Belief Free Incomplete Information Games

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Introduction

- in games of incomplete information, the private information of each agent is represented by his type, (Harsanyi (1967/68)), Mertens/Zamir (1985)
- the type of each agent contains information about the preferences of the agents **and** information about the beliefs of each agent
- the type of each agent can be decomposed into a payoff type and a belief type
- each agent may have different belief types associated with the same payoff type
- prediction of play in a game of incomplete information may be sensitive to the payoff type and the belief type

Belief Free Solution Concepts

- three solution concepts for games of incomplete information which depend only on the payoff type but not on the belief type

1. Incomplete Information Rationalizability
2. Incomplete Information Correlated Equilibrium
3. Ex Post Equilibrium

- demonstrate their role in game theoretic and mechanism design settings
- establish relationships to each other
- epistemic foundations

Belief Free and Robustness to Strategic Uncertainty

- in earlier work,
  "Robust Mechanism Design" (2005), and
  "Robust Implementation in Direct Mechanisms" (2007)
  we studied the implications for mechanism design of relaxing "implicit common knowledge assumptions"
- main results:

1. a social choice function f is Bayesian incentive compatible in every type space if and only if it is ex post incentive compatible
2. a social choice function f is Bayesian implentable in every type space if and only if it satisfies robust monotonicity

- this talk describes foundational results for pursuing this agenda in general strategic settings beyond mechanism design environments

Setting

- I players
- action $a_i \in A_i$
- action profile $a = (a_1, ..., a_I) \in A$
- payoff type $\theta_i \in \Theta_i$
- payoff type profile $\theta = (\theta_1, ..., \theta_I) \in \Theta$
- interdependent payoff functions $u_i : A \times \Theta \to \mathbb{R}$
- no beliefs over types

Example 1: Oligopoly

- action (= quantity): $a_i \in \mathbb{R}_+$
- payoff types (= private cost): $\theta_i \in [0, 1]$
- inverse demand (= price):

$$1 - \sum_j a_j$$

- payoff functions:

$$u_i(a, \theta) = a_i \left(1 - \sum_j a_j - \theta_i \right)$$

- best response:

$$a_i = \frac{1}{2} \left(1 - \theta_i - \mathbb{E}_i \sum_{j \neq i} a_j \right)$$

Example 2: Public Good Mechanism

- action (= report): $a_i \in \mathbb{R}_+$
- payoff type: $\theta_i \in [0, 1]$ and payoff function:

$$u_i(a, \theta) = y(a)(\theta_i + \gamma \sum_{j \neq i} \theta_j) - t_i(a)$$

- with cost function $c(y) = \frac{1}{2} y^2$ and efficient provision:

$$y(\theta) = \sum_j \theta_j$$

- incentive compatible transfer $t_i(\theta)$:

$$t_i(\theta) = \theta_i(\frac{1}{2}\theta_i + \gamma \sum_{j \neq i} \theta_j)$$

- best response:

$$a_i = \theta_i + \gamma \mathbb{E} \sum_{j \neq i} (\theta_j - a_j)$$

Incomplete Information Rationalizability

- set profile $R = (R_i)_{i=1}^{I}$, each $R_i : \Theta_i \to 2^{A_i} / \varnothing$
- initialize

$$R_i^0(\theta_i) = A_i$$

and inductively for each $k = 1, 2, ...,$

$$R_i^k(\theta_i) = \left\{ a_i \;\middle|\; \begin{array}{l} \exists\ \mu_i \in \Delta(A_{-i} \times \Theta_{-i}) \quad s.th. \\ (1)\ \mu_i(a_{-i}, \theta_{-i}) > 0 \Rightarrow a_j \in R_j^{k-1}(\theta_j),\ \forall j \neq i \\ (2)\ a_i \in \underset{a_i'}{\arg\max} \sum_{a_{-i}, \theta_{-i}} u_i((a_i', a_{-i}), \theta)\, \mu_i(a_{-i}, \theta_{-i}) \end{array} \right\}$$

