Robust Strategic Analysis

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- report joint work with Stephen Morris (Princeton University)

1. "Robust Predictions in Games with Incomplete Information"
2. "Correlated Equilibrium in Games with Incomplete Information"

in economic theory, we often distinguish....

- exogenous variables...
- endogenous variables...

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- exogenous variables...
  - preferences, endowments, technology
- endogenous variables...
  - consumption and production choices, prices

in game theory with incomplete information,
and economic applications such as auctions, entry/exit games,
we often distinguish....

- exogenous variables...
- endogenous variables...

in game theory with incomplete information,
and economic applications such as auctions, entry/exit games,
we often distinguish....

- - actions, states, payoff functions, information structure
- endogenous variables...
  - strategies
- find equilibrium strategy profiles of fixed game

Robust Strategic Analysis

- examine all (or many) information structures at once...
- characterize the set of equilibria that might result for all information structures

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- examine all (or many) information structures at once...
- characterize the set of equilibria that might result for all information structures
- why is this useful (and feasible)?

1. Robust Predictions

- information structure = type space
- includes description of beliefs and higher order beliefs, which are often hard to observe
- what predictions about outcomes (i.e., actions and states) can we make if we do not know some aspects of the information structure?
- in addition, game theoretic predictions are very sensitive to the information structure, the nature of the higher order beliefs as famously observed in Rubinstein's email game

- analyst/outside observer is thus tpyically uncertain about the exact nature of the information structure
- mapping from non-informational structural parameters (actions, payoff functions, states) ...
  ... to observables (distribution over actions and states) ...
  ... independent of information structure

2. Robust Identification

- reverse mapping from observables (distribution over actions and states)...

  ... to structural parameters... .

  ... independent of information structure
- but informational assumptions are frequently key to identifying assumptions (empirical auction work)...
- ...suggest attempt to understand the role of informational assumptions more fully

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- set of outcomes that could arise if a mediator who knew the payoff state could privately make action recommendations
- set of incomplete information correlated equilibrium outcomes

3. Correlated Equilibrium Characterization

- Robust predictions =
- set of outcomes consistent with equilibrium given any additional information the players may observe
- set of outcomes that could arise if a mediator who knew the payoff state could privately make action recommendations
- set of incomplete information correlated equilibrium outcomes
- we shall call this very permissive version of incomplete information correlated equilibrium "Bayes correlated equilibrium"

4. Value of Information in Strategic Settings

- set of all information structures appears like a complicated object to characterize or even to optimize over
- the correlated equilibrium characterization gives a very natural way of doing this
- valuable tool for optimizing over all information structures (not just a parameterized class)

Robust Strategic Analysis

- examine all (or many) information structures at once...
- characterize the set of equilibria that might result for all information structures
- why is this useful (and feasible)?
  1. Robust Predictions
  2. Robust Identification
  3. Correlated Equilibrium Characterization
  4. Value of Information in Strategic Settings

1. consider a continuum player linear best response normal setting environment widely used in applications
   - key "interaction parameter," sensitivity of individual's action to aggregate behavior

Today's Talk

1. consider a continuum player linear best response normal setting widely used in applications
2. correlated equilibrium characterization:
    - directly construct symmetric normal Bayes correlated equilibria
    - compare with explicit construction of what could happen in all equilibria for all symmetric normal information structures

Today's Talk

1. consider a continuum player linear normal payoff environment widely used in applications
2. correlated equilibrium characterization
3. value of information in strategic settings
   - which information structure maximizes profit of representative firm?
   - would firms in competitive (or oligopolistic market) like private information about demand to be shared?
     we show that noisy information sharing is optimal

Today's Talk

1. consider a continuum player linear normal payoff environment widely used in applications
2. correlated equilibrium characterization
3. value of information in strategic settings
4. robust predictions and identification
    - interaction parameter has few robust predictions and cannot be robustly identified with zero knowledge about the true information structure)
    - robust prediction and partial identification improve smoothly with partial knowledge of the information structure

- continuum of agents: $i \in [0, 1]$
- utility of agent i depends on own action $a_i \in \mathbb{R}$, average action $A \in \mathbb{R}$ and state of the world $\theta \in \mathbb{R}$,

$$u(a, A, \theta) = -(1-r)(a-\theta)^2 - r(a-A)^2$$

- best response:

$$a_i = (1-r) E_i(\theta) + r E_i(A)$$

- the state of the world θ is normally distributed (common prior)

$$\theta \sim N\left(\mu_\theta, \sigma_\theta^2\right)$$

Generalizations referred to in this talk....

