$ is affine. Let us denote $S(\mu) = S(\hat{\mathfrak{h}}^-) \otimes e^\mu$ and consider any irreducible representation U of $\mathfrak{g}$ and its intersections with the spaces $S(\lambda + \rho - w\rho)$. We let

$$n_{\lambda+\rho-w\rho} = \dim(U \cap S(\lambda + \rho - w\rho)). \tag{3.3}$$

The character formula (1.12) for $\mathfrak{g}$ implies that, except in the case when $U \approx V(\lambda)$,

$$\sum_{w\in W} \det w \cdot n_{\lambda+\rho-w\rho} = 0. \tag{3.4}$$

In the case when $U \approx V(\lambda)$, $n_\lambda = 1$, and $n_{\lambda+\rho-w\rho} = 0$, for $w \neq 1$. We multiply the equality (3.4) by q^k, where k is the level of the highest weight vector in $U \subset V$. Then summing up these equalities for all irreducible $U \neq V(\lambda)$ we obtain

$$\operatorname{ch}\Omega(\lambda) = \sum_{w\in W} \det w \cdot \operatorname{ch} S(\lambda + \rho - w\rho). \tag{3.5}$$

Now, using the fact that

$$\operatorname{ch} S(\mu) = \varphi(q)^{-l}q^{\langle\mu,\mu\rangle/2} \tag{3.6}$$

and the specialized form of Weyl's identity (1.14) we obtain the result.

It is a remarkable fact that in the same space $V = V_P$ we can study not only all the standard representations of $\mathfrak{g}$ but also all the standard representations of $\hat{\mathfrak{g}}$. Let us define a subalgebra $\hat{\mathfrak{g}}_{[n]} \subset \hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}}_{[n]} = \mathfrak{g} \otimes \mathbf{C}[t^n, t^{-n}] + \mathbf{C}c. \tag{3.7}$$

Then clearly $\hat{\mathfrak{g}}_{[n]} \approx \hat{\mathfrak{g}}$ where the isomorphism is given by

$$i_n(x \otimes t^{kn}) = x \otimes t^k, \qquad i_n(nc) = c \tag{3.8}$$

where $k \in \mathbf{Z}$, $x \in \mathfrak{g}$. Therefore, if we restrict the representation V to $\hat{\mathfrak{g}}_{[n]}$, then taking into account isomorphism (3.8) we obtain a representation of level n. Moreover, this representation is reducible (the only exception is when $n = 1$, and $\mathfrak{g}$ is of type E_8), and any irreducible level n standard representation $V(\lambda)$ with the highest weight λ occurs in V, e.g. such subrepresentation is generated by the highest weight vector $1 \otimes e^{\lambda_0} \in V$, where $\lambda = nd + \lambda_0 - \langle\lambda_0\lambda_0\rangle c/2$. We define $\Omega(\lambda)$ as in the finite-dimensional case and we get the decomposition

$$V \approx \sum_{\substack{\lambda\in\hat{P}^{++}\\ \text{level }\lambda=n}} V(\lambda)\otimes\Omega(\lambda). \tag{3.9}$$

In [8] we formulated a conjecture about the multiplicites of $V(\lambda)$ in V which we will now prove.

THEOREM 3.2. *Let $\hat{\mathfrak{g}}$ be an affine Lie algebra acting as the algebra $\mathfrak{g}_{[n]}$ in the vertex representation V. For any dominant λ the multiplicities of $V(\lambda)$ in V are given by the generating function*

$$\operatorname{ch}\Omega(\lambda) = q^{\langle\lambda,\lambda\rangle/2}\frac{\prod_{\alpha\in\hat{\Delta}_+}\left(1-q^{\langle\alpha,\lambda+\hat{\rho}\rangle}\right)}{\varphi(q)^l}. \tag{3.10}$$

PROOF. The proof is a literal repetition of the proof of Theorem 3.1 where we only change ρ to $\hat{\rho}$, W to $\hat{W}$ and we use (1.15) instead of (1.14).

Note that Theorem 3.2 implies immediately Theorem 1.6 of [8].

We remarked in [8] about the miraculous coincidence

$$\operatorname{ch}\Omega(\lambda) = \operatorname{ch}_q\Omega(\lambda)' \tag{3.11}$$

where $\Omega(\lambda)'$ and ch_q are defined as follows (see [22]). Let $\hat{\mathfrak{h}}'$ be a principal Heisenberg subalgebra [19]; then the restriction of any irreducible level 1 representation of $\hat{\mathfrak{g}}$ to $\hat{\mathfrak{h}}'$ is still irreducible and isomorphic to the canonical Fock space V_0. The general standard representation $V(\lambda)$ of $\hat{\mathfrak{g}}$ has the decomposition

$$V(\lambda) = V_0\otimes\Omega(\lambda)' \tag{3.12}$$

where $\Omega(\lambda)'$ is the vector space of vacuum vectors in $V(\lambda)$ for $\hat{\mathfrak{h}}'$. One can define a character ch_q of the spaces in (3.12) with respect to a differentiation d' so that

$$\operatorname{ch}_q V(\lambda) = \operatorname{ch}_q V_0\cdot\operatorname{ch}_q\Omega(\lambda)' \tag{3.13}$$

where

$$\operatorname{ch}_q V(\lambda) = \operatorname{ch}V(\lambda)\big|_{e^{-\alpha_i}=q,\, i=1,\ldots,l}$$

is the so-called principally specialized character. Then the character $\operatorname{ch}_q\Omega(\lambda)'$ is given by the right side of (3.10). Theorem 3.2, (3.11) and (3.13) imply that

$$\operatorname{Ch} = \sum_{\substack{\lambda\in\hat{P}_{++}\\ \text{level }\lambda=n}} \operatorname{ch}V(\lambda)\big|_{e^{-c}=q^n}\cdot\operatorname{ch}_q V(\lambda) \tag{3.14}$$

does not depend on level n, and thanks to (1.16) has a very simple form. An explanation of this fact from the represenation theory point of view would be very important.

