Dynamic Marginal Contribution Mechanism

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Intertemporal Efficiency with Private Information

- random arrival of buyers, sellers and/or objects
  - selling seats for an airplane with random arrival of buyers
  - bidding on ebay
  - bidding for construction projects with uncertain arrival of new projects
- bidding for links in sponsored search (Google, Yahoo, etc.)
  - uncertainty about click-through probability
  - uncertainty about conversion probability
- leasing resource over time
  - auction of renewable license, right, capacity over time
  - web serving, computational resource (bandwidth, CPU)

Static Efficiency with Private Information

private value environment

- Vickrey (1961): single or multiple unit discriminatory auctions implement socially efficient allocation
  - in private value environments
  - in (weakly) dominant strategies
- Clarke (1971) and Groves (1973) extend to general allocation problems in private value environments
  - agent *i* internalizes the social objective and is led to report her type truthfully

Pivot Mechanism

- Green & Laffont (1977) analyze specific VCG mechanism
- i internalizes social objective if i pays her externality cost
- externality cost:

utility of $I \backslash i$ given i is present - utility of $I \backslash i$ given i is absent

- marginal contribution of i = utility of i - externality cost of i
- in Pivot mechanism:

1. payoff of i is her marginal contribution to social value
2. participation constraint holds ex post and no budget deficit

Dynamic Marginal Contribution Mechanism

- marginal contribution = payoff in Pivot mechanism
- develop marginal contribution mechanism in intertemporal environments with new arrival of information regarding:
  - preferences
  - agents
  - allocations
- design sequence of payments so that each agent receives flow marginal contribution in every period

...but wait...

- solve intertemporal problem as a completely contigent plan
- embed intertemporal problem in a static problem (as in an Arrow Debreu economy) ...
- ... and then appeal to the classic VCG results.
- but the contingent view fails to account for strategic possibilities of the agents in the sequential model

Sequential Incentive and Participation Constraints

- information arrives over time
- report of agent i in period t responds to private information of agent i, but may also respond to past reports of other agents (possibly inferred from allocative decisions)
- truthtelling (generally) fails to be a weakly dominant strategy
- with forward looking agents, participation constraint is required to be satisfied at every point in time (and not only in the initial period)

Results

- marginal contribution mechanism is dynamically efficient
- periodic ex post: with respect to information available at period t
- satisfies (periodic) ex post incentive constraints
- satisfies (periodic) ex post participation constraints
- adding efficient exit condition (weak “online” condition): if agent i does not impact future decisions, then agent i does not receive future payments, uniquely identifies marginal contribution mechanism

Literature

- Dolan (RAND 1978):
  priority queuing
- Parkes et al. (2003):
  delayed VCG without participation or budget balance constraints
- Bergemann & Valimaki (JET 2006):
  complete information, repeated allocation of single object over time, first price bidding
- Athey & Segal (2007):
  balanced budget rather than participation constraints

- scheduling tasks
- discrete time, infinite horizon: $t = 0, 1, ...$
- common discount factor δ
- finite number of agents: $i \in \{0, 1, ..., I\}$
- each agent i has a single task
- value of task for i is:

$$v_i > 0$$

- quasilinear utility: $v_i - p_i$

- values are given wlog in descending order:

$$v_0 > v_1 > \cdots > v_I > 0$$

- marginal contribution of task i :
  difference in welfare with i and without i
- efficient task assignment policy:

policy without i	0	1	$\cdots$	$i-1$	i+1	i+2	$\cdots$		I
					$\searrow$	$\searrow$			$\searrow$
policy with i	0	1	$\cdots$	$i-1$	i	i+1	i+2	$\cdots$	I
					$\uparrow$				

Marginal Contribution

$$\begin{array}{lcccccccccc}
\text{policy without } i & 0 & 1 & \cdots & i-1 & \text{i+1} & \text{i+2} & \cdots & & \text{I} & \\
 & & & & & \searrow & \searrow & & & \searrow & \\
\text{policy with } i & 0 & 1 & \cdots & i-1 & \text{i} & \text{i+1} & \text{i+2} & \cdots & & \text{I} \\
 & & & & & \uparrow & & & & &
\end{array}$$