- general linear best responses...

$$a_i = u + cE_i(\theta) + rE_i(A)$$

  includes competitive economies with linear demand and quadratic costs
- finite players....
- general games and uncertainty....

- each agent has linear best response function

$$a = (1 - r)\, E_i\,(\theta) + rE_i\,(A)$$

- we will call r the "interaction parameter" and focus on this payoff parameter in this talk
- $r < 0$: strategic substitutes
- $r = 0$: no strategic interaction, single person decision problem
- $r > 0$: strategic complementarities

- assume $r < 1$
- necessary for model to be well behaved
- unique Nash (and correlated) equilibrium if there is complete information about θ:

$$a^* (\theta) = \theta$$

Statistical Model of Observables

- we are interested in what could happen in a normal symmetric equilibrium for any information structurethe players might have
- before imposing optimality, how to statistically describe normal symmetric behavior?
- multivariate normal distribution
  1. mean: $\mu_\theta, \ \mu_a, \ \mu_A$
  2. variance-covariance: $\sigma_\theta, \sigma_a, \sigma_A, \rho_{a\theta}, ...$

Statistical Model of Observables

- exogenous parameters: $\mu_\theta, \sigma_\theta$
- endogenous parameters: $\mu_a, \sigma_a, \rho_a, \rho_{a\theta}$
- symmetry imposes restrictions:

$$\mu_A = \mu_a, \quad \sigma_A^2 = \rho_a \sigma_a^2, \quad \dots$$

- multivariate normal distribution

$$\begin{pmatrix} a \\ A \\ \theta \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_a \\ \mu_a \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma_a^2 & \rho_a \sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_a \sigma_a^2 & \rho_a \sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\theta}\sigma_a\sigma_\theta & \sigma_\theta^2 \end{pmatrix} \right)$$

- restriction from positive semidefiniteness of variance covariance matrix:

$$\rho_a \geq \rho_{a\theta}^2$$

- suppose the distribution was

$$\begin{pmatrix} a_i \\ A \\ \theta \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_a \\ \mu_a \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma_a^2 & \rho_a \sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_a\sigma_a^2 & \rho_a\sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\theta}\sigma_a\sigma_\theta & \sigma_\theta^2 \end{pmatrix} \right)$$

- what additional restrictions are imposed on the four endogenous parameters

$$\{\mu_a, \ \sigma_a, \ \rho_a, \ \rho_{a\theta}\}$$

by the assumption that agents (who may know something that we don't) are setting

$$a_i = (1 - r) E_i(\theta) + r E_i(A) \, ?$$

- in particular, their action could be informed beyond common prior

- a Bayes correlated equilibrium, BCE, is a joint distribution

$$\begin{pmatrix} a_i \\ A \\ \theta \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_a \\ \mu_a \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma_a^2 & \rho_a\sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_a\sigma_a^2 & \rho_a\sigma_a^2 & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\theta}\sigma_a\sigma_\theta & \sigma_\theta^2 \end{pmatrix}\right)$$

satisfying

$$a_i = (1-r)\,\mathbb{E}_i\,[\theta|a_i] + r\mathbb{E}_i\,[A|a_i]$$

everywhere

- an "obedience" constraint formed by conditional expectation

- impose

$$a_i = (1-r)\,\mathbb{E}_i\left[\theta|a_i\right] + r\mathbb{E}_i\left[A|a_i\right]$$

- updating rules

$$\begin{aligned}\mathbb{E}_i\left[\theta|a_i\right] &= \mu_\theta + \frac{\rho_{a\theta}\sigma_\theta}{\sigma_a}\left(a_i - \mu_\theta\right) \\ \mathbb{E}_i\left[A|a_i\right] &= \mu_a + \rho_a\left(a_i - \mu_a\right)\end{aligned}$$