3.2. We know from (2.11) and (2.12) that the Virasoro algebra commutes with the action of $\mathfrak{g}$. Therefore, starting from one element $1\otimes e^\lambda \in \Omega(\lambda)$, we can generate an infinite-dimensional subspace of $\Omega(\lambda)$ by the action of the Virasoro algebra. However, only in one special case, when $\mathfrak{g} = \mathfrak{sl}(2)$, do we obtain the whole space $\Omega(\lambda)$, i.e., the Virasoro algebra acts irreducibly on $\Omega(\lambda)$. In order to generate $\Omega(\lambda)$ for an arbitrary $\mathfrak{g}$, we need an extension of the Virasoro algebra. Before we give the definition of this algebra we recall Wick's theorem well known in physics literature.

The product of two linear operators $a_1, a_2 \in \hat{\mathfrak{h}}$ differ from the normally ordered product by some scalar operator. We call this scalar operator contraction and we will denote it by a lower bracket

$$a_1a_2 = \,:a_1a_2:\, + \underbrace{a_1a_2}. \tag{3.15}$$

We define a normally ordered product with pairing by

$$:a_1\cdots\underbrace{a_j\cdots a_k}\cdots a_n: = \underbrace{a_ja_k}\,:a_1\cdots a_{j-1}a_{j+1}\cdots a_{k-1}a_{k+1}\cdots a_n:. \tag{3.16}$$

Now we can give a simple formulation of Wick's theorem.

THEOREM 3.3. *The product of normally ordered products of linear operators*

$$(:a_1\cdots a_{k_1}:)(:a_{k_1+1}\cdots a_{k_2}:)\cdots(:a_{k_{n-1}+1}\cdots a_{k_n}:)$$

is equal to the sum of all normally ordered products of these operators

$$:a_1\cdots a_{k_1}a_{k_1+1}\cdots a_{k_n}:$$

with all possible pairings between elements of the n sets

$$(a_1,\ldots,a_{k_1}),\ (a_{k_1+1},\ldots,a_{k_2}),\ldots,\ (a_{k_{n-1}+1},\ldots,a_{k_n})$$

including the normal product without pairings.

PROOF. The proof is a straightforward induction.

We will apply the Wick theorem to the case when $n = 2$ and $A_k = a_k(z) = \sum_{n\in\mathbf{Z}}a_k(n)z^{-n}$, $a_k\in\mathfrak{h}$. Simple calculations show that

$$\underbrace{a_1(z_1)a_2(z_2)} = \frac{z_1z_2}{(z_1-z_2)^2}\langle a_1, a_2\rangle, \qquad |z_1|>|z_2|. \tag{3.17}$$

For example we can prove, using Wick's theorem, the commutation relations (2.9) with $c = l \cdot \mathrm{Id}$. One has

$$\sum_{i=1}^{l}\sum_{j=1}^{l} :h_i(z)h_i(z): :h_j(z_0)h_j(z_0):$$

$$= \sum_{i=1}^{l}\sum_{j=1}^{l} \delta_{ij}\frac{zz_0}{(z-z_0)^2}\delta_{ij}\frac{zz_0}{(z-z_0)^2}$$

$$= l \cdot \frac{z^2z_0^2}{(z-z_0)^4}.$$

We get four terms with one contraction and two terms with two contractions; therefore,

$$\sum_{i=1}^{l}\sum_{j=1}^{l} :h_i^2(z): :h_j^2(z_0):$$

$$= \sum_{i=1}^{l}\sum_{j=1}^{l} :h_i^2(z)h_j^2(z_0):$$

$$+ \frac{4zz_0}{(z-z_0)^2}\sum_{i=1}^{l} :h_i(z)h_i(z_0): + 2l\frac{z^2z_0^2}{(z-z_0)^4}. \tag{3.18}$$

Now (2.9) follows from the Cauchy residue formula

$$[L(m), L(z_0)] = \int_{C_R\setminus C_r}\left(\frac{zz_0}{(z-z_0)^2}\sum_{i=1}^{l} :h_i(z)h_i(z_0): + \frac{l}{2}\frac{z^2z_0^2}{(z-z_0)^4}\right)z^m\frac{dz}{z}$$

$$= \left(z_0\frac{d}{dz}z^m\sum_{i=1}^{l} :h_i(z)h_i(z_0): + \frac{l}{12}z_0^2\frac{d^3}{dz^3}z^{m+1}\right)_{z=z_0}$$

$$= 2mz_0^mL(z_0) + z_0^{m+1}\frac{d}{dz_0}L(z_0) + \frac{l}{12}(m+1)m(m-1)z_0^m. \tag{3.19}$$

We define now a new class of algebras generalizing the Virasoro algebra. Let $h \in \mathfrak{h}$, we denote

$$h^{(n)}(z) = D_z^n h(z) = \left(z\frac{d}{dz}\right)^n h(z). \tag{3.20}$$

Let $P(z)$ be a W-invariant polynomial in $h_i^{(n)}(z)$, $i = 1,\ldots,l$, $n \in \mathbf{Z}_+$, and let $P(n)$, $n \in \mathbf{Z}$, be defined by

$$:P(z): = \sum_{n\in\mathbf{Z}} P(n)z^{-n}. \tag{3.21}$$

The algebra of operators $\hat{S}^W$ is by definition the linear span of all $P(n)$, $n \in \mathbf{Z}$, and Id, such that $:P(z):$ commutes with $\mathfrak{g}$. The operators $P(n)$ are well defined, and $\hat{S}^W$ has a natural grading.

