DYNAMIC COMMON AGENCY

BY

DIRK BERGEMANN

AND

JUUSO VÄLIMÄKI

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Dynamic common agency

Dirk Bergemann[a,*] and Juuso Välimäki[b]

[a] Department of Economics, Yale University, 28 Hillhouse Avenue, New Haven, CT 06520-8268, USA
[b] Department of Economics, Helsinki School of Economics, Runeberginkatu 22-24, 0100 Helsinki, Finland and University of Southampton, Southampton, UK

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Abstract

A general model of dynamic common agency with symmetric information is considered. The set of truthful Markov perfect equilibrium payoffs is characterized and the efficiency properties of the equilibria are established. A condition for the uniqueness of equilibrium payoffs is derived for the static and the dynamic game. The payoff is unique if and only if the payoff of each principal coincides with his marginal contribution to the social value of the game. The dynamic model is applied to a game of agenda setting.
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1. Introduction

Common agency refers to a broad class of problems in which a single individual, the agent, controls a decision that has consequences for many individuals with distinct preferences. The other affected parties, the principals, influence the agent's decisions by promising payments contingent on the action chosen. The static model of common agency under perfect information was introduced by Bernheim and Whinston [4] as a model of an auction where bidders are submitting a menu of offers to the auctioneer. Since then it has gained prominence in many applications, such as procurement contracting, models of political economy [6,7], as well as strategic international trade [9].

*Corresponding author. Fax: 1-203-432-5779.
E-mail addresses: dirk.bergemann@yale.edu (D. Bergemann), valimaki@hkkk.fi (J. Välimäki).

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In this paper, we examine the structure of dynamic common agency problems. The extension of the model beyond the static version is of particular interest for the applications above. Political choices are rarely made only once, and the future implications of a current policy are often more important than its immediate repercussions. If the politician and the lobbyists cannot commit to future actions and transfers, a dynamic perspective is needed. Similarly, many procurement situations involve staged development with bidding occurring at each stage of the process.

The dynamic perspective also broadens the reach of common agency models. Consider for example a dynamic matching problem where the employee works in each period for at most a single employer, but may change employers over time. In the language of common agency, the employee has only one principal in each period. If human capital is acquired over time within jobs, then future employers may have preferences over the current career choices of the employee. Thus the intertemporal element introduces a more subtle aspect of common agency to various allocations problems such as career choice and job matching models.

We start our analysis with the static common agency model of Bernheim and Whinston [4], who concentrated on a refinement of Nash equilibrium, called *truthful equilibrium*. A strategy is said to be truthful relative to a given action if it reflects accurately the principals' willingness to pay for any other action relative to the given action. For the static game, we show that the truthful equilibrium payoff is unique if and only if the marginal contributions of the principals to the value of the grand coalition are weakly superadditive. We show that in such equilibria, all principals receive their marginal contributions as payoffs. We call equilibria satisfying this property *marginal contribution equilibria*.

The second part of the paper derives conditions for the existence of a marginal contribution equilibrium in the dynamic framework. Since we assume that the players lack commitment power over periods, the interplay between the payoffs received at different stages of the game becomes important. The first model we analyze is perhaps the simplest of all dynamic common agency models and illustrates the importance of changes in the stage game. This is a two-period game where in the first period, the agent chooses the available actions for the second stage. In the second stage, the common agency game is played with the set of actions as determined in the first period. We refer to this game as the agenda game as the agent initially determines the set of actions (agenda) from which the choice will be eventually made. Depending on the context this may represent the choice of the relevant policy alternatives by a political decision maker or the choice of the auction format by an auctioneer, such as how different licenses or rights should be bundled in a multi-unit auction.

The equilibrium payoff to the agent depends on the degree of competition between the principals in the second stage. Since the competitiveness in turn depends on the set of available actions, the agent is sometimes able to increase her second period rent by excluding the efficient action. This leads to an inefficiency in the overall game unless we allow the principals to lobby the first period choices of the agent as well.

With lobbying in both periods, the overall efficiency is restored, but the payoff to the agent is higher in the two-stage game than in the static game where the first period choice of actions is ignored. In the following sections, we show that the essential features of the agenda game, in particular the aspect of intertemporal rent extraction, carry over to a general model of dynamic common agency.

In the rest of the paper, we can be quite general in formulating the states of nature governing the transitions between stage games. In particular, we can accommodate deterministic as well as stochastic transitions between states. We require that the current state depends only on the previous state and the previous action by the agent. The former of these assumptions is not crucial to the argument and is made to simplify the exposition. The second assumption is more substantial and it is made to preserve the flavor of the static model where the agent is the only player that affects the utilities directly. We concentrate on Markov strategies since we want to study the effects of changes in the stage game in isolation from the effects created by conditioning on payoff irrelevant histories. In the spirit of Bernheim and Whinston [4], we are particularly interested in truthful Markovian policies.

We start by proving the existence of truthful Markov equilibria in the dynamic model. As in the static case, all truthful Markov perfect equilibria of the dynamic game are efficient. The truthful equilibrium payoffs are also unique if and only if the game possesses a marginal contribution equilibrium. We characterize the necessary and sufficient conditions for the dynamic game to have a marginal contribution equilibrium in terms of the trade-off between efficiency and rent extraction for the agent.

The papers that are the most closely related to the current paper are those by Bernheim and Whinston [4], Dixit et al. [7] and Laussel and Le Breton [11], (as well as [10]). Our paper extends the static model of common agency in Bernheim and Whinston [4] to a dynamic setting. The second major point of departure is our focus on marginal contribution equilibria as an interesting solution concept in this class of games. Another recent extension of the basic model of common agency may be found in [7], where the assumption of quasi-linear preferences is dropped. Whereas the motivation for that extension is based on concerns relating to the distribution of the payoffs within the single period of analysis, our motivation is based on the distribution of payoffs between the players over time when commitment is precluded. The work by Laussel and Le Breton analyzes the payoffs received by the agent in a class of static common agency games. A recent paper by Prat and Rustichini [14] extends the common agency game to many common agents.

The paper is organized as follows. In Section 2 we introduce the common agency model in its dynamic version. The notion of marginal contribution is introduced here as well. Section 3 introduces the basic results for the static model of common agency. Section 4 analyzes the agenda setting game. Section 5 presents the main results for the dynamic common agency. The characterization of the truthful Markov perfect equilibrium is given here and necessary and sufficient conditions for its uniqueness are stated as well.

2. Model

2.1. Payoffs

We extend the common agency model of Bernheim and Whinston [4] to a dynamic setting. The set of players is the same in all periods, but actions available to them as well as payoffs resulting from the actions may change from period to period.

The principals are indexed by $i \in \mathscr{I} = \{1, \ldots, I\}$. Time is discrete and indexed by $t = 0, 1, \ldots, T$, where T can be finite or infinite. In period 0, the agent selects an action a_0 from a finite set of available actions $\mathscr{A}_0$, and principal i offers a reward schedule $r_i(a_0) \in \mathbb{R}_+$. The stage game may change from period to period and the payoff relevant state of the world (in the sense of Maskin and Tirole [13]) in period t is denoted by $\theta_t \in \Theta$. For simplicity, we assume that $\Theta = \{\theta_1, \ldots, \theta_K\}$ for some $K < \infty$.[1] In the spirit of Bernheim and Whinston [4], only the agent makes directly payoff relevant choices in any of the periods. To that effect, we assume that the transition function, $q(\theta_{t+1}|a_t, \theta_t)$ is Markovian in the sense that the distribution of the payoff relevant state in period $t+1$ depends only on the current action, a_t, and the current state θ_t. By setting $h_1 = (\theta_0, a_0, r_0, \theta_1)$, the histories for $t > 1$ in the game are given by

$$h_t = (h_{t-1}, a_{t-1}, \mathbf{r}_{t-1}, \theta_t),$$

where $\boldsymbol{r}_t$ is the profile of reward schedules in period t and $a_t \in \mathscr{A}(\theta_t)$, the set of available actions in state θ_t. For simplicity, we assume that $\bigcup_{\theta \in \Theta} \mathscr{A}(\theta)$ is finite. We maintain the assumption made in the common agency literature that the principals can commit in every period to the reward schedules.