with

$$R_i(\theta_i) = \bigcap_{k \geq 0} R_i^k(\theta_i).$$

- Battigalli and Siniscalchi (2003), distinct from interim rationalizability

Rationalizability with Oligopoly

- best response

$$a_i = \frac{1}{2}\left(1 - \theta_i - \mathbb{E}_i \sum_{j \neq i} a_j\right)$$

consider $I \geq 3$

- $R_i^0(\theta_i) = \mathbb{R}_+$
- $R_i^1(\theta_i) == \left[0, \frac{1}{2}(1-\theta_i)\right]$
- $R_i^k(\theta_i) = \left[0, \frac{1}{2}(1-\theta_i)\right], \forall k \geq 1$
- $R_i(\theta_i) = \left[0, \frac{1}{2}(1-\theta_i)\right]$

Rationalizability with Public Good

- best response

$$a_i = \theta_i + \gamma \mathbb{E}_i \sum_{j \neq i} (\theta_j - a_j)$$

- $\theta_j - a_j \neq 0$ is misreport relative to thruthelling $\theta_j - a_j = 0$
- initialize

$$R_i^0(\theta_i) = [0, 1]$$

- largest misreports - given R_i^0 - by each agent are $\{-1, +1\}$, so:

$$R_i^1(\theta_i) = [\theta_i - |\gamma| (I - 1), \;\; \theta_i + |\gamma| (I - 1)]$$

- respecting $A = [0, 1]$:

$$R_i^1(\theta_i) = [\max(0, \theta_i - |\gamma| (I - 1)), \min(1, \theta_i + |\gamma| (I - 1))]$$

Inductive Steps

$$R_i^1(\theta_i) = [\max(0, \theta_i - |\gamma|(I-1)), \min(1, \theta_i + |\gamma|(I-1))]$$

and for all $k \geq 1$:

$$R_i^k(\theta_i) = \left[\max\left(0, \theta_i - (|\gamma|(I-1))^k\right), \min\left(1, \theta_i + (|\gamma|(I-1))^k\right)\right]$$

- the limit set is

$$R_i(\theta_i) = \begin{cases} \{\theta_i\}, & \text{if } |\gamma| < \frac{1}{I-1} \\ [0,1], & \text{if } |\gamma| \geq \frac{1}{I-1} \end{cases}$$

- mor generally, unique rationalizable outcome with moderate interdependence, see “Robust Implementation in Direct Mechanisms" (2007)

Incomplete Information Correlated Equilibrium

Definition

$\mu \in \Delta(A \times \Theta)$ is an incomplete information correlated equilibrium (ICE) if $\forall i, \theta_i, a_i$ and a_i':

$$\sum_{a_{-i},\theta_{-i}} u_i\left((a_i, a_{-i}), (\theta_i, \theta_{-i})\right) \mu\left((a_i, a_{-i}), (\theta_i, \theta_{-i})\right)$$
$$\geq \sum_{a_{-i},\theta_{-i}} u_i\left((a_i', a_{-i}), (\theta_i, \theta_{-i})\right) \mu\left((a_i, a_{-i}), (\theta_i, \theta_{-i})\right).$$

- Forges (1993)

$C_i(\theta_i)$ is set of actions that could be played by type θ_i in an ICE:

$$C_i(\theta_i) = \left\{ a_i \in A_i \;\middle|\; \begin{array}{c} \exists \text{ ICE } \mu \text{ and } (a_{-i}, \theta_{-i}) \in A_{-i} \times \Theta_{-i} \\ \text{such that } \mu\left((a_i, a_{-i}), (\theta_i, \theta_{-i})\right) > 0 \end{array} \right\}.$$

Correlated Equilibrium with Oligopoly

- best response

$$a_i = \frac{1}{2}\left(1 - \theta_i - \mathbb{E}_i \sum_{j \neq i} a_j\right)$$

- $C_i(\theta_i) = R_i(\theta_i) \ldots$
- ... but fix a probability distribution over payoff types $\psi \in \Delta\left([0,1]^I\right)$ and find Bayesian Nash equilibrium s of "naive" incomplete information game...
- ... in every ICE with marginal ψ on θ, the expected action of each agent i is $\mathbb{E}_{\psi}[s_i(\theta_i)]$.