PROPOSITION 3.4. *The algebra of operators $\hat{S}^W$ is closed under the Lie bracket.*

PROOF. We have to show that for any $P(m), Q(n) \in \hat{S}^W$, $m, n \in \mathbf{Z}$,

$$[P(m), Q(n)] = \sum_{j\in J} f_j(m,n)R_j(m+n) + f(m,n)\delta_{m,-n} \tag{3.22}$$

where $R_j(m+n) \in \hat{S}^W$, f_j, f are some functions on $\mathbf{Z}\times\mathbf{Z}$, and J is a finite set. This form follows from the Wick theorem and (3.21). In fact, the typical term of

$$\left[\int_C :P(z): z^m\frac{dz}{z}, \int_{C_0} :Q(z_0): z_0^n\frac{dz_0}{z_0}\right]$$

will be

$$\int_{C_0}\left(\int_{C_R\setminus C_r}\prod_i\left(D_z^{m_i}D_{z_0}^{n_i}\frac{zz_0}{(z-z_0)^2}\right) :P_1(z)Q_1(z_0): z^m\frac{dz}{z}\right)z_0^n\frac{dz_0}{z_0} \tag{3.23}$$

where i runs through a finite number of contractions, and P_1, Q_1 are contracted P and Q. The typical term in (3.22) will be

$$\int_{C_0}\left(\int_{C_R\setminus C_r}\frac{z^{1+M}z_0^{1+N}}{(z-z)^{2+M+N}} :P_1(z)Q_1(z_0): z^m\frac{dz}{z}\right)z_0^n\frac{dz_0}{z_0}$$

$$= \int_{C_0}\frac{1}{(M+N+1)!}\frac{d^{M+N+1}}{dz^{M+N+1}}\left(z^{M+m}:P_1(z)Q_1(z_0):\right)_{z=z_0}z_0^{n+N}\,dz_0,$$

which is clearly of the form in (3.22).

The Virasoro algebra certainly is a subalgebra of $\hat{S}^W$, and spanned by $P(n)$, $n \in \mathbf{Z}$, and Id, where P is the Casimir element in $S(\mathfrak{h})^W$. In order to illustrate what type of elements occur in $\hat{S}^W$ we will introduce a generalization of Segal's operators (see [7]).

Let $\{x_j\}$ be an orthonormal basis of $\mathfrak{g}$, and let $[x_i, x_j] = \sum_k C_{ij}^k x_k$. For any $\mathfrak{g}$-module V from the category $\mathfrak{M}$ [15, p. 102] we introduce the operator

$$L(z_1, z_2) = \sum x_j(z_1)x_j(z_2) = \sum_j :x_j(z_1)x_j(z_2): + \dim \mathfrak{g} \frac{z_1 z_2}{(z_1 - z_2)^2}, \tag{3.24}$$

where $|z_1| > |z_2|$. The second expression in (3.24) allows us to extend the definition of the operator $L(z_1, z_2)$ for every $z_1 \neq z_2$. Clearly the homogeneous components of $L(z_1, z_2)$ are well-defined operators in V.

PROPOSITION 3.5. The operator $L(z_1, z_2)$ commutes with $\mathfrak{g}$. If, in addition, $z_1^n = z_2^n$, $n = 2, 3, \ldots$, then $L(z_1, z_2)$ commutes with $\hat{\mathfrak{g}}_{[n]}$.

PROOF.

$$\begin{aligned}\left[x_i(m), \sum_j x_j(z_1)x_j(z_2)\right] &= \sum_j \left(\left[x_i(m), x_j(z_1)\right]x_j(z_2) + x_j(z_1)\left[x_i(m), x_j(z_2)\right]\right) \\ &= \sum_{jk} \left(C_{ij}^k x_k(z_1)x_j(z_2)z_1^m + C_{ij}^k x_k(z_1)x_k(z_2)z_2^m\right) \\ &= \sum_{ji} C_{ij}^k x_k(z_1)x_j(z_2)(z_1^m - z_2^m),\end{aligned}$$

which obviously implies the result.

In the vertex representation we have

$$\begin{aligned}L(z_1, z_2) &= \sum_{i=1}^{l} h_i(z_1)h_i(z_2) - \sum_{\alpha\in\Delta} X^\epsilon(\alpha_1 z_1)X^\epsilon(-\alpha_1 z_2) \\ &= \sum_{i=1}^{l} :h_i(z_1)h_i(z_2): + l\frac{z_1 z_2}{(z_1 - z_2)^2} \\ &\quad - \frac{z_1 z_2}{(z_1 - z_2)^2} \sum_{\alpha\in\Delta} :X^\epsilon(\alpha_0 z_1)X^\epsilon(-\alpha_1 z_2):.\end{aligned} \tag{3.25}$$

Therefore the operators

$$-D_{z_1}^{2n}\left(\frac{(z_1 - z_2)^2}{z_1 z_2} L(z_1, z_2)\right)\bigg|_{z_1=z_2=z} = \sum_{\alpha\in\Delta} :\alpha^{2n}(z): + Q \tag{3.26}$$

commute with $\mathfrak{g}$, belong to $\hat{S}^W$, and $\deg Q < 2n$. We formulate without proof a generalization of this result.