The cost of action a_t in period t to the agent is given by $c(a_t, \theta_t)$. The benefit to principal i is $v_i(a_t, \theta_t)$, which may again depend on θ_t. After a history h_t the aggregate reward paid by a subset of principals $S \subset \mathscr{I}$ for an action a_t is

$$r_S(a_t, h_t) \triangleq \sum_{i \in S} r_i(a_t, h_t),$$

and the aggregate benefits for the principals are

$$v_S(a_t, \theta_t) \triangleq \sum_{i \in S} v_i(a_t, \theta_t).$$

For $S = \mathscr{I}$, the aggregate rewards and benefits are denoted by $r(a_t, h_t) \triangleq r_{\mathscr{I}}(a_t, h_t)$ and $v(a_t, \theta_t) \triangleq v_{\mathscr{I}}(a_t, \theta_t)$, respectively. Without loss of generality we shall assume that $v_i(a_t, \theta_t) \geqslant 0$ and $c(a_t, \theta_t) \geqslant 0$ for all a_t and θ_t. We also assume the existence of a (default) action $a_t \in \mathscr{A}_t(\theta_t)$ such that $c(a_t, \theta_t) = 0$ for all θ_t.

All players maximize expected discounted value and their common discount factor for future periods is $\delta \in (0, 1)$.

[1] The only place that uses this finiteness assumption is the existence proof in Section 5. Under appropriate continuity properties for state transitions, more general state spaces can be incorporated as well. An example with a continuous state space is the job matching model given in an earlier version of this paper.

2.2. Social values

With transferable utility between the agent and the principals, Pareto efficiency coincides with surplus maximization. The value of the socially efficient program is denoted by

$$W(\theta_t) \triangleq W_{\mathscr{I}}(\theta_t),$$

and the value of the efficient program with a subset S of principals and the agent is denoted by $W_S(\theta_t)$. These values are obtained from a familiar dynamic programming equation:

$$W_S(\theta_t) = \max_{a_t \in \mathscr{A}(\theta_t)} \mathbb{E}\{v_S(a_t, \theta_t) - c(a_t, \theta_t) + \delta W_S(\theta_{t+1})\}.$$

The efficient action for the set S in state θ_t is denoted by $a_S^* \triangleq a_S^*(\theta_t)$ and for the entire set $\mathscr{I}$, it is $a^* \triangleq a_{\mathscr{I}}^*(\theta_t)$. The value of a set of firms $\mathscr{I} \setminus S$ is similarly denoted by $W_{-S}(\theta_t)$.

The marginal contribution of principal i is defined by

$$M_i(\theta_t) \triangleq W(\theta_t) - W_{-i}(\theta_t). \tag{1}$$

The marginal contribution of a subset of principals $S \subset \mathscr{I}$ is by extension

$$M_S(\theta_t) \triangleq W(\theta_t) - W_{-S}(\theta_t). \tag{2}$$

The marginal contribution of a set of principals S measures the increase in the total value to the grand coalition which results from adding the set S of principals. We emphasize that for all social values $W_S(\theta_t)$ the agent is always implicitly included in the set S of principals, whereas for all marginal contributions $M_S(\theta_t)$, the agent is always excluded from set S.

3. Static common agency

3.1. Equilibrium characterization

This section presents the equilibrium concept and new characterization results for the static common agency game. As we discuss the static model here, the state variable θ_t is omitted in this section. The basic model and the equilibrium notions were first introduced by Bernheim and Whinston [4].

A strategy for principal i is a reward function $r_i : \mathscr{A} \to \mathbb{R}_+$ by which the principal offers a reward to the agent contingent on the action chosen by her. The net benefit from action a to principal i is $n_i(a) \triangleq v_i(a) - r_i(a)$. The vector of net benefits is $n(a) = (n_1(a), \ldots, n_I(a))$ and the aggregate benefits for a subset S is $n_S(a) \triangleq \sum_{i \in S} n_i(a)$. The net benefit to the agent is given by $r(a) - c(a)$.

Definition 1 (Best response).

1. An action a is a best response to the rewards $r(\cdot)$ if
$$a \in \arg\max_{a' \in \mathscr{A}} (a') - c(a').$$
2. A reward function $r_i(\cdot)$ is best response to the rewards $r_{-i}(\cdot)$, if there does not exist another reward function $r_i'(\cdot)$ and action a' such that
$$v_i(a') - r_i'(a') > v_i(a) - r_i(a),$$
where a and a' are best responses to $(r_i(\cdot), r_{-i}(\cdot))$ and $(r_i'(\cdot), r_{-i}(\cdot))$, respectively.

Definition 2 (Nash equilibrium). A *Nash equilibrium* of the common agency game is an I-tuple of reward functions $\{r_i(\cdot)\}_{i=1}^I$ and an action a such that $r_i(\cdot)$ and a are best responses.

Bernheim and Whinston suggest that the focus be put on a subset of the Nash equilibria where all strategies satisfy an additional restriction, called truthfulness.

Definition 3 (Truthful strategy).

(1) A reward function $r_i(\cdot)$ is said to be truthful relative to a if for all $a' \in \mathscr{A}$, either
  (a) $n_i(a') = n_i(a)$, or,
  (b) $n_i(a') < n_i(a)$, and $r_i(a') = 0$.

(2) The strategies $\{\{r_i(\cdot)\}_{i=1}^I, a\}$ are said to be a truthful Nash equilibrium if they form a Nash equilibrium and $\{r_i(\cdot)\}_{i=1}^I$ are truthful strategies relative to a.

The main result of Bernheim and Whinston [4] describes the set of truthful equilibrium payoffs as follows.

Theorem 1 (Bernheim and Whinston). *A vector* $\mathbf{n} \in \mathbb{R}^I$ *is a vector of net payoffs for some truthful equilibrium if and only if*

1. *for all* $S \subseteq \mathscr{I}$,
$$\sum_{i \in S} n_i \leqslant M_S, \quad \textit{and}$$
2. *for all* $i \in \mathscr{I}$, *there exists an* $S \subseteq \mathscr{I}$ *such that* $i \in S$ *and*
$$\sum_{j \in S} n_j = M_S.$$

Using these (in-)equalities, Bernheim and Whinston [4] prove that all truthful equilibria are efficient and coalition-proof. Next we provide an additional characterization of the set of truthful Nash equilibrium payoffs for the static game.

Definition 4 (Marginal contribution equilibrium). A *marginal contribution equilibrium* of the common agency game is a truthful Nash equilibrium with $n_i = M_i$ for all i.

In other words, all principals receive their marginal contribution to the social welfare as their equilibrium net payoff in a marginal contribution equilibrium. The following theorem characterizes the games that have marginal contribution equilibria and their truthful equilibrium payoffs.

Theorem 2 (Existence and uniqueness).

1. *A marginal contribution equilibrium exists if and only if*

$$\forall S \subseteq \mathscr{I}, \quad \sum_{i \in S} M_i \leqslant M_S. \tag{3}$$

2. *The truthful Nash equilibrium payoff set is a singleton if and only if the game has a marginal contribution equilibrium.*[2]
3. *If M_S is superadditive*:

$$\forall S, T, S \cap T = \emptyset, \;\; M_S + M_T \leqslant M_{S \cup T}, \tag{4}$$

then the truthful equilibrium is unique.

Proof. See appendix. □

Condition (3) requires that the sum of the marginal contributions of each principal $i \in S$ to $\mathscr{I}$ is less than the marginal contribution of the entire set S to $\mathscr{I}$. Condition (3) is referred to as *weak superadditivity* of the marginal contributions. Superadditivity condition (4) is a sufficient condition for (3) and it agrees with (3) if $|\mathscr{I}| = 2$. Laussel and LeBreton [11] give the following equivalent form of the sufficient condition (4). If for $\forall S, T \subset \mathscr{I}$, such that $S \cap T = \emptyset$,

$$W \leqslant W_{-S} + W_{-T} - W_{-(S \cup T)},$$

then the truthful equilibrium is unique. This is, however, not a necessary condition for uniqueness.