Correlated Equilibrium with Public Good

- best response

$$a_i = \theta_i + \gamma \mathbb{E}_i \sum_{j \neq i} \left(\theta_j - a_j\right).$$

- $C_i\left(\theta_i\right) = \begin{cases} \{\theta_i\}, & \text{if } -\frac{1}{I-1} < \gamma < 1 \\ [0,1], & \text{if otherwise} \end{cases}$
- for $|\gamma| < \frac{1}{I-1}$, this follows from $C_i\left(\theta_i\right) \subseteq R_i\left(\theta_i\right)$
- we will report new “potential” argument for $\gamma \in \left[\frac{1}{I-1}, 1\right)$ extending Neyman (1997)

Ex Post Equilibrium

- A *payoff type strategy* for player i is a function $s_i : \Theta_i \to A_i$.
- A payoff type strategy profile $s^* = \left(s_i^*\right)_{i=1}^{I}$ is an *ex post equilibrium* if for all i and all θ, we have

$$u_i\left(\left(s_i^*\left(\theta_i\right), s_{-i}^*\left(\theta_{-i}\right)\right), \theta\right) \geq u_i\left(\left(a_i, s_{-i}^*\left(\theta_{-i}\right)\right), \theta\right)$$

  for all $a_i \in A_i$.
- Holmstrom and Myerson (1983) introduced it as "uniform incentive compatibility" and in subsequent mechanism design literature; Kalai (2004) in large games

Ex Post Equilibrium with Oligopoly

- best response

$$a_i = \frac{1}{2}\left(1 - \theta_i - \mathbb{E}_i \sum_{j \neq i} a_j\right)$$

- no ex post equilibrium

Ex Post Equilibrium with Public Good

- best response

$$a_i = \theta_i + \gamma \mathbb{E}_i \sum_{j \neq i} (\theta_j - a_j) \, .$$

- unique ex post equilibrium: $s_i^* (\theta_i) = \theta_i$ (truthtelling)

Incomplete Information Correlated Equilibrium

- $\psi \in \Delta(\Theta)$ is a distribution over payoff type profiles
- ψ_μ is marginal distribution over payoff types generated by μ:

$$\psi_\mu(\theta) \triangleq \sum_{a \in A} \mu(a, \theta)$$

- for any $\psi \in \Delta(\Theta)$ and payoff type strategy profile s, write $\mu^{\psi,s}$ as the induced probability distribution over $A \times \Theta$

$$\mu^{\psi,s}(a, \theta) \triangleq \begin{cases} \psi(\theta), & \text{if } a = s(\theta), \\ 0, & \text{if otherwise.} \end{cases}$$

Potential Game I

- u has *weighted potential* $v : A \times \Theta \to \mathbb{R}$ if there exist $w \in \mathbb{R}^I_{++}$ such that

$$u_i\left(\left(a_i, a_{-i}\right), \left(\theta_i, \theta_{-i}\right)\right) - u_i\left(\left(a_i', a_{-i}\right), \left(\theta_i, \theta_{-i}\right)\right)$$
$$=$$
$$w_i \cdot \left[v\left(\left(a_i, a_{-i}\right), \left(\theta_i, \theta_{-i}\right)\right) - v\left(\left(a_i', a_{-i}\right), \left(\theta_i, \theta_{-i}\right)\right)\right]$$

  for all i, $a_i, a_i' \in A_i$, $a_{-i} \in A_i$ $\theta_i \in \Theta_i$ and $\theta_{-i} \in \Theta_{-i}$.
- v is a strictly concave potential if $v\left(\cdot, \theta\right)$ is a strictly concave function of a for all $\theta \in \Theta$.