THEOREM 3.6. (i) *For every $P \in S(\mathfrak{h})^W$ there exist Q, $\deg Q < \deg P$, so that : $P(z)$: + : $Q(z)$: commutes with $\mathfrak{g}$.*

(ii) *For any dominant highest weight λ, $\Omega(\lambda)$ is an irreducible representation of $\hat{S}^W$ satisfying the following properties*:

$$P(n)v_0 = 0, \quad n > 0, \qquad P(0)v_0 = \langle P, \lambda\rangle v_0, \tag{3.27}$$

where $v_0 = 1 \otimes e^\lambda \in V$, and $P(n) \in \hat{S}^W$, $n \in \mathbf{Z}$.

Here we will note only that the proof of Theorem 3.6 for $\mathfrak{g} = \mathfrak{sl}(n)$ follows easily from Theorem 1.6 in **[8]**.

We remarked already that the decomposition (3.1) displays some kind of duality between the representations of the algebras $\mathfrak{g}$ and $\hat{S}^W$. In fact, the irreducible finite-dimensional representations $V(\lambda)$ of $\mathfrak{g}$ naturally correspond to the irreducible representations $\Omega(\lambda)$ of $\hat{S}^W$. The duality can be extended further if one considers another natural class of representations, namely the induced representations called Verma modules. By definition a Verma module of $\mathfrak{g}$ is defined by any $\lambda \in \mathfrak{h}^*$ as the induced module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbf{C}_\lambda \tag{3.28}$$

where U denotes the universal enveloping algebra and $\mathbf{C}_\lambda$ is a $\mathfrak{b}^+$-module via the action

$$(h + x)z = \langle\lambda, h\rangle z, \qquad h \in \mathfrak{h}, x \in \mathfrak{n}^+, z \in \mathbf{C}. \tag{3.29}$$

One can define an analogue of Verma modules for $\hat{S}^W$ by fixing a polarization

$$\hat{S}^W = \hat{S}_+^W + \left(\hat{S}_0^W + \mathbf{C}\,\mathrm{Id}\right) + \hat{S}_-^W \tag{3.30}$$

where S_+^W is spanned by $P(n)$, $n > 0$, and similarly for $\hat{S}_0^W$, $\hat{S}_-^W$. Then for any $\lambda \in \mathfrak{h}^*$,

$$\mathfrak{M}(\lambda + \rho) = U(\hat{S}^W) \otimes_{U(\hat{S}_+^W + \hat{S}_0^W)} \mathbf{C}^\lambda \tag{3.31}$$

where $\mathbf{C}^\lambda$ is a $(\hat{S}_+^W + \hat{S}_0^W)$-module via the action

$$(h + x)z = \langle\lambda, h\rangle z, \qquad h \in \hat{S}_0^W, x \in \hat{S}_+^W, z \in \mathbf{C}, \tag{3.32}$$

and $\langle\lambda, h\rangle$ is the value of the polynomial h on λ. Using the Chevalley theorem that $S(\mathfrak{h})^W \approx \mathbf{C}[P_1, \ldots, P_l]$ and Theorem 3.6, one has

$$\mathrm{ch}\,\mathfrak{M}(\lambda + \rho) = q^{\langle\lambda,\lambda\rangle/2}/\varphi(q)^l. \tag{3.33}$$

Therefore Theorem 3.2 implies that

$$\mathrm{ch}\,\Omega(\lambda) = \sum_{w\in W} \det w\, \mathrm{ch}\,\mathfrak{M}(w(\lambda + \rho)) \tag{3.34}$$

which is similar to the Weyl formula (1.11)

$$\operatorname{ch} V(\lambda) = \sum_{w \in W} \det w \operatorname{ch} M(w(\lambda + \rho)). \tag{3.35}$$

We know from the results of Bernstein, Gelfand and Gelfand [1], that (3.35) reflects a much stronger result about the existence of the exact sequence of $\mathfrak{g}$-modules

$$0 \leftarrow V(\lambda) \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_s \leftarrow 0 \tag{3.36}$$

where $s = \dim \mathfrak{n}_-$, $C_k = \oplus_{w \in W^{(k)}} M(w(\lambda + \rho))$. Therefore, one is led to conjecture that the formula (3.34) also is not just a coincidence.

Conjecture 3.7. *There exists an exact sequence of $\hat{S}^W$-modules*

$$0 \leftarrow \Omega(\lambda) \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_s \leftarrow 0 \tag{3.37}$$

where s is as in (3.35), $C_k = \oplus_{w \in W^{(k)}} \mathfrak{M}(w(\lambda + \rho))$.

In particular, for $\mathfrak{g} = \mathfrak{sl}(2)$ one should have, according to Conjecture 3.7, the exact sequence

$$0 \leftarrow \Omega(\lambda) \leftarrow \mathfrak{M}(\lambda + \rho) \leftarrow \mathfrak{M}(-\lambda - \rho) \leftarrow 0 \tag{3.38}$$

where $\Omega(\lambda)$, $\mathfrak{M}(\ )$ are modules of $\hat{S}^W$ of rank 1. We noted above that in this case the Virasoro subalgebra contains the main information about $\hat{S}^W$. Rocha-Caridi has proved that (3.38) is in fact an exact sequence of Virasoro modules.