4. Agenda setting

The static common agency model is almost exclusively a game among the principals. The outcome of the bidding game decides which of the feasible actions the agent should pursue. As the agent herself is made a take-it-or-leave-it offer, her

[2] The truthful Nash equilibrium is also unique in a generic set of games. To see that this cannot be always the case, consider an example where $\mathscr{I} = \{1,2\}$, $A = \{1,2\}$, $c(a) = 0$ for all a, $u_i(a) = v$ if $a = i$ and $u_i(a) = 0$ if $a \neq i$ for $i \in \{1,2\}$. The set of truthful equilibria for this game is $r_1 = r_2 = v$, $a \in \{1,2\}$.

strategic role in the game is minimal, and under truthful bidding strategies, she will indeed select the socially efficient action.

In this section, we change the static common agency minimally by allowing the set of actions available to the agent to be endogenous. More precisely, we allow the agent to set the agenda in the initial period by selecting a subset of the exogenously given set of feasible actions. In the subsequent period, the principals bid on the actions in the selected subset. In order to preserve the spirit of the common agency game, we allow the principals to influence the selection in the initial stage. The resulting game, referred to as agenda setting game, extends over two-periods and is arguably the minimal extension from a static to a dynamic problem of common agency. The initial choice has no direct payoff consequences for any player and with no discounting between the periods, the set of feasible payoffs in the agenda game is the same as in the static common agency game.

The game proceeds as follows. In period 0, each principal bids on the subset A chosen by the agent from the set of feasible actions, $\mathscr{A}$. The agent receives a reward $r_i(A)$ from principal i if she selects the subset A for the second stage. The choice of A is costless to the agent and has no immediate payoff consequences for the principals. The eventual choice of the agent in period 1 is, however, restricted to the subset A. The payoffs in period 1 game are as in the previous section.

This simple model is sufficient to introduce the conceptual difference between static and dynamic common agency. By selecting an action today, the agent can change the nature of competition among the principals tomorrow. The agent prefers early stage actions that increase competition between the principals and therefore give rise to higher equilibrium payoffs to the agent in the later stage. In general, the socially optimal action does not have to be included in the subset of actions that maximize the equilibrium payoff to the agent in the final stage. Since the principals' future payoffs are reflected in their bid schedules in the initial period, the agent faces a trade-off between efficiency and rent extraction in period 0.

The set of available actions in the initial period is the set of all subsets of $\mathscr{A}$, or $2^{\mathscr{A}}$. As we want to focus on the interaction between period 0 and period 1 outcomes, we assume that a marginal contribution equilibrium in period 1 exists for all subsets A. The payoffs in period 1 then given directly by their marginal contribution relative to the set A and by the residual rent for the agent. For this purpose, let

$$W_S(A) \triangleq \max_{a \in A} \{v_S(a) - c(a)\},$$

and

$$M_S(A) \triangleq W(A) - W_{-S}(A).$$

By assumption, every set A induces a marginal contribution equilibrium in the second stage with payoffs given by $M_i(A)$ for principal i and by $W(A) - \sum_{i \in \mathscr{I}} M_i(A)$ for the agent. By backward induction, we may then take the equilibrium payoffs tomorrow as the gross payoffs associated with the selection of set A today and analyze the agenda game as a single-period common agency game with the payoff structure as just defined. This intertemporal structure suggests a recursive notion of marginal contribution. The social value the set S of principals can achieve jointly with the agent is denoted by

$$\widehat{W}_S \triangleq \max_{A \in 2^{\mathscr{A}}} \left\{ W(A) - \sum_{i \notin S} M_i(A) \right\}.$$

It is a recursive notion as it incorporates the fact that every principal outside of S will be able to claim his marginal contribution in the future. The *recursive contribution* of a set S of principals is then defined as

$$\widehat{M}_S \triangleq \widehat{W} - \widehat{W}_{-S}.$$

When we compare $\widehat{W}_{-S}$ with W_{-S}, where the former is social value of set $\mathscr{I} \backslash S$ in the static game, we observe that they differ as the contribution of the set S in the static game is computed directly from the payoffs, whereas in the agenda game, it is computed from the (future) marginal contributions. It is easy to verify that $W = \widehat{W}$, $W_{-i} = \widehat{W}_{-i}$, and $M_i = \widehat{M}_i$, but the equalities do not hold in general for other $S \subseteq \mathscr{I}$. The discrepancy arises as the contributions of the coalitions are now computed on the basis of their equilibrium continuation payoffs rather than their gross payoffs. In particular, if $\sum_{i \in S} M_i < M_S$, then $\widehat{M}_S < M_S$.

The agenda game can now be analyzed as a static game of common agency. The first result is that all truthful equilibria of the agenda game are efficient. As all subsets A which include the efficient action a^* permit the realization of the efficient surplus tomorrow, the equilibrium choice of A is not unique, but rather includes all subset which include a^*. Denote by $\mathscr{A}^*$ the set of all such subsets:

$$\mathscr{A}^* = \{A \in 2^{\mathscr{A}} | a^* \in A\}.$$

Whether each principal gets his marginal contribution, which is his equilibrium payoff in the static common agency game, depends on the ability of the agent to structure the agenda to her advantage.

Theorem 3 (Agenda game).

1. *Every truthful equilibrium of the agenda game is efficient*: $A \in \mathscr{A}^*$, $a = a^*$.
2. *The following three statements are equivalent*:
   (a) *the agenda game has a marginal contribution equilibrium*;
   (b) *for all* $S \subseteq \mathscr{I}$:

$$\sum_{i \in S} \widehat{M}_i \leqslant \widehat{M}_S \tag{5}$$

   (c) *for all* $A \subseteq \mathscr{A}$ *and all* $S \subseteq \mathscr{I}$:

$$W(\mathscr{A}) - \sum_{i \in S} M_i(\mathscr{A}) \geqslant W(A) - \sum_{i \in S} M_i(A). \tag{6}$$

Proof. See appendix. □

Condition (2b) in Theorem 3 is the familiar condition of superadditivity, but now stated in terms of the recursive contributions for the agenda game. Of more interest is the equivalent condition (2c). The inequality fails to hold if there exist subsets A and S such that

$$W(\mathscr{A}) - W(A) < \sum_{i \in S} M_i(\mathscr{A}) - \sum_{i \in S} M_i(A). \tag{7}$$

Observe first that a subset A which satisfies this inequality cannot be an element of $\mathscr{A}^*$. If it were the case that $A \in \mathscr{A}^*$, then $W(\mathscr{A}) = W(A)$, and moreover $M_i(\mathscr{A}) \leqslant M_i(A)$. The last inequality holds as a strict subset $A \subset \mathscr{A}$ reduces the choice set and therefore the social values display in general $W_S(A) \leqslant W_S(\mathscr{A})$, and in particular, $W_{-i}(A) \leqslant W_{-i}(\mathscr{A})$. In consequence, the marginal contribution of i satisfies $M_i(A) \geqslant M_i(\mathscr{A})$ as long as $A \in \mathscr{A}^*$.

Consider therefore a set $A \notin \mathscr{A}^*$ and relative to this set A, partition the set of principals into two groups, with

$$S = \{i \in \mathscr{I} | M_i(\mathscr{A}) > M_i(A)\},$$

and its complement, S^{c}:

$$S^{\mathrm{c}} = \{i \in \mathscr{I} | M_i(\mathscr{A}) \leqslant M_i(A)\}.$$

If inequality (7) is to hold for the set A and some set S', then it certainly holds for the set S as just defined. By the truthfulness of the equilibrium strategies, all principals $i \in S^{\mathrm{c}}$, bid for agenda A in such a way that they are indifferent between agenda A and $\mathscr{A}$. In consequence, the agent acts as if she were the residual claimant after conceding surplus to all principals in the set S. In other words, she acts as if she were to maximize the joint objective of all principals outside of S and her private objective.[3] Hence if (7) holds, there exist sets $A \in 2^{\mathscr{A}}$ and $S \subset \mathscr{I}$, such that the efficiency losses caused by the restriction to A are smaller than the increase in the rent extraction from the subset S of principals.