- insert valuable task i:
- raise the value of all future tasks: $t > i$
- marginal contribution M_i:

$$M_i = \sum_{t=0}^{I} \delta^t v_t - \left(\sum_{t=0}^{i-1} \delta^t v_t + \sum_{t=i+1}^{I} \delta^{t-1} v_t \right)$$

or

$$M_i = \sum_{t=i}^{I} \delta^t \left(v_t - v_{t+1} \right) \geq 0$$

Externality

- from marginal contribution to externality pricing:

$$M_i = v_i - p_i$$

- externality cost of task i is:

$$p_i = v_{i+1} - \sum_{t=i+1}^{I} \delta^{t-i} \overbrace{(v_t - v_{t+1})}^{>0}$$

- task i directly replaces task $i+1$, but also:
- task i raises the value of all future tasks

Incomplete Information

- suppose v_i is private information to agent i at $t = 0$
- incentive compatibility and efficient sorting
- when would agent i like to win against j versus $j + 1$:

$$(\mathrm{v_i} - v_j) - \sum_{t=j}^{I} \delta^{t-(j-1)} (v_t - v_{t+1}) \geq \delta (\mathrm{v_i} - v_{j+1}) - \sum_{t=j+1}^{I} \delta^{t-j} (v_t - v_t$$

- reduces to cost of delay:

$$(1 - \delta)\, \mathrm{v_i} \geq (1 - \delta)\, v_j.$$

- report thruthfully if others report truthfully: ex post equilibrium

Bidding vs Direct Revelation Mechanism

- an ascending (English) auction in every period
- winning bidder i pays bid of second highest bidder
- bid by agent i in period t:

$$b_i^t$$

- bid should reflect value of task but ...
- value of task today versus value of task tomorrow
- value = utility - option value

Option Value

- bidding strategy b_i^t determined recursively in i and t
- option value is value of realizing task tomorrow

$$\delta \left(v_i - p_i^{t+1} \right)$$

and the price tomorrow is

$$p_i^{t+1} \triangleq \max_{j \neq i} \left\{ ..., b_j^{t+1}, ... \right\}$$

- net value of realizing task today is

$$v_i - \delta \left(v_i - p_i^{t+1} \right)$$

Dynamic Bidding

- bidding strategy of agent *i* is given

$$b_i^t = v_i - \delta \left(v_i - p_{i+1}^t \right) = (1 - \delta) \, v_i + \delta b_{i+1}^{t+1}$$

- ascending auction gives efficient assignment in all periods
- Bergemann and Valimaki (JET 2006):
  dynamic price competition, complete information, first price bidding
- Edelmann, Ostrovsky and Schwarz (AER 2007):
  static price competition, incomplete information, second price bidding

Information Arrival: Licensing

- sequential allocation of a single indivisible object with initially uncertain value to the bidders
- bidder *i* receives additional information only in periods in which *i* is assigned the object
- license to use facility or to explore resource for a limited time

Single Unit Auction

- single unit auction repeated over time
- discrete time, infinite horizon: $t = 0, 1, ...$
- finite number of bidders: $i \in \{1, ..., I\}$
- realized value of object for winning bidder in period t is

$$v_{i,t} = \omega_i + \varepsilon_{i,t}$$

- $\varepsilon_{i,t}$ is i.i.d. over time with $\mathbb{E}\left[\varepsilon_{i,t}\right] = 0$
- ω_i is true value of object
- $\varepsilon_{i,t}$ is random noise