Finally, we will note that the strange duality between Lie algebras $\mathfrak{g}$ and $\hat{S}^W$ becomes more transparent in the affine case. Let us consider the decomposition (3.9). The vector space $V(\lambda)$ is by definition the representation space of the affine Lie algebra $\hat{\mathfrak{g}}$. The vector space $\Omega(\lambda)$, thanks to the miracle in (3.11), can be considered as the representation space of the Z-algebras studied in [**23**]. It is shown in [**23**] that some categories of representations (which include standard representations) of Z-algebras and the corresponding affine Lie algebras are equivalent. This implies immediately the equivalence of the affine analogues of (3.35) and (3.36).

4. The dual model in the critical dimension and hyperbolic algebras of rank 26. One of the main achievements of dual models is the no-ghost theorem, which reveals the critical number 26. This beautiful result implies the existence of one special representation of hyperbolic Lie algebras of rank 26 and allows us to obtain an upper bound for root multiplicities. For the sake of completeness we recall here the main steps of the proofs of the no-ghost theorem.

We define V_n, $n \in \mathbf{Z}$, to be the space of highest weight vectors of the Virasoro algebra, i.e. $v \in V_n$ iff

$$L(m)v = 0, \quad m \geqslant 1, \qquad L(0)v = nv. \tag{4.1}$$

The space V_1 plays the most important role in the dual theories. Let $V_1^{\mathfrak{h}} = \{v \in V_1 \colon v = h(-1) \otimes 1,\ h \in \mathfrak{h}\}$. Clearly $V_1^{\mathfrak{h}}$ contains elements of negative norm if and only if $\mathfrak{h}_{\mathbf{R}}$ contains such elements. Let us denote $V_1^{\circ} = V_1 \ominus V_1^{\mathfrak{h}}$. Then there is no reason to expect that there are no other elements with negative norm in V_1°. However, physicists obtained the following remarkable result:

THEOREM 4.1 (NO-GHOST THEOREM). *Let V be a vertex representation of a hyperbolic Lie algebra $\mathfrak{g}$, then V_1° is a positive semidefinite space if and only if* rank $\mathfrak{g} \leqslant 26$.

PROOF. We will indicate the main steps of the proof. (See details in [**25**].)

Step 1. We have to prove that for every $\alpha \in Q \backslash 0$, the space $V_1^{\circ} \cap S(\alpha)$ is positive semidefinite. We consider $S(\alpha)$ naturally graded of $S(\alpha)$ with respect to the degree operator

$$S(\alpha) = \sum_{M=0}^{\infty} S(\alpha)_M. \tag{4.2}$$

Then $v \in S(\alpha)_M$ belongs to V_1° only if $\langle \alpha, \alpha \rangle = 2(1 - M)$. Now let us fix any isotropic vector $c \in \mathfrak{h}$, normalized by the condition $\langle c, \alpha \rangle = -1$, and let us define the transverse space $T(\alpha, c)$ by the condition that $v \in T(\alpha, c)$ if and only if

$$L(n)v = c(n)v = 0, \qquad n \geqslant 1. \tag{4.3}$$

Let us define the subspace $G(\alpha, c)$ of $S(\alpha)$ as the linear span of the elements of type

$$v_{(\lambda_j, \mu_j)} = L(-1)^{\lambda_1} \cdots L(-n)^{\lambda_n} c(-1)^{\mu_1} \cdots c(-m)^{\mu_m} v \tag{4.4}$$

where $v \in T(\alpha, c)$, $\lambda_j, \mu_j \in \mathbf{Z}_+$, and $\Sigma\lambda_j + \Sigma\mu_j > 0$.

LEMMA 4.2. (i) *The space $T(\alpha, c)$ is positive definite.*

(ii) *The subspaces $T(\alpha, c)$ and $G(\alpha, c)$ have zero intersection.*

(iii) $S(\alpha) = T(\alpha, c) \oplus G(\alpha, c)$.

Step 2. Now we can construct an operator which projects the space $S(\alpha)_N$ to $T(\alpha, c)_N$. Let $c(z) = \Sigma_{n \in \mathbf{Z}} c(n) z^{-n}$ be the generating function; then the operator

$$D(n) = \int_C c(z)^{-1} z^n \frac{dz}{z} \tag{4.5}$$

is well defined in $S(\alpha)$ thanks to the condition $\langle c, \alpha \rangle = -1$, namely, $c(z)^{-1} = (1 + c_0(z))^{-1} = 1 - c_0(z) + c_0^2(z) - \cdots$. Moreover,

$$(D(n) - \delta_{0,n})v = 0, \qquad n \geqslant 0,\ v \in T(\alpha, c), \tag{4.6}$$

because $D(n) - \delta_{0,n}$, $n \geqslant 0$, is given by a series of terms each of which contains at least one $c(m)$, $m > 0$. We define now an important operator

$$E = (D(0) - 1)(L(0) - 1) + \sum_{n=1}^{\infty} (D(-n)L(n) + L(-n)D(n)). \quad (4.7)$$

It is clear that $Ev = 0$ for $v \in T(\alpha, c)$. However, only when rank $\mathfrak{g}$ is 26 do all solutions come from $T(\alpha, c)$.

LEMMA 4.3. *Let* rank $\mathfrak{g}$ *be* 26, *then*
(i) *the eigenvectors of E span all of $S(\alpha)$,*
(ii) *the eigenvalues of E are nonpositive integers,*
(iii) *the eigenvalue zero corresponds to, precisely, the transverse subspace $T(\alpha, c)$.*

The critical dimension appears in the formula

$$[L(n), E] = -nL(n) + \frac{\dim \mathfrak{h} - 26}{12}(n^3 - n)D(n),$$

which provides the key to the proof of the lemma.