Yet, in the truthful equilibrium of the agenda game, agenda and action choice will be efficient. But the option to increase rent extraction tomorrow allows the agent to extract more surplus from the principals than she could in the static game. The next corollary identifies the principals who will see a decrease in their equilibrium payoff due to the possibility of agenda setting by the agent. It suffices to describe the payoffs associated with $A = \mathscr{A}$, as truthfulness implies that for all $A, A' \in \mathscr{A}^*$, the net return for each principal is constant, or

$$M_i(A) - r_i(A) = M_i(A') - r_i(A'),$$

and we recall that $M_i(\mathscr{A}) = M_i$. The characterization of the equilibrium transfers follows directly from the properties of static equilibrium established in Theorems 1 and 2, after using the recursive notion of marginal contribution, $\widehat{M}_i$ and $\widehat{M}_S$.

[3] A different line of argument, based on the notion of coalition-proof equilibrium, would lead to the same conclusion.

Corollary 1 (Equilibrium payoffs in agenda game). *The equilibrium transfers* $\{r_i(\mathscr{A})\}_{i\in\mathscr{I}}$ *have the properties*:

1. $\forall S, \sum_{i\in S}(M_i - r_i(\mathscr{A})) \leqslant \widehat{M}_S$;
2. $\forall i, \exists S$ *s.th.* $i \in S$ *and* $\sum_{j\in S}\ (M_j - r_j(\mathscr{A})) = \widehat{M}_S$.

Notice the close analogy of properties 1 and 2 to the characterization of static payoffs in Theorem 1. Note, however that since $\widehat{M}_S < M_S$ in some games, property (2) of Theorem 1 does not hold in general.

The equilibrium transfers in the initial period are such that the superadditivity of the net payoffs is maintained in the agenda game. If principal i does not belong to any subset S of principals for which the agent's gain in rent extraction exceeds the efficiency losses, then i receives his marginal contribution and offers no rewards to the agent for keeping the choice set unrestricted, or

$$\forall i,\ \text{such that } \forall S \text{ with } i \in S,\ \sum_{j\in S} M_j \leqslant \widehat{M}_S \Rightarrow r_i(\mathscr{A}) = 0.$$

On the other hand if a subset S exists where rent extraction would be sufficiently increased by an inefficient restriction of the agenda, then in equilibrium the principals belonging to the set S, have to jointly provide transfers so as to make the agent indifferent between $\mathscr{A}$ and an inefficient agenda $A' \notin \mathscr{A}^*$, or

$$\forall S,\ \text{such that } \sum_{j\in S} M_j > \widehat{M}_S,\ \exists i \in S, r_i(\mathscr{A}) > 0.$$

The preceding results are illustrated by a simple example. There are two principals, 1 and 2. The agent can choose among three actions $\mathscr{A} = \{a, b, c\}$ at zero cost. The actions a and b are most favored by principals 1 and 2, respectively, whereas the action c represents a compromise. The gross payoffs are (with slight abuse of notation):

$$v_1(a) = a, \quad v_1(b) = 0, \quad v_1(c) = c,$$

and

$$v_2(a) = 0, \quad v_2(b) = b, \quad v_2(c) = c.$$

The ranking of the payoffs is $a \geqslant b > c > 0$, and the social payoffs are ranked $2c > a \geqslant b$. The static common agency has a unique equilibrium in which the efficient action c is selected and supported by the equilibrium transfers $r_1(c) = b - c$ and $r_2(c) = a - c$. However, by excluding the compromise from the set of available actions, the agent could extract a larger rent from the principals. Indeed, inequality (6) fails to hold for $A = \{a, b\}$ and $S = \mathscr{I}$. In the agenda game, the principals therefore have to lobby the agent to keep the compromise c on the agenda. In consequence, their transfers in period 0 have to be large enough to make the agent forego the sharper conflict between the principals. This requires that

$$r_1(\mathscr{A}) + r_2(\mathscr{A}) = 2c - a.$$

The sharing of the burden to maintain c as a feasible solution is subject to the participation constraints:

$$M_1(\mathscr{A}) - r_1(\mathscr{A}) \geqslant M_1(\{a,b\}) = a - b,$$

and

$$M_2(\mathscr{A}) - r_2(\mathscr{A}) \geqslant M_2(\{a,b\}) = 0.$$

The ability to set the agenda then allows the agent to extract higher rents, b rather than $a+b-2c$. Remarkably, the differential value of setting the agenda increases as a decreases, or as the competition between 1 and 2 intensifies.

5. Dynamic common agency

The agenda game introduced some of the new aspects of the common agency game which arise in a dynamic setting. In this section we present the main results for a general dynamic model of common agency. The equilibrium of the dynamic game is defined in Section 5.1, where we also prove its existence. The general characterization is given in Section 5.2, and necessary and sufficient conditions for the uniqueness of the truthful equilibrium are given in Section 5.3.

5.1. *Truthful equilibrium*

In the dynamic game, a reward strategy for principal i is a sequence of reward mappings

$$r_i : A_t \times H_t \to \mathbb{R}_+$$

assigning to every action $a_t \in \mathscr{A}_t$ a nonnegative reward, possibly contingent on the entire past history of the game. A strategy by the agent is a sequence of actions over time

$$a_t : \mathop{\times}_{i=1}^{I} \mathbb{R}_+^{|\mathscr{A}_t|} \times H_t \to \mathscr{A}_t,$$

depending on the profile of reward schedules in period t and history until period t. Strategies that depend on h_t only through θ_t are called Markov strategies. Since our main objective in this paper is to analyze how changes in the stage game influence the strategic positions of the players, we restrict our attention to Markov equilibria. Under this modeling choice, behavior cannot depend on payoff irrelevant features of the past path of play. Notice also that as θ_t depends on a_t, but not directly on r_t, the principals influence the future state of the world only through inducing different choices by the agent.

The expected discounted payoff with a history h_t for a given sequence of reward policies $\mathbf{r}$ and action profiles $\mathbf{a}$ is denoted by $V_0(h_t)$ for the agent and $V_i(h_t)$ for principal i. When $\mathbf{a}$ and $\mathbf{r}$ are Markov policies, then the values are given by $V_0(\theta_t)$ and $V_i(\theta_t)$ if the state is θ_t in period t. In this context, $\mathbb{E}V_i(a_t, \theta_t)$ represents the expectation of the continuation value in period $t+1$ if in period t the action was a_t and the state was θ_t. While the transition from θ_t to θ_{t+1} may be stochastic, we shall omit the expectations operator $\mathbb{E}[\cdot]$ for simplicity and all values are henceforth understood to represent expected values.

Definition 5 (Markov perfect equilibrium). The strategies $\{r_i(a_t, \theta_t)\}_{i\in\mathcal{I}}$ and $a(r(\cdot), \theta_t)$ form a Markov perfect equilibrium (MPE) if

1. $\forall\theta_t, \forall r'(\cdot), a(r'(\cdot), \theta_t)$ is a solution to

$$\max_{a_t\in\mathcal{A}_t}\{r'(a_t, \theta_t) - c(a_t, \theta_t) + \delta V_0(a_t, \theta_t)\},$$

2. $\forall i, \forall\theta_t$, there is no other reward function $r'_i(a_t, \theta_t)$ such that

$$v_i(a', \theta_t) - r'_i(a', \theta_t) + \delta V_i(a', \theta_t) > v_i(a, \theta_t) - r_i(a, \theta_t) + \delta V_i(a, \theta_t),$$

where a and a' are best responses to $(r_i(\cdot), r_{-i}(\cdot))$ and $(r'_i(\cdot), r_{-i}(\cdot))$, respectively.

Truthful strategies are defined as in the static game by the property that they reflect accurately each principal's net willingness to pay. The major difference to the static definition is that the allocation relative to which truthfulness is defined is now an action a_t *and* a state θ_t. The intertemporal net benefit $n_i(a_t, \theta_t)$ of an allocation a_t in the state θ_t is the flow benefit $v_i(a_t, \theta_t) - r_i(a_t, \theta_t)$ and the continuation benefit $\delta V_i(a_t, \theta_t)$:

$$n_i(a_t, \theta_t) \triangleq v_i(a_t, \theta_t) - r_i(a_t, \theta_t) + \delta V_i(a_t, \theta_t). \tag{8}$$

With this extension to the dynamic framework, the definition of a truthful (Markov) strategy and an associated MPE in truthful strategies is immediate.