Information Flow

- at $t = 0$: common prior distribution $F_i\left(\omega_i\right)$ for each agent i
- at $t \geq 0$: **winning bidder** receives informative signal $s_{i,t+1}$:

$$s_{i,t+1} = v_{i,t} = \omega_i + \varepsilon_{i,t}$$

- realized value in period t constitutes private information for period $t + 1$
- at $t \geq 0$: **loosing bidders** don't receive additional information:

$$s_{i,t+1} = s_{i,t}$$

Histories

- private history of bidder *i*:

$$s_i^t = \left(s_{i,0}, ..., s_{i,t}\right)$$

- **expected value** for bidder *i* in perid *t* :

$$v_{i,t}\left(s_i^t\right) \triangleq \mathbb{E}\left[\omega_i \left| s_i^t \right.\right]$$

Interpretation

- renting store space in mall
  - current winner (lessee) gets traffic data, purchase behavior
  - current looser does not get traffic data, purchase behavior
- bidding for keywords
  - current winner gets information about click-through rate, sales conversion rate
  - current looser doesn’t get information about click-through rate, sales conversion rate

Dynamic Direct Mechanism

- bidder i is asked to report her signal in every period t
- initial reports:

$$r_0 = \left(r_{1,0}, \ldots, r_{I,0}\right)$$

- inductively, a history of reports:

$$r^t = \left(r^{t-1},\ r_{1,t}, \ldots, r_{I,t}\right) \in R^t$$

- allocation rule:

$$x_t : R^{t-1} \times R_t \to [0,1]^I$$

- transfer (or pricing) rule is given by:

$$p_t : R^{t-1} \times R_t \to \mathbb{R}^I$$

Strategies

- reporting strategy for agent i:

$$r_{i,t} : R^{t-1} \times S_i \to S_i.$$

- expected payoff for bidder i :

$$\mathbb{E} \sum_{t=0}^{\infty} \delta^t \left[x_{i,t}\left(r^t\right) v_i\left(s_i^t\right) - p_{i,t}\left(r^t\right)\right].$$

- reporting strategy of i solves sequential optimization problem $V_i(s_i^t, r^{t-1})$:

$$\max_{r_{i,t} \in S_i} \mathbb{E}\left\{ x_{i,t}\left(r^t\right) v_{i,t}\left(s_i^t\right) - p_{i,t}\left(r^t\right) + \delta V_i\left(s_i^{t+1}, r^t\right)\right\}$$

- taking expectation $\mathbb{E}$ wrt $\left(s_{-i,t}, r_{-i,t}\right)$

Equilibrium

- denote by $s^t_{-(i,t)} \triangleq s^t \setminus s_{i,t}$
- *Bayesian incentive compatible* if $r_{i,t} = s_{i,t}$ solves

$$\max_{r_{i,t} \in S_i} \mathbb{E} \left\{ x_{i,t} \left(r_{i,t}, s^t_{-(i,t)} \right) v_i \left(s^t_i \right) - p_{i,t} \left(r_{i,t}, s^t_{-(i,t)} \right) + \delta V_i \left(r_{i,t}, s^t_{-(i,t)} \right)\right.$$

- periodic ex post: with respect to all the information available at period t
- (periodic) *ex post incentive compatible* if $r_{i,t} = s_{i,t}$ solves

$$\max_{r_{i,t} \in S_i} \left\{ x_{i,t} \left(r_{i,t}, s^t_{-(i,t)} \right) v_i \left(s^t_i \right) - p_{i,t} \left(s_{i,t}, s^t_{-(i,t)} \right) + \delta V_i \left(r_{i,t}, s^t_{-(i,t)} \right)\right.$$

for all $s_{-i,t} \in S_{-i}$

Social Efficiency

- socially efficient assignment policy

$$W\left(s^u\right) = \max_{\{x_t(s^t)\}_{t=u}^{\infty}} \mathbb{E} \sum_{t=u}^{\infty} \sum_{i=1}^{N} \delta^{t-u} x_{i,t}\left(s^t\right) v_i\left(s_i^t\right)$$

- optimal assignment is a multi–armed bandit problem
- optimal policy is an index policy:

$$\gamma_i\left(s_i^u\right) = \max_{\tau} \mathbb{E} \left\{ \frac{\sum_{t=0}^{\tau} \delta^t v_i\left(s_i^{u+t}\right)}{\sum_{t=0}^{\tau} \delta^t} \right\}$$