Let us define the projection operator onto $T(\alpha, c)_N$

$$P = \int_C z^E \frac{dz}{z}.$$

Then the definition (4.7) of E implies that $\langle v, v \rangle = \langle P_v, P_v \rangle \geqslant 0$. This ends the proof of the theorem for rank $\mathfrak{g} = 26$.

We also note how to construct an explicit basis of the space $T(\alpha - Nc, c)_N$. Let us denote

$$A_i(n) = \int_C \alpha_i(z) X^{\varepsilon}(nc, z) \frac{dz}{z} \quad (4.8)$$

where $i = 1, \ldots, l$, α_i orthogonal to c (and not proportional to c). Then the elements of type

$$\prod_i A_{m_i}(-n_i) 1 \otimes e^{\alpha}, \qquad \sum_i n_i = N, \quad n_i > 0, \quad (4.9)$$

provides a basis of $T(\alpha - Nc, c)_N$. We denote by $\overline{V}_{1,c,\alpha}$ the space spanned by the elements (4.9) with arbitrary N.

Before we use the no-ghost theorem for hyperbolic Lie algebras of rank 26, we will prove one general fact about the subspace V_1. We note that when $\mathfrak{h}_{\mathbf{R}}$ is positive definite, the subspace V_1 is isomorphic to the adjoint representation of $\mathfrak{g}$. It is no longer true in general; however, one has

PROPOSITION 4.4. *Let A be a subspace of V_1 generated by the action of Kac-Moody algebra $\mathfrak{g}$ acting on $1 \otimes e^{\alpha}$, $\alpha \in \Delta_R$. Then there is an epimorphism of representations of $\mathfrak{g}$,*

$$p: A \to \mathfrak{g}, \quad (4.10)$$

preserving the weight subspaces.

PROOF. Let $\{\alpha_i\}_{i=1}^{l}$ be a root basis of Δ_R. Any root vector of $\mathfrak{g} \setminus \mathfrak{h}$ is of the form

$$\begin{aligned} &\operatorname{ad} X_0^{\varepsilon}(\alpha_{i_1}) \cdots \operatorname{ad} X_0^{\varepsilon}(\alpha_{i_{n-1}}) \cdot X_0^{\varepsilon}(\alpha_{i_n}) \quad \text{or} \\ &\operatorname{ad} X_0^{\varepsilon}(-\alpha_{i_1}) \cdots \operatorname{ad} X_0^{\varepsilon}(-\alpha_{i_{n-1}}) \cdot X_0^{\varepsilon}(-\alpha_{i_n}). \end{aligned} \quad (4.11)$$

We will consider the following elements of A:

$$\begin{aligned} &X_0^{\varepsilon}(\alpha_{i_1}) \cdots X_0^{\varepsilon}(\alpha_{i_{n-1}}) \cdot 1 \otimes e^{\alpha_{i_n}} \quad \text{or} \\ &X_0^{\varepsilon}(-\alpha_{i_1}) \cdots X_0^{\varepsilon}(-\alpha_{i_{n-1}}) \cdot 1 \otimes e^{-\alpha_{i_n}}. \end{aligned} \quad (4.12)$$

By definition, the map p sends the elements of the form (4.12) to the elements of the form (4.11), respectively. Also we set

$$p: \alpha(-1) \otimes 1 \to \alpha(0), \qquad \alpha \in \mathfrak{h}. \quad (4.13)$$

We will prove by induction on n that p is a linear map. This is true for $n = 1$; suppose it is true for $n - 1$. Suppose that for n we have a linear dependence of elements (4.12) (denote it by Y) but the corresponding linear combination of elements (4.11) (denote it by Z) does not vanish. There exists $i \in \{1, \ldots, l\}$ so that $[X_0^{\varepsilon}(-\alpha_i), Z] \neq 0$, because otherwise $\mathfrak{g} \cdot Z$ would be a nontrivial ideal in the simple Lie algebra $\mathfrak{g}$. But $X_0^{\varepsilon}(-\alpha_i) \cdot Y = 0$, which contradicts the induction assumption. Finally the homomorphism property of p follows from simple calculations

$$p: X_0^{\varepsilon}(-\alpha) \cdot 1 \otimes e^{\alpha} = \alpha(-1) \otimes 1 \to \operatorname{ad} X_0^{\varepsilon}(-\alpha) \cdot X_0^{\varepsilon}(\alpha) = \alpha(0),$$

$$p: \left(X_0^{\varepsilon}(-\alpha)\right)^2 \cdot 1 \otimes e^{\alpha} = 2 \cdot 1 \otimes e^{-\alpha}$$
$$\to \left(\operatorname{ad} X_0^{\varepsilon}(-\alpha)\right)^2 \cdot X_0^{\varepsilon}(\alpha) = 2X_0^{\varepsilon}(-\alpha).$$

The no-ghost theorem and Proposition 4.4 allow us to get an upper bound for root multiplicities of hyperbolic algebras of rank 26.