Definition 6 (Truthful (Markov) strategy).

1. A reward function $r_i(a_t, \theta_t)$ is truthful relative to (a, θ_t) if for all $a_t \in \mathcal{A}(\theta_t)$, either:
   (a) $n_i(a_t, \theta_t) = n_i(a, \theta_t)$, or,
   (b) $n_i(a_t, \theta_t) < n_i(a, \theta_t)$, and $r_i(a_t, \theta_t) = 0$.
2. The strategies $\{r_i(\cdot)\}_{i=1}^{I}$ and $a(r(\cdot), \theta_t)$ are an MPE in truthful strategies if they are an MPE and $\{r_i(\cdot)\}_{i=1}^{I}$ are truthful strategies relative to $a(r(\cdot), \theta_t)$.

With this definition at hand, we can prove the existence result for our solution concept.

Theorem 4. *A Markov perfect equilibrium in truthful strategies exists.*

Proof. See appendix. □

The proof runs along familiar lines in games with discounted payoffs. In the first step, the existence of an MPE in truthful strategies is proved for finite games. In the second step, it is shown that because of continuity at infinity, the limit of finite equilibria forms an equilibrium in the infinite game. As the limit of finite equilibria may still depend on calendar time, we also provide an independent proof, suggested by Michel LeBreton, to prove the existence of a stationary MPE in the infinite horizon game.

5.2. Characterization

The characterization of the set of truthful equilibria relies as in the static model on the marginal contribution of each principal. The marginal contribution of principal i is, as defined earlier,

$$M_i(\theta_t) \triangleq W(\theta_t) - W_{-i}(\theta_t). \tag{9}$$

We are now in a position to develop the recursive argument sketched in the agenda game to its full extent. The recursion developed in the agenda game can be extended from terminal payoffs to arbitrary continuation payoffs. By the principle of optimality, this allows us to show in the next theorem that all truthful equilibria have to be efficient. To this effect, we define $W(\theta_t|a_t)$ to be the social value of the program which starts with an arbitrary and not necessarily efficient action a_t, but thereafter chooses an intertemporally optimal action profile. Similarly, let $M_i(\theta_t|a_t) \triangleq W(\theta_t|a_t) - W_{-i}(\theta_t|a_t)$.

If the equilibrium continuation play is indeed efficient, the distribution of the surplus along the equilibrium path can also be determined recursively by the sequence of residual claims the agent can establish. The maximal value the agent and a subset $\mathscr{I}\backslash S$ of principals can achieve along the equilibrium path is obtained by selecting a_t so as to solve

$$\max_{a_t}\left\{ W(\theta_t|a_t) - v_S(a_t,\theta_t) - \sum_{i\in S} \delta V_i(a_t,\theta_t)\right\}. \tag{10}$$

The net value $n_S(\theta_t)$ of the set S of principals in truthful equilibrium must then satisfy the following inequality in every period:

$$n_S(\theta_t) \leqslant W(\theta_t) - \max_{a_t\in\mathscr{A}_t}\left\{ W(\theta_t|a_t) - v_S(a_t,\theta_t) - \sum_{i\in S} \delta V_i(a_t,\theta_t)\right\}.$$

By relating the equilibrium continuation values $V_i(\theta_t)$ recursively to the marginal contributions $M_i(\theta_t)$, we obtain the following:

Theorem 5 (Efficiency).

1. *All MPE in truthful strategies are efficient.*
2. *For all* $S \subseteq \mathscr{I}$,

$$\sum_{i\in S} V_i(\theta_t) \leqslant M_S(\theta_t). \tag{11}$$

Proof. See appendix. □

An important qualification for the efficiency result is the participation issue which did not arise in the static game. For the equilibrium to be efficient, it has to be the case that every principal i who might realize some nontrivial payoff $v_i(a_\tau, \theta_\tau)$ at some future time τ, participates in the game in all periods t prior to τ. For if he were absent in some period, the agent and the remaining principals would only seek to maximize their current and future payoff, and fail to internalize the impact of their decision on principal i. For the same reason, the theorem cannot accommodate a change in the identity of the agent, unless of course the sequence of agents would have perfectly dynastic preferences. The efficiency failure with varying participation is related to the observation made in Aghion and Bolton [1], where the collusion between an incumbent and a buyer against a potential future entrant may result in welfare losses.

In the static game, there is an additional result relating the equilibrium payoffs to the marginal contributions. For every i, there is a set S, with $i \in S$ such that the joint equilibrium payoff of the set S of principals equals their marginal contribution, or

$$\sum_{j \in S} V_j = M_S. \tag{12}$$

The analysis of the agenda game in the previous section provided an example of a dynamic environment where the above equality failed to hold for every i. As the principals lobbied to keep the option of a compromise open, neither a single principal nor the principals jointly could realize their marginal contribution.

While the focus of our analysis is on the efficiency and the existence of a unique truthful equilibrium, Bernheim and Whinston [4] also gave a complete description of the set of equilibrium payoffs in the absence of a unique marginal contribution equilibrium. The characterization relied on a set of inequalities relating the social payoffs of various subsets of principals (see their Theorem 2). For all finite horizon games, such as the agenda game discussed in the previous section, a straightforward recursive extension of their inequalities would give a similar characterization for dynamic games.

5.3. *Marginal contribution equilibrium*

The intertemporal aspects of the game weakens the position of the principals as neither an individual principal nor any group of principals can receive their marginal contribution in general. In this section, we give necessary and sufficient conditions for a marginal contribution equilibrium to exist. A *marginal contribution equilibrium* is a Markov perfect equilibrium in truthful strategies where the payoff of each principal coincides with his marginal contribution, or for all i and all θ_t,

$$V_i(\theta_t) = M_i(\theta_t).$$

The previous efficiency theorem already indicated that the weak superadditivity of the marginal contributions $M_i(\theta_t)$ remains a necessary condition in the dynamic game as the inequality

$$\sum_{i \in S} V_i(\theta_t) \leqslant M_S(\theta_t),$$

can only hold with $V_i(\theta_t) = M_i(\theta_t)$ if indeed

$$\sum_{i \in S} M_i(\theta_t) \leqslant M_S(\theta_t).$$

However the agenda game already illustrated that it cannot be a sufficient condition anymore. The analysis of the agenda game also suggested that if a marginal contribution equilibrium is to exist, then the current decision by the agent should not be biased too strongly by her interest to depress the future shares of the principals relative to the shares along the efficient path. Since the agent is the residual claimant after the principals receive their marginal contribution, a formal statement of this requirement is that the social loss from a deviation from the efficient policy exceeds the loss in the marginal contributions of the principals, or $\forall a_t \in \mathscr{A}(\theta_t), \forall S \subseteq \mathscr{I}$:

$$W(\theta_t|a_t) - W(\theta_t) \leqslant \sum_{i \in S} (M_i(\theta_t|a_t) - M_i(\theta_t)).$$

Recall that $W(\theta_t|a_t)$ is the social value of the program which starts with an arbitrary action a_t, but thereafter chooses an intertemporally optimal action profile, and it follows that $W(\theta_t|a_t) - W(\theta_t) < 0$ for all $a_t \neq a^*$.

Theorem 6 (Marginal contribution equilibrium). *The marginal contribution equilibrium exists if and only if*

$$\sum_{i \in S} (M_i(\theta_t) - M_i(\theta_t|a_t)) \leqslant W(\theta_t) - W(\theta_t|a_t), \quad \forall a_t, \forall \theta_t, \forall S. \tag{13}$$

Proof. See appendix. □

Inequality (13) can be directly interpreted as the trade-off between rent extraction and efficiency gains. The lhs of the inequality describes the opportunities of the agent to extract additional rents from the principals by deviating from the efficient path, whereas the rhs describes the social losses associated with such a deviation. The rent of the agent here is not due to informational asymmetries, but rather to the changing nature of the competition between the principals. In the case of a repeated common agency game, condition (13) reduces to the condition of weak superadditivity of the marginal contributions in the static game as the transition from period t to $t+1$ is of course independent of the action chosen in period t.