- so best response becomes

$$a_i = (1-r)\left[\mu_\theta + \frac{\rho_{a\theta}\sigma_\theta}{\sigma_a}\left(a_i - \mu_\theta\right)\right] + r\left[\mu_a + \rho_a\left(a_i - \mu_a\right)\right]$$

holds everywhere

- best response holds for all a_i and by "law of iterated (or total) expectation", replace a_i by μ_a :

$$\mu_a = (1-r)\left[\mu_\theta + \frac{\rho_{a\theta}\sigma_\theta}{\sigma_a}(\mu_a - \mu_\theta)\right] + r\left[\mu_a + \rho_a(\mu_a - \mu_a)\right]$$

Theorem (First Moment)

In all Bayes correlated equilibria, the mean action is given by:

$$\mu_a = \mu_\theta$$

- hence, we can focus on higher moments, in particular second moments

- we still need to determine:

$$\{\sigma_a,\ \rho_a,\ \rho_{a\theta}\}$$

- and best response has to hold everywhere, best response imposes restriction on covariance of actions, "law of total variance"

Theorem (Second Moment)

The triple $(\sigma_a, \rho_a, \rho_{a\theta})$ forms a Bayes correlated equilibrium iff:

$$\rho_a - \rho_{a\theta}^2 \geq 0,$$

and

$$\sigma_a = \frac{(1-r)\,\rho_{a\theta}}{1-\rho_a r}\sigma_\theta .$$

- must have $\rho_{a\theta} > 0$ and statistical requirement $\rho_a \geq \rho_{a\theta}^2$ but no other restrictions on correlation coefficients....

Properties of BCE

- two dimensional array of BCE parameterized by $\rho_a, \rho_{a\theta}$ with

$$\rho_a \geq \rho_{a\theta}^2$$

- ANY $r \in (-\infty, 1)$ is consistent with ANY $(\rho_a, \rho_{a\theta})$
  - (and observing σ_θ does not allow r to be identified with general linear payoffs)

Standard (Fixed Information Structure) Approach

- fix arbitrary information structure, e.g. bivariate information structure:

1. every agent i observes a public signal y about θ :

$$y \sim N\left(\theta, \frac{1}{\tau_y}\right)$$

2. every agent i observes a private signal x_i about θ :

$$x_i \sim N\left(\theta, \frac{1}{\tau_x}\right)$$

- precision of private and public signal

$$\tau_x = \frac{1}{\sigma_x^2}, \quad \tau_y = \frac{1}{\sigma_y^2}$$

- different information structures parameterized by (τ_x, τ_y)

Unique Bayes Nash Equilibrium

- precision of common prior is

$$\tau_\theta = \frac{1}{\sigma_\theta^2}$$

- unique Bayes Nash equilibrium, BNE, is:

$$a^*(x,y) = \frac{\tau_\theta \mu_\theta + \tau_y y + (1-r)\tau_x x}{\tau_\theta + \tau_y + (1-r)\tau_x}$$

- Each information structure (τ_x, τ_y) will imply unique equilibrium and corresponding values of

$$(\mu_a,\ \sigma_a,\ \rho_a,\ \rho_{a\theta})$$

- In particular, we have
  - restriction on r, ρ_a and $\rho_{a\theta}$ via private signal τ_x:

$$\frac{\tau_x}{\tau_\theta}\left(1-\rho_{a\theta}^2 - r\left(\rho_a - \rho_{a\theta}^2\right)\right)^2 = \left(1-\rho_a\right)\rho_{a\theta}^2 \tag{1}$$

  - restriction on r, ρ_a and $\rho_{a\theta}$ via public signal τ_y :

$$\frac{\tau_y}{\tau_\theta}\left(\frac{1-\rho_a}{1-r} + \rho_a - \rho_{a\theta}^2\right)^2 = \left(\rho_a - \rho_{a\theta}^2\right)\rho_{a\theta}^2 \tag{2}$$

- consider fixed precision τ_x of private signal
- moderate strategic complementarities, $r = \frac{1}{4}$

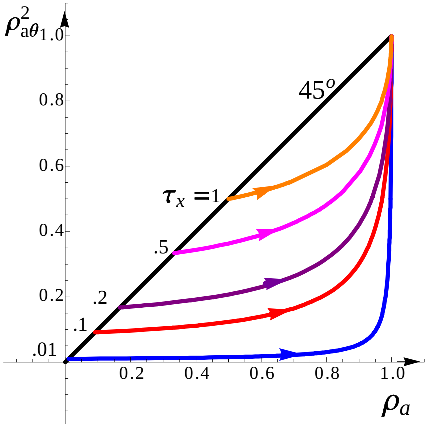


Figure:

- consider given precision τ_y of public signal
- moderate strategic complementarities, $r = \frac{1}{4}$

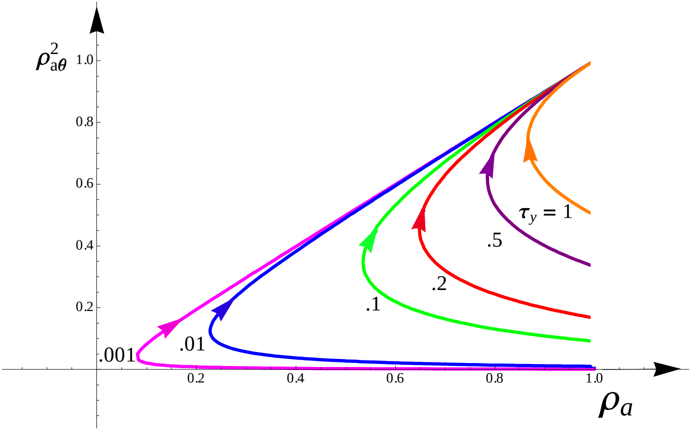


Figure:

BCE and Robust Predictions: Normal Example

- For each $r \in (-\infty, 1)$ and $(\rho_a, \rho_{a\theta})$, there is a unique value of τ_x solving (1) and a unique value of τ_y solving (2)

Theorem

There is BCE with $(\rho_a, \rho_{a\theta})$ iff there is a BNE with (τ_x, τ_y).

- Any other symmetric normal information structure, but with possibly multi-dimensional and dependent signals, would also have a unique Bayes Nash Equilibrium which would give rise to a BCE within same class
- Recall that:
  - BCE $=$ set of symmetric normal distributions over actions and states satisfying "obedience"
  - robust predictions $=$ set of distributions over actions and states that could arise in a BNE for any symmetric normal information structure
- BCE $=$ robust predictions

BCE and Robust Predictions: General Games

- same thing is true in general games (without any symmetry or normality restrictions)
- recall that:
  - BCE = set of distributions over actions and states satisfying "obedience"
  - robust predictions = set of distributions over actions and states that could arise in a BNE for any information structure
- BCE = robust predictions
- incomplete information analogue of Aumann (1987)
- BCE weaker than standard definitions of correlated equilibrium in incomplete information games
  - impose additional feasibility or incentive constraints

Value of Information in Strategic Settings

- Often interested in identifying "best" information structure
- Our results suggest an approach for analyzing this problem:
  - find the set of BCE
  - find the best BCE
  - find the information structure that supports this BCE as a BNE

Value of Information in Strategic Settings

- If $r < 0$ (strategic substitutes), our model is isomorphic to competitive economy with linear demand curve and quadratic costs with uncertainty about the demand intercept
- What information structure would firms prefer?
- large literature: Novshek and Sonnenschein (1982), Clarke (1983), Vives (1984) and huge literature since
- We will answer this question using our BCE methodology (and get a new answer)

- Agent payoff in BCE is increasing in

$$\frac{\rho_{a\theta}^2}{(1-\rho_a r)^2}$$

- In market interpretation,
    - $|r|$ measures slope of demand curve
    - firms want high correlation of output with demand (high $\rho_{a\theta}$) but low correlation of output among firms (low ρ_a)
- This is maximized (subject to $\rho_a \geq \rho_{a\theta}^2$) setting

$$\rho_a = \rho_{a\theta}^2 = \begin{cases} 1, \text{ if } r \geq -1 \\ \left|\frac{1}{r}\right|, \text{ if } r \leq -1 \end{cases}$$

Value of Information in Strategic Settings

- Best information structure if $r \geq -1$ is complete information
- Best information structure if $r < -1$ (demand is elastic) is zero public signal

$$\tau_y = 0$$

and noisy private signal

$$\tau_x = \frac{\tau_\theta}{|r|}$$

- noise is increasing in the elasticity of demand

Finite Number of Firms

- with continuum agents, not possible to have actions negatively correlated ($\rho_a < 0$)
- our analysis extends straightforwardly to finite instead of continuum of agents, with a few interesting changes
- with N agents, optimal to have actions as negatively correlated as possible in a symmetric normal distribution, so that $\rho_a = -\frac{1}{N}$
- cannot be attained by public and conditionally independent private signals