- socially efficient allocation policy $\mathbf{x}^* = \{x_t^*\}_{t=0}^{\infty}$:

$$x_{i,t}^* > 0 \text{ if } \gamma_i\left(s_i^t\right) \geq \gamma_j\left(s_j^t\right) \text{ for all } j.$$

Marginal Contribution

- value of social program after removing bidder i

$$W_{-i}\left(s^{u}\right)=\max_{\left\{x_{-i,t}\left(s^{t}\right)\right\}_{t=u}^{\infty}}\mathbb{E}\sum_{t=u}^{\infty}\sum_{j\neq i}\delta^{t-u}x_{j}^{t}\left(s^{t}\right)v_{j}\left(s_{j}^{t}\right)$$

- *marginal contribution* $M_i\left(s^t\right)$ of bidder i at history s^t is:

$$M_i\left(s^t\right)=W\left(s^t\right)-W_{-i}\left(s^t\right)$$

- value M conditional on history s^u and allocation x_u:

$$M\left(s^u,x_u\right)$$

- *flow marginal contribution* $m_i\left(s^t\right)$:

$$M_i\left(s^t\right)=m_i\left(s^t\right)+\delta M_i\left(s^t,x_t^*\right)$$

Flow Marginal Contribution

- flow marginal contribution:

$$m_i\left(s^t\right) = M_i\left(s^t\right) - \delta M_i\left(s^t, x_t^*\right)$$

- expanding flow expression with respect to time

$$m_i\left(s^t\right) = \overbrace{\left(W\left(s^t\right) - W_{-i}\left(s^t\right)\right)}^{\text{M}_\text{i}\text{ starting at t}} - \overbrace{\delta\left(W\left(s^t, x_t^*\right) - W_{-i}\left(s^t, x_t^*\right)\right)}^{\text{M}_\text{i}\text{ starting at t+1 and x}_\text{t}^*}$$

- expanding flow expression with respect to identity

$$m_i\left(s^t\right) = \overbrace{\left(W\left(s^t\right) - \delta W\left(s^t, x_t^*\right)\right)}^{\text{current value with i}} - \overbrace{\left(W_{-i}\left(s^t\right) - \delta W_{-i}\left(s^t, x_t^*\right)\right)}^{\text{current value without i but x}_\text{t}^*}$$

- note $W_{-i}\left(s^t\right) \geq W_i\left(s^t, x_t^*\right)$

Efficient Assignment

$$m_i\left(s^t\right) = \left(W\left(s^t\right) - \delta W\left(s^t, x_t^*\right)\right) - \left(W_{-i}\left(s^t\right) - \delta W_{-i}\left(s^t, x_t^*\right)\right)$$

- consider efficient assignment $x_t^* = i$:
- information about $x_t^* = i$ is worthless without i:

$$W_{-i}\left(s^t, i\right) = W_{-i}\left(s^t\right)$$

leads to

$$m_i\left(s^t\right) = v_i\left(s_i^t\right) - (1-\delta) W_{-i}\left(s^t\right)$$

- consider inefficient bidder: $x_t^* \neq j$:

$$x_{-j,t}^* = x_t^*$$

leads to

$$m_j\left(s^t\right) = 0$$

Dynamic Second Price Auction

- match net payoff to flow marginal contribution
- for winner i:
$$m_i\left(s^t\right) = v_i\left(s^t\right) - p_i\left(s^t\right)$$
- for losers, $j \neq i$:
$$m_j\left(s^t\right) = -p_j\left(s^t\right)$$

Result

Theorem (Dynamic Second Price Auction)

The socially efficient allocation rule $\mathbf{x}^*$ *satisfies ex post incentive and participation constraints with payment* $\mathbf{p}^*$*:*

$$p_j^*\left(s^t\right) = \begin{cases} (1-\delta)\,W_{-j}\left(s^t\right) & \text{if } x_{j,t}^* = 1, \\ 0 & \text{if } x_{j,t}^{t*} = 0. \end{cases}$$