The equilibrium characterization by the inequality is particularly useful in applications. Since all values entering the inequality can be obtained from appropriate efficient (continuation) programs, the inequality can be established independently of any equilibrium considerations. As efficient programs are in general easier to analyze than dynamic equilibrium conditions, the technique suggested here may be usefully applied to a wide class of dynamic bidding models. In an earlier version of this paper [3a] we analyzed a job-matching model with n firms. The value of the match between the worker and any of the firms is initially uncertain and the issue is whether equilibrium wages can induce intertemporally efficient matching. The technique developed here, and in particular the theorem above allows us to prove efficiency and characterize the equilibrium payoffs. Earlier work on this class of models by Bergemann and Välimäki [2] and Felli and Harris [8] could prove efficiency without the use of the common agency framework only for two firms.

In a recent contribution Bergemann and Välimäki [3b] show how the equilibrium argument for spot prices rather than menus can be extended to many firms by using the marginal contributions as value functions in the dynamic programming equations.

A reformulation of the rent extraction inequality (13) provides a link between the static and the dynamic conditions for the existence of a marginal contribution equilibrium. For any state θ_t define:

$$\widehat{M}_S(\theta_t) \triangleq W(\theta_t) - \max_{a_t \in \mathscr{A}_t} \left\{ W(\theta_t | a_t) - v_S(a_t, \theta_t) - \sum_{i \in S} \delta M_i(a_t, \theta_t) \right\}$$

as the *recursive contribution* of a subset S of principals. For all singleton subsets, we have $M_i(\theta_t) = \widehat{M}_i(\theta_t)$, but in general the equality fails to holds for sets S with a cardinality exceeding one. The main difference between the marginal contribution, $M_S(\theta_t)$, and the recursive contribution, $\widehat{M}_S(\theta_t)$, is the different treatment of the continuation value associated with a subset S. While $M_S(\theta_t)$ attributes the entire future marginal contribution of coalition S to its members, $\widehat{M}_S(\theta_t)$ attributes only the sum of individual marginal contributions. These two notions are equivalent if and only if the marginal contributions are additive. Likewise, if the marginal contributions are (strictly) superadditive, or $\sum_{i \in S} M_i(\theta_t) < M_S(\theta_t)$, then using the definition given above, one can show that $\widehat{M}_S(\theta_t) < M_S(\theta_t)$. We obtain necessary and sufficient conditions for a marginal contribution equilibrium more in the spirit of the static condition as follows:

Corollary 2. *A marginal contribution equilibrium exists if and only if* $\forall \theta_t, \forall S$:

$$\sum_{i \in S} \widehat{M}_i(\theta_t) \leqslant \widehat{M}_S(\theta_t). \tag{14}$$

Proof. See appendix. □

The disadvantage of condition (14), when compared to the rent extraction inequality, is that it is based on two nested optimization problems rather than a single one based entirely on the social value of the program. An immediate implication of the reformulation in terms of the recursive contribution is that the relation between the uniqueness of the truthful equilibrium and the marginal contribution equilibrium is still valid in the dynamic game.

Corollary 3 (Uniqueness). *A MPE in truthful strategies is unique if and only if it is a marginal contribution equilibrium.*

As the marginal contribution equilibrium is by definition a truthful equilibrium, the 'if' part of the corollary states that if there is a marginal contribution then it is also the unique MPE in truthful strategies.

6. Conclusion

This paper considered common agency in a general class of dynamic games with symmetric information. By focusing on Markovian equilibria, a detailed characterization of the equilibrium strategies and payoffs was possible for this class of games. As in the static analysis by Bernheim and Whinston [4], the link between truthful strategies and the social value of various coalitions was central in obtaining the results. In the dynamic context the link is even more valuable. The continuation payoffs which determine the current bidding strategies, are themselves endogenous to the equilibrium and hence of little help in determining the equilibrium strategies. In contrast, the marginal contributions are defined independently of equilibrium considerations.

As in the static game, a connection can be made between the truthful equilibria and coalition proof equilibria. It is relatively straightforward to extend the notion of coalition-proof equilibrium period by period in a finite horizon game. The notion becomes a bit more problematic in an infinite horizon model. We refer the reader to the previous version of this paper [3a] for a minimal notion of coalition proofness in an infinite horizon model.

We restricted our analysis to symmetric information environments. Bernheim and Whinston [4], however, observed that in the static context with two bidders for a single good, the principals net payoffs are equivalent to the equilibrium net payoffs of the Vickrey–Clarke–Groves mechanism with incomplete information. In this paper, we showed that in fact whenever the principals receive their marginal contribution, their equilibrium net payoff is equal to the Vickrey–Clarke–Groves payoff. This suggests that the techniques presented here could possible be extended to sequential allocation problems with asymmetric information, or more generally, dynamic efficient mechanism design problems. In this context, it should be noted that the case of private information for the principals is distinct from the analysis of Bernheim and Whinston [5], Martimort [12] or Stole [15], where moral hazard or adverse selection is due to a better informed agent.

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Appendix

Proof of Theorem 2. (1) (if) Suppose that $\sum_{i\in S} M_i \leqslant M_S$. Set $n_i = M_i$ for all i, and by hypothesis $n_S \leqslant M_S$. Thus the vector $\mathbf{n} = (n_1, \ldots, n_I)$ satisfies the condition of Theorem 1 and hence a marginal contribution equilibrium exists.

(only if) Suppose $n_i = M_i$ is the net payoff for every principal i in a truthful Nash equilibrium. Then by Theorem 1.1, the sum of the net payoffs have to satisfy for all S, $\sum_{i\in S} n_i \leqslant M_S$. But as $n_i = M_i$, this implies that $\sum_{i\in S} M_i \leqslant M_S$.

(2) (if) By part 1 of this theorem, the existence of a marginal contribution equilibrium is equivalent to: $\forall S$, $\sum_{i\in S} M_i \leqslant M_S$. In a marginal contribution equilibrium we set $n_i = M_i$ for all i, and by hypothesis $n_S \leqslant M_S$. Thus if there were to be a different equilibrium payoff vector $\mathbf{n} = (n_1, \ldots, n_I)$, there would have to be a subset S such that $\forall i$, $i \in S$, $n_i < M_i$, as $n_i \leqslant M_i$ has to hold by Theorem 1.1. But notice that the decrease in the equilibrium net payoff for all i in some subset S does not permit the increase of any other n_j, $j \notin S$, as $n_j = M_j$ is a binding constraint and hence the uniqueness of the payoffs follows. By the definition of a truthful strategy, this also determines uniquely the equilibrium strategies.

(only if) We prove the contrapositive. Suppose for some $S \subseteq \mathscr{I}$, $\sum_{i\in S} M_i > M_S$, then we show that the equilibrium net payoff vector cannot be unique. The proof is by construction using the greedy algorithm. Define $n_1 \triangleq M_1$, and in general,

$$n_k \triangleq \min_{\{S \mid k\in S \wedge S\subseteq\{1,2,\ldots,k\}\}} M_S^k,$$

where

$$M_S^k \triangleq M_S - \sum_{j\in S\setminus k} n_j.$$

It can be verified that the induced allocation $\{n_1, n_2, \ldots, n_I\}$ is an equilibrium allocation with, by hypothesis, $n_{i'} < M_{i'}$, for some $i' > 1$. Consider next a permutation $\sigma : \mathscr{I} \to \mathscr{I}$ such that $i' \mapsto 1$. By applying the greedy algorithm to the new ordering, we again obtain an equilibrium allocation, but clearly $n_{\sigma(i')} = M_{\sigma(i')}$, which is distinct from the previous allocation.