Uniqueness I

Theorem

If u has a strictly concave smooth potential function and an ex post equilibrium s, then μ is an incomplete information correlated equilibrium of u if and only if there exists $\psi \in \Delta(\Theta)$ such that $\mu = \mu^{\psi,s}$.

- public good model has a smooth concave potential iff

$$\gamma \in \left[-\frac{1}{I-1}, 1\right]$$

- uniqueness for a larger set of interdependencies than incomplete information rationalizability

Payoff Environment I

(u, ψ)

- recall ψ is a distribution over payoff types $\psi \in \Delta(\Theta)$

Definition (ICE of (u, ψ))

A probability distribution $\mu \in \Delta(A \times \Theta)$ is an incomplete information correlated equilibrium (ICE) of (u, ψ) if

$$\int_A d\mu = d\psi$$

and for each i and each measurable $\phi_i : A_i \times \Theta_i \to A_i$

$$\int_{a,\theta} u_i\left(\left(a_i, a_{-i}\right), \left(\theta_i, \theta_{-i}\right)\right) d\mu \geq \int_{a,\theta} u_i\left(\left(\phi_i\left(a_i, \theta_i\right), a_{-i}\right), \left(\theta_i, \theta_{-i}\right)\right) d\mu.$$

Oligopoly with Complete Information I

- complete information: type is average cost of i:

$$\overline{\theta}_i = \mathbb{E}_{\psi}\left[\theta_i\right]$$

- complete information Nash equilibrium $(\overline{a}_1, ..., \overline{a}_I)$, given average cost $\left(\overline{\theta}_1, ..., \overline{\theta}_I\right)$, solves:

$$\overline{a_i} = \frac{1}{2}\left(1 - \sum_{j \neq i} \overline{a_j} - \overline{\theta}_i\right).$$

Characterization of ICE I

Theorem (Characterization)

In all incomplete information correlated equilibria μ *of* (u, ψ)*:*

$$\mathbb{E}_{\mu}[a_i] = \bar{a}_i.$$

- Liu (1996) and Neyman (1997) show with complete information that the correlated equilibrium actions is unique
- with incomplete information average action is unique, but is not true for conditional average action $\mathbb{E}_{\mu}[a_i | \theta_i]$

Beyond the Mean I

- assume independent prior distributions:

$$\psi = \Pi_i \psi_i$$

Theorem

In all incomplete information correlated equilibria μ of (u, ψ):

1. *The correlation between a_i and θ_i is strictly positive and bounded away from zero;*
2. *The correlation between a_i and a_j is weakly negative;*
3. *The correlation between a_i and θ_j is weakly negative.*

- partial identification in Chwe (2005)

An Example with Normal Distribution

- θ_i is independently normally distributed with mean θ and standard deviation τ^2
- describe normally distributed symmetric (across players) incomplete information correlated equilibrium:

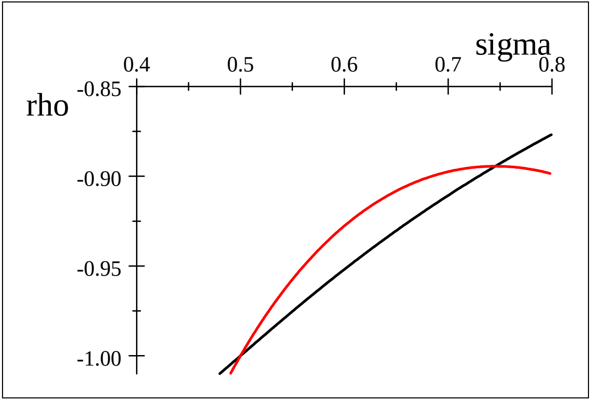
$$\begin{pmatrix} a_1 \\ a_2 \\ \theta_1 \\ \theta_2 \end{pmatrix} \sim N\left(\begin{pmatrix} a \\ a \\ \theta \\ \theta \end{pmatrix}, \Sigma\right)$$

with interaction matrix:

$$\Sigma = \begin{pmatrix} \sigma^2 & \psi\sigma^2 & \rho\sigma\tau & z\sigma\tau \\ \psi\sigma^2 & \sigma^2 & z\sigma\tau & \rho\sigma\tau \\ \rho\sigma\tau & z\sigma\tau & \tau^2 & 0 \\ z\sigma\tau & \rho\sigma\tau & 0 & \tau^2 \end{pmatrix}$$