PROPOSITION 4.5. *Let $\mathfrak{g}$ be a hyperbolic algebra of rank 26 and $\alpha \in \Delta$. Then*

$$\dim \mathfrak{g}_{\alpha} \leqslant \mu(-\langle \alpha, \alpha \rangle / 2) \quad (4.14)$$

where

$$\sum_{n=-1}^{\infty} \mu(n) q^n = \frac{1}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}}. \quad (4.15)$$

PROOF. Let us consider the null radical $\{0\}$ of the form $\langle\, ,\rangle$ in V_1. Clearly it is a subrepresentation of $\mathfrak{g}$. We denote by $\overline{V}_1$ the factor representation $V_1/\{0\}$ of $\mathfrak{g}$.

We will prove that for $\alpha \neq 0$,

$$\dim\left(\overline{V}_1\right)_\alpha = \mu(-\langle \alpha, \alpha\rangle/2). \tag{4.16}$$

Lemma 4.2 and the no-ghost theorem imply that $(V_1)_\alpha \approx S(\alpha)_M$ and $(\overline{V}_1)_\alpha \approx T(\alpha, c)_M$, where $\langle \alpha, \alpha\rangle = 2(1-M)$. Let us define $p^{(l)}(n)$ by

$$\sum_{n=0}^{\infty} p^{(l)}(n) q^n = \varphi(q)^{-l}. \tag{4.17}$$

We will show by induction that

$$\dim T(\alpha, C)_N = p^{(24)}(N). \tag{4.18}$$

It follows from the definition of $S(\alpha)$ that $\dim S(\alpha)_N = p^{(26)}(N)$. Also we deduce from Lemma 4.2 that $\dim G(\alpha, c)_N = \Sigma_{M<N} \dim T(\alpha, c)_M \cdot p^{(2)}(N-M) = p^{(26)}(N) - p^{(24)}(N)$ by induction (obviously $T(\alpha, c)_0 = 1$). This implies that $\dim T(\alpha, c)_N = p^{(24)}(N)$.

Now we consider the epimorphism $p\colon A \to \mathfrak{g}$ from Proposition 4.4. The subrepresentation $A \cap \{0\}$ is mapped to zero because $\mathfrak{g}$ does not contain nontrivial ideals. Thus we have an epimorphism $\bar{p}\colon \overline{A} = A/A \cap \{0\} \to \mathfrak{g}$. Thus $\dim \mathfrak{g}_\alpha \leqslant \dim \overline{A}_\alpha \leqslant \dim(\overline{V}_1)_\alpha = \mu(-\langle \alpha, \alpha\rangle/2)$. Q.E.D.

Proposition 4.5 and the results of [5] allow us to conjecture that for every hyperbolic algebra $\mathfrak{g}$ of rank l one has

$$\dim \mathfrak{g}_\alpha \leqslant p^{(l-2)}(-\langle \alpha, \alpha\rangle/2 + 1). \tag{4.19}$$

Moreover, the upper bound (4.16) is the best possible (when the function depends only on $\langle \alpha, \alpha\rangle$). In fact, Theorem 1.1 and Proposition 1.3 imply that for $\alpha \in \Delta_R$, isotropic $c \in \Delta$, $\langle c, \alpha\rangle = 1$,

$$\dim \mathfrak{g}_{\alpha - nc} = p^{(l-2)}(n) = p^{(l-2)}\left(-\frac{\|\alpha - nc\|^2}{2} + 1\right). \tag{4.20}$$

The no-ghost theorem distinctly displays the critical dimension 26 for dual resonance models. It is amazing that the same critical number appears in a theory which seemed to be absolutely remote from any physics, namely, the theory of finite simple groups. Recently Conway and collaborators [2] defined one infinite-dimensional Lie algebra, which they called the Monster Lie algebra, with the hope that it would "explain" the biggest sporadic group F_1 of Fisher and Griess. We will give another

definition of this algebra based on the vertex representation. Let $\mathfrak{h}_{\mathbf{R}} \approx \mathbf{R}^{25,1}$; then there is a unique even unimodular lattice Q in $\mathfrak{h}_{\mathbf{R}}$. We set $\Delta_R = \{\alpha \in Q\colon \langle \alpha, \alpha\rangle = 2\}$. Then we can construct the space V as in subsection 2.1. The Monster Lie algebra $\mathfrak{g}$ is by definition the Lie algebra generated by $\{X_0^\varepsilon(\alpha),\ \alpha \in \Delta_R\}$. Clearly $\mathfrak{g}$ contains all the hyperbolic algebras of rank 26.

Let us fix a light-cone element $c \in \Delta$ such that there are no real roots orthogonal to it. Such a vector exists and the set $L = \{\alpha \in \Delta_R\colon \langle \alpha, c\rangle = 1\}$ is isomorphic to the unique even unimodular lattice of rank 24, which does not contain elements of length $\sqrt{2}$ [2]. We denote by $\overline{V}_{1,c}$ the space $\Sigma_{\alpha \in L} \overline{V}_{1,c,\alpha}$. Then the character of $\overline{V}_{1,c}$ is

$$j(q) = \theta_L(q)/\eta(q)^{24} = q^{-1} + 24 + 196884q + \cdots. \tag{4.21}$$

It was noticed by McKay that the number 196884 exceeds by only one the dimension of the minimal representation of F_1. Conway and Norton [3] conjectured that there is a natural graded representation of F_1 with the character (4.21) minus 24. First Garland [12] and Kac [17] independently tried to construct F_1 in a space isomorphic to $\overline{V}_{1,c}$. The first problem was to obtain a representation of one important subgroup $C = 2_+^{24+1} \cdot (\cdot 0)/\pm 1$, where $\cdot 0$ is the automorphism group of the Leech lattice. It is easy to construct another group $C' = 2^{24} \cdot (\cdot 0)$ $(= (2_+^{24+1}/\pm 1) \cdot (\cdot 0))$. Using one observation of Griess, Kac [18] succeeded in passing from C' to C. The last question is: Where is the whole group F_1? Recently, important progress has been made in answer to this question [10]. Turning again to the dual resonance models gives a hint as to the answer. Physicists know that in the continuous version of $\overline{V}_{1,c}$ the obvious action of the group $O(24)$ can be extended to the bigger group $O(25)$. This extension becomes apparent only if we return to the bigger space $\overline{V}_1$. Whether this unusual phenomenon corresponds to the extension of C to F_1 will become clear in the future.