(3) It is enough to prove that (4) is a sufficient condition for (3). Consider sets $S, T \subset \mathscr{I}$, and suppose that (4) holds, then we have for any $S_1, S_2 \subset S$, $S_1 \cap S_2 = \emptyset, S \cap T = \emptyset$,

$$M_{S_1} + M_{S_2} \leqslant M_S,$$

and thus

$$M_{S_1} + M_{S_2} + M_T \leqslant M_{S \cup T}.$$

As we continue to split up S and T until we have singleton sets consisting of single principals on the left-hand side, we obtain

$$\sum_{i \in S \cup T} M_i \leqslant M_{S \cup T},$$

which completes the claim. □

Proof of Theorem 3. (1) The efficiency of the action choice $a = a^*$ in period 1 is immediate if $A \in \mathscr{A}^*$. By assumption every $A \in 2^{\mathscr{A}}$ induces a unique truthful equilibrium in period 1. By backward induction, we take the continuation payoffs following an agenda choice A to be the static payoffs in period 1 associated with agenda A. In consequence we can analyze the game in period 0 as a static common agency game with the action set being the set of all possible agendas, $2^{\mathscr{A}}$. By Theorem 2 of Bernheim and Whinston [4], it then follows that $A \in \mathscr{A}^*$.

(2) By backward induction and the uniqueness of the continuation payoffs in period 1, the equilibrium net payoffs are decided by the choice of an agenda A in period 0. The gross payoffs in period 0 can then taken to be $M_i(A)$ and $W(A) - \sum_{i \in \mathscr{I}} M_i(A)$. By the definition of $\widehat{M}_S$, the necessary and sufficient condition for a marginal contribution equilibrium of Theorem 2, can then be written as $\forall S \subseteq \mathscr{I}$, $\sum_{i \in S} \widehat{M}_i \leqslant \widehat{M}_S$.

As we have $\widehat{M}_i = M_i(\mathscr{A})$, and $\widehat{M}_S = W(\mathscr{A}) - \max_{A \in 2^{\mathscr{A}}} \{W(A) - \sum_{i \in S} M_i(A)\}$, we can in turn write the inequality (5) as

$$W(\mathscr{A}) - \sum_{i \in S} M_i(\mathscr{A}) \geqslant \max_{A \in 2^{\mathscr{A}}} \left\{ W(A) - \sum_{i \in S} M_i(A) \right\},$$

and since the rhs has to hold for all $A \in 2^{\mathscr{A}}$, the equivalence between (5) and (6) follows directly. □

Proof of Theorem 4. The existence of a truthful MPE for finite horizon games follows from the existence result in [4] by backwards induction.

Consider next the case of $T = \infty$. The existence of a truthful MPE follows by a limiting argument from the case of $T < \infty$ along the lines of Maskin and Tirole [13]. Unfortunately, the equilibrium thus obtained may fail to be in stationary strategies and it could be time dependent. As a result, we prove the existence of a stationary equilibrium directly without using the previous result. For this, we need some preliminary definitions.

Let $\mathscr{A}$ be an arbitrary finite action set and denote an arbitrary static common agency game on the action set $\mathscr{A}$ by Γ, with

$$\Gamma = \{\mathscr{I}, \{v_i(a)\}_{i \in \mathscr{I}, a \in \mathscr{A}}, \{c(a)\}_{a \in \mathscr{A}}\}.$$

For the static common agency game Γ, define $W_\Gamma(S)$ as follows for all $S \subseteq \mathscr{I}$:

$$W_\Gamma(S) = \max_{a \in \mathscr{A}} \left\{ \sum_{i \in S} v_i(a) - c(a) \right\}.$$

Let $W_\Gamma = (W_\Gamma(S))_{S \subseteq \mathscr{I}}$ and thus W_Γ is a vector in $\mathbb{R}^{2^I - 1}$. Let $V(W_\Gamma) = (V_0(W_\Gamma), \ldots, V_I(W_\Gamma))$ be the vector of net payoffs for the agent and the principals, where the payoffs for the principals satisfy the Bernheim–Whinston inequalities and the payoffs are obtained by the greedy algorithm defined in the proof of Theorem 2 for the order $1, 2, \ldots, I$. The payoff for the agent is then defined by $V_0(W_\Gamma) = W_\Gamma(\mathscr{I}) - \sum_{i \in \mathscr{I}} V_i(W_\Gamma)$.

Lemma 1. *The mapping* $V : \mathbb{R}^{2^I - 1} \to \mathbb{R}^{I+1}$ *is continuous.*

Proof. The greedy algorithm assigns payoff

$$V_1(W_\Gamma) = W_\Gamma(\mathscr{I}) - W_\Gamma(\mathscr{I} \backslash \{1\})$$

to $i = 1$. This is clearly continuous in W_Γ. Since

$$V_i(W_\Gamma) = W_\Gamma(\mathscr{I}) - W_\Gamma(\mathscr{I} \backslash \{1, \ldots, i\}) - \Sigma_{j=1}^{i-1} V_j(W_\Gamma),$$

an inductive argument establishes continuity for all i. The continuity of the agent's payoff follows similarly. □

Next suppose that we augment the fixed static payoff functions $c(a)$ and $v_i(a)$, for agent and principals respectively, with a nonnegative vector $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_I)$ and define

$$c(a; \lambda) = c(a) - \lambda_0,$$

and

$$v_i(a; \lambda) = v_i(a) + \lambda_i.$$

Let $\Gamma(\lambda)$ denote the resulting common agency game parametrized by λ.

Lemma 2. *The mapping* $\lambda \to W_{\Gamma(\lambda)}$ *is continuous in* λ.

Proof. Since

$$W_{\Gamma(\lambda)}(S) = \max_{a \in \mathscr{A}} \left\{ \sum_{i \in S} v_i(a; \lambda) - c(a; \lambda) \right\}$$

for all $S \in \mathscr{I}$, the continuity property follows from the theorem of the maximum. □

The role of the additive term λ in the next step is to represent the continuation values in a dynamic common agency game. To see this consider next K different common agency games, denoted by Γ_k for every state θ_k, with $k \in \{1, \ldots, K\}$. We assume that the action set $\mathscr{A}$ does not depend on θ,[4] and that the payoffs are parametrized by a matrix

$$v = [v_i(\theta_k)]_{0 \leqslant i \leqslant I, 1 \leqslant k \leqslant K}.$$

The gross payoffs in the common agency game Γ_k are defined by the following payoff functions:

$$v_0(a, \theta_k; v) = -c(a, \theta_k) + \delta \sum_{k'=1}^{K} q(\theta_{k'} | a, \theta_k) v_0(\theta_{k'}),$$
$$v_i(a, \theta_k; v) = v_i(a, \theta_k) + \delta \sum_{k'=1}^{K} q(\theta_{k'} | a, \theta_k) v_i(\theta_{k'}), \quad \forall i = 1, \dots, I, \tag{A.1}$$

where $q(\theta_{k'} | a, \theta_k)$ are the state transition probabilities defined earlier. The payoff functions in all of these games are clearly continuous in v. For every given matrix v we can then look at the net payoffs of the common agency game Γ_k as described for Lemma 1 by $V(W_{\Gamma_k})$.

Lemma 3. *The mapping* $\Phi : \mathbb{R}^{(I+1)K} \to \mathbb{R}^{(I+1)K}$ *defined by*

$$\Phi(v) = (V(W_{\Gamma_1}), \dots, V(W_{\Gamma_K}))$$

is continuous in v.

Proof. This follows from Lemmas 1 and 2. □

Finally observe that the image under the mapping Φ of the hypercube, where the later is defined by

$$\left[0, \max_{\{i,k,a\}} \frac{v_i(a, \theta_k)}{1 - \delta}\right]^{(I+1)K}$$

is contained in the hypercube itself, or

$$\Phi\left(\left[0, \max_{\{i,k,a\}} \frac{v_i(a, \theta_k)}{1 - \delta}\right]^{(I+1)K}\right) \subset \left[0, \max_{\{i,k,a\}} \frac{v_i(a, \theta_k)}{1 - \delta}\right]^{(I+1)K}.$$

Hence $\Phi(v)$ satisfies the conditions for Brouwer's fixed point theorem and we have:

Lemma 4. *There is a* v *such that* $v = \Phi(v)$.