Sharing Information

- Classical literature asks different but closely related question:
  - suppose firms start with some private information;
  - would they like to share some or all of that information (if they could commit to do so)?
- Our answer to this question:
  - if demand is elastic ($r < -1$) and each firm does not have too much information about demand, they would like to pass their information to an intermediary who would aggregate it and then send conditionally independent (negatively correlated in the finite firm case) signals about demand back to the firms
- by contrast, literature concluded that zero or complete disclosure is optimal

Robust Predictions and Identification

- in the general linear best response setting, limited results for robust prediction and negative result for robust identification:
  - Strategic Interaction parameter (r) has no implications for $(\rho_a, \rho_{a\theta})$
  - observing $(\rho_a, \rho_{a\theta})$ says nothing about r
- Negative result because we assume NOTHING known about the information structure

- the analyst may not know how much private and public information the agents have, yet he may have a lower bound on how much information the agents have
- what can happen in any BNE if agents know this but possibly more?
- generalized version of BCE when agents also have known private signals
- assume public and private signal:

$$y \sim N\left(\theta, \frac{1}{\tau_y}\right), \quad x_i \sim N\left(\theta, \frac{1}{\tau_x}\right)$$

- analysis extends in similar way to general games and general information structures

- look at five variables (a_i, x_i, y, A, θ)
- impose exogenous distribution on (x_i, y, θ)
- impose best response condition now augmented from for all "a" to "for all a, x, y" as additional "incentive constraints"

$$a = (1-r)\mathbb{E}\left[\theta \,|a, x, y\right] + r\mathbb{E}\left[A \,|a, x, y\right], \quad \forall a, x, y.$$

- new variables ρ_{ax}, ρ_{ay}

- new equality, e.g.,

$$\rho_{ay} = \frac{\sigma_\theta}{\sigma_y \rho_{a\theta}} \left(\frac{1-\rho_a r}{1-r} - \rho_{a\theta}^2 \right)$$

and inequality

$$\rho_a - \rho_{a\theta}^2 - \rho_{ay}^2 \geq 0,$$
$$1 - \rho_a - \rho_{ax} \geq 0,$$

conditions

- if signals are **not** observed, implications for old endogenous variables $(\mu_a, \ \sigma_a, \ \rho_a, \ \rho_{a\theta})$ are inequality versions of (1) and (2):

$$\frac{\tau_x}{\tau_\theta} \left(1 - \rho_{a\theta}^2 - r\left(\rho_a - \rho_{a\theta}^2\right)\right)^2 \leq \left(1-\rho_a\right)\rho_{a\theta}^2 \qquad (3)$$

$$\frac{\tau_y}{\tau_\theta} \left(\frac{1-\rho_a}{1-r} + \rho_a - \rho_{a\theta}^2 \right)^2 \leq \left(\rho_a - \rho_{a\theta}^2\right)\rho_{a\theta}^2$$

- the interior of each level curve (or the intersection thereof) describes the correlated equilibria for a given lower bound on private and public information
- more information reduces set of possible distributions: adds incentive constraints, does not remove correlation possibilities

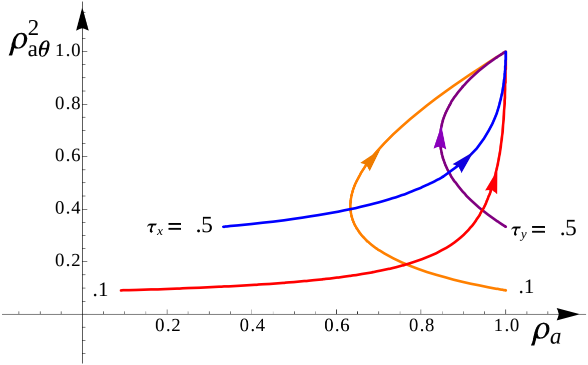


Figure:

Partial Identification of Interaction Parameter

- interaction parameter $r \in (-\infty, 1)$
- extreme case 1: with zero knowledge information structure, cannot say anything about the value of r
- extreme case 2: in BNE with known exogenous information structure we can infer r *either* from precision of private signal τ_x, (1), or from precision of public signal τ_y, (2)

Testing for Strategic Complementarities

- candidate test for strategic complementarities
- *high correlation of actions for a given action state correlation:*

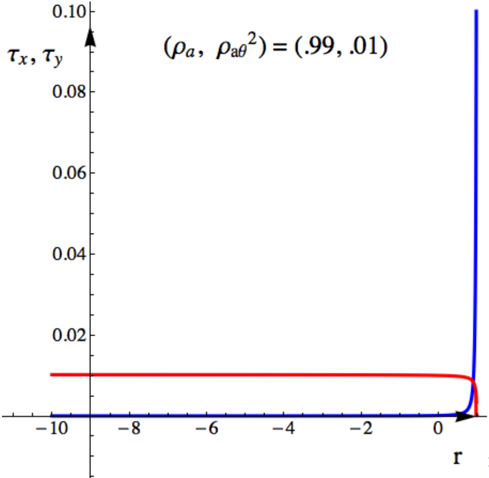
$$\rho_a >> \rho_{a\theta}^2$$

suggests strategic complementarities (i.e., positive value of r)

- test fails in general: we will have $\rho_a >> \rho_{a\theta}^2$ if there is no private signal....
- actual proof of strategic complementarities: *if players are known to have at least precision τ_x, then sufficiently high correlation of actions given action state correlation proves strategic complementarity ($r > 0$):*

$$\rho_a > 1 - \frac{\left(1 - \rho_{a\theta}^2\right)^2 \tau_x}{\rho_{a\theta}^2 \tau_\theta}$$

- as a function of interaction parameter r, what precision of private and public signal would support $\rho_a = 0.99$ and $\rho_{a\theta} = 0.01$?
- blue line is private precision τ_x, red line is public precision τ_y



Testing for Strategic Substitutes

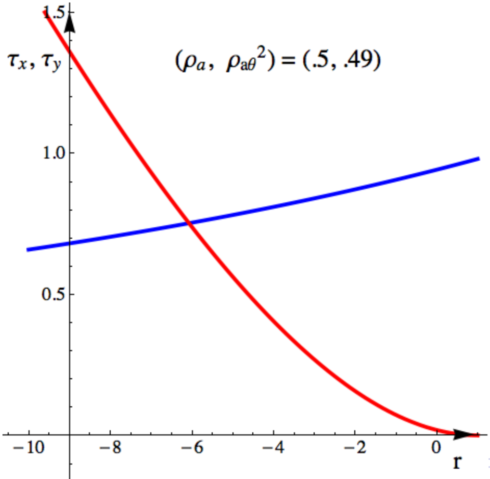
- candidate test for strategic substitutes
- *low correlation of actions for a given action state correlation:*

$$\rho_a \approx \rho_{a\theta}^2$$

  suggests strategic substitutes (i.e., negative value of r)
- test fails in general: we will have $\rho_a \approx \rho_{a\theta}^2$ if there is no public signal....
- actual proof of strategic complementarities: *if players are known to have at least precision τ_y, then sufficiently low correlation of actions given action state correlation proves strategic substitutes ($r < 0$):*

$$\rho_a < \rho_{a\theta}^2 + \frac{\left(1 - \rho_{a\theta}^2\right)^2 \tau_y}{\rho_{a\theta}^2 \tau_\theta}$$

- as a function of interaction parameter r, what precision of private and public signal would support $\rho_a = 0.50$ and $\rho_{a\theta} = 0.49$?
- blue line is private precision τ_x, red line is public precision τ_y



Conclusion

- instead of analyzing an incomplete information game with a fixed information structure...
- ... it is easier - because correlated equilibrium is well behaved - and insightful - for robustness and value of information structure questions - to ask what happens for lots of information structures at once
- compare information structures $\mathcal{I}$ and $\mathcal{I}'$: additional, or better information, generate additional incentive constraints, restrict the set of BCE
- BCE($\mathcal{I}$) $\subset$ BCE($\mathcal{I}'$) $\Leftrightarrow$ $\mathcal{I}$ is more informative than $\mathcal{I}'$
- equivalent to multi-agent generalization of Blackwell's sufficiency condition