ACKNOWLEDGMENTS. I am grateful to A. Feingold, J. Lepowsky and A. Meurman for stimulating discussions. I thank A. Feingold for reading the manuscript and A. Rocha-Caridi for proving the exactness of (3.38).

ADDED IN PROOF. I have learned from P. Goddard and D. Olive that they independently discovered the vertex representation of the Lie algebra associated with an even unimodular lattice (see Section 2). I am grateful to P. Goddard for correction of Proposition 4.4.

The representation of F_1 in the graded space with the character $j(q)$–24 has been recently constructed by J. Lepowsky, A. Meurman and myself. However, the relation of this construction to the dual resonance model in the critical dimension 26 is still waiting for its explanation.

References

1. I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Differential operators on the base affine space and a study of* $\mathfrak{g}$*-modules, Lie groups and their representations*, Proc. Summer School on Group Representations (I. M. Gelfand, Ed.), Bolyai János Math. Soc., Wiley, New York, 1975, pp. 39–64.

2. J. H. Conway, L. Queen and N. J. A. Sloane, *A Monster Lie algebra?* (to appear).

3. J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), 308–339.

4. E. Del Guidice, P. DiVecchia and S. Fubini, *General properties of the dual resonance model*, Ann. Physics **70** (1972), 378.

5. A. J. Feingold and I. B. Frenkel, *A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus* 2, Math. Ann. **263** (1983), 82–144.

6. A. J. Feingold, and J. Lepowsky, *The Weyl-Kac character formula and power series identities*, Adv. in Math. **29** (1978), 271–309.

7. I. B. Frenkel, *Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory*, J. Funct. Anal. **44** (1981), 259–327.

8. ______, *Representations of affine Lie algebras, Hecke modular forms and Korteweg-de Vries type equations* (Proc. 1981 Rutgers Conf. Lie Algebras and Related Topics), Lecture Notes in Math., vol. 933; Springer-Verlag, Berlin and New York, 1982, pp. 71–110.

9. I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980), 23–66.

10. I. B. Frenkel, J. Lepowsky and A. Meurman, *An* E_8*-approach to* F_1, Proc. 1982 Montreal Conf. on Finite Group Theory (to appear).

11. O. Gabber and V. G. Kac, *On defining relations of certain infinite-dimensional Lie algebras*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), 185–189.

12. H. Garland, *Lectures on loop algebras and Leech lattice*, Yale University, Spring 1980.

13. V. G. Kac, *Simple irreducible graded Lie algebras of finite growth*, Math. USSR-Izv. **2** (1968), 1271–1311.

14. ______, *Infinite-dimensional Lie algebras and Dedekind's* η*-function*, J. Funct. Anal. Appl. **8** (1974), 68–70.

15. ______, *Infinite-dimensional algebras, Dedekind's* η*-function, classical Möbius function and the very strange formula*, Adv. in Math. **30** (1978), 85–136.

16. ______, *Contravariant form for infinite dimensional Lie algebras and superalgebras*, Lecture Notes in Physics, vol. 94, Springer-Verlag, Berlin and New York, 1979, pp. 441–445.

17. ______, *An elucidation of "Infinite-dimensional algebras... and the very strange formula."* $E_8^{(1)}$ *and the cube root of the modular invariant j*, Adv. in Math. **35** (1980), 264–273.

18. ______, *A remark on the Conway-Norton conjecture about the "Monster" simple group*, Proc. Nat. Acad. Sci. U.S.A. **77** (1980), 5048–5049.

19. V. G. Kac, D. A. Kazhdan, J. Lepowsky and R. L. Wilson, *Realization of the basic representations of the Euclidean Lie algebras*, Adv. in Math. **42** (1981), 83–112.

20. J. Lepowsky, *Application of the numerator formula to k-rowed plane partitions*, Adv. in Math. **35** (1980), 179–194.

21. J. Lepowsky and R. L. Wilson, *Construction of the affine Lie algebra* $A_1^{(1)}$, Comm. Math. Phys. **62** (1978), 43–63.

22. ______, *A Lie theoretic interpretation and proof of the Rogers-Ramanujan identities*, Adv. in Math. **43** (1982).

23. ______, *The structure of standard modules. I. Universal algebras and the Rogers-Ramanujan identities* (to appear).

24. R. V. Moody, *A new class of Lie algebras*, J. Algebra **10** (1968), 211–230.

25. J. Scherk, *An introduction to the theory of dual models and strings*, Rev. Modern Physics **47** (1975), 123–164.

26. J. H. Schwarz, *Dual-resonance theory*, Phys. Rep. **8** (1973), 269–335.

27. ______, *Mathematical issues in superstring theory*, these Proceedings.

28. G. Segal, *Unitary representations of some infinite dimensional groups*, Comm. Math. Phys. **80** (1981), 301–342.

Department of Mathematics, Yale University, New Haven, Connecticut 06520

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903