Thus the proof of the theorem is complete if we can show that v is a truthful and stationary Markov perfect equilibrium payoff of the dynamic game. Observe that by the construction of Φ and the payoff functions $v_i(a, \theta_k)$ for $1 \leqslant i \leqslant I$ and $v_0(a, \theta_k)$, all strategies obtained in the construction of the truthful equilibria of the Γ_k game are

[4] This is without loss of generality since the action set for each state could always be taken to be the union of all possible actions where the payoffs to the actions not available at that state are assigned arbitrarily large negative payoffs.

stationary Markovian as well as truthful for all $\theta \in \Theta$ and furthermore there are no profitable one-shot deviations in the dynamic game. As the dynamic game satisfies continuity at infinity, this is sufficient for establishing that v is indeed a truthful stationary Markov perfect equilibrium payoff of the dynamic game. □

Proof of Theorem 5. (1) By the assumption of Markovian strategies, the continuation values for the agent and the principals depend only on the action a_t inducing the transition from θ_t to θ_{t+1}. This implies by Theorems 2 and 3 of Bernheim and Whinston [4] efficiency.

(2) The equilibrium value function $V_i(\theta_t)$ of principal i are required to satisfy the following set of equalities, $\forall i$,

$$V_i(\theta_t) \leqslant \max_{a_t}\left\{v(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k=0}^{I} \delta V_k(a_t,\theta_t)\right\} - \max_{a_t}\left\{v_{-i}(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k\neq i} \delta V_k(a_t,\theta_t)\right\}, \tag{A.2}$$

and inequalities $\forall S \subseteq \mathscr{I}$,

$$\sum_{i\in S} V_i(\theta_t) \leqslant \max_{a_t}\left\{v(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k=0}^{I} \delta V_k(a_t,\theta_t)\right\} - \max_{a_t}\left\{v_{-S}(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k\notin S} \delta V_k(a_t,\theta_t)\right\}. \tag{A.3}$$

Since all truthful equilibria are efficient by part 1 of this theorem, we have the identity:

$$W(\theta_t) = \max_{a_t}\left\{v(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k=0}^{I} \delta V_k(a_t,\theta_t)\right\}.$$

Next we argue by contradiction. Suppose inequality (11) does not hold for some S, but inequalities (A.2) and (A.3) are still satisfied. Then there $\exists \varepsilon > 0$ such that

$$\sum_{i\in S} V_i(\theta_t) - M_S(\theta_t) > \varepsilon, \tag{A.4}$$

and a fortiori

$$W(\theta_t) - \max_{a_t}\left\{v_{-S}(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k\notin S} \delta V_k(a_t,\theta_t)\right\} - M_S(\theta_t) > \varepsilon. \tag{A.5}$$

Since the inequality in (A.5) holds for the maximizing a_t in (A.5), it has to hold for a^*_{-S} as well, so that (A.5) may be rewritten in this instance as

$$\delta W(a^*,\theta_t) - \sum_{k\notin S} \delta V_k(a^*_{-S},\theta_t) - \delta W_{-S}(a^*,\theta_t) + \delta W_{-S}(a^*_{-S},\theta_t) - \delta M_S(a^*,\theta_t) > \varepsilon, \tag{A.6}$$

and since

$$\delta W(a^*, \theta_t) = \delta W_{-S}(a^*, \theta_t) + \delta M_S(a^*, \theta_t),$$

it follows from (20) that

$$W_{-S}(a_{-S}, \theta_t) - \sum_{k \notin S} V_k(a_{-S}, \theta_t) > \frac{\varepsilon}{\delta},$$

which is equivalent to

$$\sum_{i \in S} V_i(a^*_{-S}, \theta_t) - M_S(a^*_{-S}, \theta_t) > \frac{\varepsilon}{\delta}.$$

But by repeating the argument, which we started at (A.4), it then follows that the equilibrium value for the set S of principals increases without bound along some path $(\theta_t, \theta_{t+1}, \ldots)$ which delivers the contradiction as the value of the game is finite. $\square$

Proof of Theorem 6. It suffices to show that $\{M_i(\theta_t)\}_{i \in \mathscr{I}}$ satisfy the following set of equalities, $\forall i$,

$$M_i(\theta_t) = \max_{a_t}\left\{ v(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k=0}^{I} \delta V_k(a_t, \theta_t) \right\} - \max_{a_t}\left\{ v_{-i}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \neq i} \delta V_k(a_t, \theta_t) \right\}, \tag{A.7}$$

and inequalities $\forall S \subseteq \mathscr{I}$,

$$\sum_{i \in S} M_i(\theta_t) \leqslant \max_{a_t}\left\{ v(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k=0}^{I} \delta V_k(a_t, \theta_t) \right\} - \max_{a_t}\left\{ v_{-S}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \notin S} \delta V_k(a_t, \theta_t) \right\}, \tag{A.8}$$

if and only if the inequalities represented by (13) hold. By hypothesis $V_k(a_t, \theta_t) = M_k(a_t, \theta_t)$ for all $k > 0$. For notational ease, we omit that a_t is restricted to $a_t \in \mathscr{A}(\theta_t)$. We start with the set of equalities (A.7). Since all truthful equilibria are efficient by Theorem 5 we have the identity:

$$W(\theta_t) = \max_{a_t}\left\{ v(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k=0}^{I} \delta V_k(a_t, \theta_t) \right\}.$$

Consider next the term

$$\max_{a_t}\left\{ v_{-i}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \neq i} \delta V_k(a_t, \theta_t) \right\},$$

which can be written as

$$\max_{a_t}\{v_{-i}(a_t,\theta_t)-c(a_t,\theta_t)+\delta V(a_t,\theta_t)-\delta M_i(a_t,\theta_t)\}=W_{-i}(\theta_t),$$

where the equality follows from the definition of the marginal contribution in (1) and hence the equality in (A.7) is satisfied. Consider next the set of inequalities (A.8)

$$\sum_{i\in S} M_i(\theta_t)\leqslant W(\theta_t)-\max_{a_t}\left\{v_{-S}(a_t,\theta_t)-c(a_t,\theta_t)+\sum_{k\notin S}\delta V_k(a_t,\theta_t)\right\}. \tag{A.9}$$

If for any set S, $a^*_{-S}=a^*$, it follows that

$$\sum_{i\in S} M_i(\theta_t)-\delta\sum_{i\in S} M_i(a_t,\theta_t)=v_S(a^*,\theta_t),$$

and hence the set as an aggregate is not making any net contributions to $\mathscr{I}\backslash S$, and (A.9) is satisfied. Suppose next that $a^*_{-S}\neq a^*$, then (A.9) is equivalent to

$$\sum_{i\in S} M_i(\theta_t)\leqslant W(\theta_t)-\max_{a_t}\left\{W(\theta_t|a_t)-v_S(a_t,\theta_t)-\sum_{i\in S}\delta M_i(a_t,\theta_t)\right\}. \tag{A.10}$$

Since the inequality has to hold for the action a^*_{-S} which maximizes the payoff inside the braces, it follows a fortiori that the inequality has to hold for an arbitrary action a_t. Then we may write (A.10) as

$$\sum_{i\in S} M_i(\theta_t)-v_S(a_t,\theta_t)-\sum_{i\in S}\delta M_i(a_t,\theta_t)\leqslant W(\theta_t)-W(\theta_t|a_t),$$

or equivalently

$$\sum_{i\in S}(M_i(\theta_t)-M_i(\theta_t|a_t))\leqslant W(\theta_t)-W(\theta_t|a_t),$$

which completes the proof. □

Proof of Corollary 2. It is sufficient to show the equivalence between (13) and (14). Starting with (14), we can write the inequality as

$$W(\theta_t)-\sum_{i\in S} M_i(\theta_t)\geqslant\max_{a_t}\left\{W(\theta_t|a_t)-v_S(a_t,\theta_t)-\sum_{i\in S}\delta M_i(a_t,\theta_t)\right\}.$$

As

$$M_i(\theta_t|a_t)=v_i(a_t,\theta_t)+\delta M_i(a_t,\theta_t),$$

it follows that

$$W(\theta_t)-\sum_{i\in S} M_i(\theta_t)\geqslant\max_{a_t}\left\{W(\theta_t|a_t)-\sum_{i\in S} M_i(\theta_t|a_t)\right\}. \tag{A.11}$$

As inequality (13) has to hold for all S and all a_t, it has to hold in particular for the maximand of the rhs of (A.11), which establishes the result. □

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