- price equals intertemporal opportunity cost
- delay $(1-\delta)$ of the optimal program for all but j

Dominant versus Ex Post Incentive Compatibility

- with private values, static mechanism satisfies incentive compatibility in weakly dominant strategies
- in dynamic mechanism, dominant incentive compatibility fails to hold in private value environment
- truthtelling after all histories fails to be a weakly dominant strategy as it removes the ability to respond to past announcements
- yet ex post incentive compatibility can be satisfied in dynamic mechanism

General Allocation Problems

- description of a dynamic Vickrey-Clarke-Groves mechanism
- general specification of utility of each agent and arrival of private information over time
- dynamic VCG mechanism is time consistent
  - social choice function can be implemented by a sequential mechanism without ex ante commitment by the designer
  - thruthtelling strategy in the dynamic setting forms an ex-post equilibrium rather than an equilibrium in weakly dominant strategies

General Allocation Problem

- extend single unit auction to general allocation model
- net expected flow utility of agent i in period t :

$$v_i\left(x_t, s_i^t\right) - p_{i,t}$$

- private signal of agent i in period $t+1$ is generated by conditional distribution function:

$$s_{i,t+1} \sim G_i\left(\cdot \left| x_t, s_i^t \right.\right).$$

- generalize information flow by allowing signal $s_{i,t+1}$ of agent i in period $t+1$ to depend on current decision x_t and entire past history of private signals of i

Dynamic VCG Mechanism

- efficiency
- marginal contribution pricing

Theorem (Dynamic VCG Mechanism)

The socially efficient allocation rule $\{x^*\}$ satisfies ex post incentive and ex post participation constraint with payment p^*:

$$p^*_{i,t}\left(x^*\left(s^t\right), s^t_{-i}\right) = v_i\left(x^*\left(s^t\right), s^t_i\right) - m_i\left(s^t\right).$$

- characterization of transfer prices via marginal contribution
- in specific environments additional insights from observing how social policies are affected by removal of agent *i*

Efficient Exit

- many transfer rules support ex post incentive and ex post participation constraints in dynamic setting
- temporal separation between allocative influence and monetary payments may be undesirable for may reasons:
  - agent i could be tempted to leave the mechanisms and break her commitment *after* she ceases to have a pivotal role but *before* her payments come due
  - if arrival and departure of agents is random, then an agent could falsely claim to depart to avoid future payments
- in intertemporal environment if agent i ceases to influence current or future allocative decisions in period t, then she also ceases to have monetary obligations

Efficient Exit

- given state s^{τ} the presence of i is immaterial for the efficient decision x_u^* if

$$\tau = \min \left\{ t \;\middle|\; \begin{array}{l} x_u^*\left(s^u\right) = x_{-i,u}^*\left(s^u\right), \forall u \geq t, \\ \forall s^u = \left(s^{\tau}, s_{\tau+1}, ..., s_t, ..., s_u\right) \end{array} \right\}$$

Definition (Efficient Exit)

A mechanism satisfies the efficient exit condition if for all i, s^{τ} and τ :

$$p_{i,u}\left(s^u\right) = 0, \quad \text{for all } u \geq \tau.$$

Uniqueness

- weak online requirement: decisions regarding agent *i* have to be made in the presence of agent *i*

Theorem (Uniqueness)

If a dynamic direct mechanism is efficient, satisfies ex post incentive and participation constraints and the efficient exit condition, then it is the dynamic marginal contribution mechanism.

Conclusion

- direct dynamic mechanism in private value environments with transferable utility
- design of monetary transfers relies on notions of marginal contribution and flow marginal contribution
- transfer the insights of VCG mechanism from static to dynamic settings
- many interesting questions are left open
  - current contribution is silent on issue of revenue maximizing mechanisms
  - characterization of implementable allocations in dynamic setting will first be necessary
  - restriction to private value environments