Set of Incomplete Information Correlated Equilibria

- symmetric equilibrium conditions determine two parameters of the interaction matrix Σ
- normal distribution gives determine two inequalities for the set of correlated equilibria in terms of σ and ρ:



Epistemic Foundations

A type space $\mathcal{T}$ is defined as $\mathcal{T} \triangleq \left(T_i, \widehat{\pi}_i, \widehat{\theta}_i\right)_{i=1}^{I}$ where

1. T_i is a finite set of types
2. $\widehat{\pi}_i : T_i \to \Delta\left(T_{-i}\right)$ describes the beliefs of i's types
3. $\widehat{\theta}_i : T_i \to \Theta_i$ describes the payoff types of agent i's types

- type space encodes information about payoff relevant information, but also about strategically relevant information

Interim Equilibrium

A behavioral strategy of player i in type space $\mathcal{T}$ is given by a function $\sigma_i : T_i \to \Delta(A_i)$.

Strategy profile σ is an *interim equilibrium* of $(u, \mathcal{T})$ if for each i, $t_i \in T_i$, $a_i \in A_i$ with $\sigma_i(a_i|t_i) > 0$, and $a_i' \in A_i$,

$$\sum_{a_{-i},t_{-i}} \widehat{\pi}_i(t_i)[t_{-i}] \left(\prod_{j \neq i} \sigma_j(a_j|t_j) \right) u_i\left((a_i, a_{-i}), \left(\widehat{\theta}_i(t_i), \widehat{\theta}_{-i}(t_{-i}) \right) \right)$$

$$\geq \sum_{a_{-i},t_{-i}} \widehat{\pi}_i(t_i)[t_{-i}] \left(\prod_{j \neq i} \sigma_j(a_j|t_j) \right) u_i\left((a_i', a_{-i}), \left(\widehat{\theta}_i(t_i), \widehat{\theta}_{-i}(t_{-i}) \right) \right)$$

Actions Played in Equilibrium

Actions played by agent i with payoff type θ_i in some interim equilibrium on some type space $\mathcal{T}$:

$$S_i(\theta_i) = \left\{ a_i \;\middle|\; \begin{array}{l} \exists\, \mathcal{T}, \text{ an eq. } \sigma, \text{ of } (u, \mathcal{T}), \text{ and a type } t_i \\ \text{s. t. } \widehat{\theta}_i(t_i) = \theta_i \text{ and } \sigma_i(a_i | t_i) > 0 \end{array} \right\}.$$

Actions played by agent i with payoff type θ_i in some interim equilibrium on some common prior type space $\mathcal{T}$:

$$S_i^{CP}(\theta_i) = \left\{ a_i \;\middle|\; \begin{array}{l} \exists \text{ a c.p. } \mathcal{T}, \text{ an eq. } \sigma, \text{ of } (u, \mathcal{T}), \text{ and a type } t_i \\ \text{s. t. } \widehat{\theta}_i(t_i) = \theta_i \text{ and } \sigma_i(a_i | t_i) > 0 \end{array} \right\}.$$

Epistemic Results

Theorem

For all i and θ_i,

1. $R_i(\theta_i) = S_i(\theta_i)$,
2. $C_i(\theta_i) = S_i^{CP}(\theta_i)$.

In words:

1. "an action a_i is incomplete information rationalizable for type θ_i if and only if it could be played in equilibrium for some beliefs and higher order beliefs about others' types";
2. "an action a_i is an ICE action for type θ_i if and only if it could be played in equilibrium for some common prior beliefs and higher order beliefs about others' types"

Write $\sigma^{s,\mathcal{T}}$ for the strategy profile in $(u, \mathcal{T})$ induced by s, so that

$$\sigma^{s,\mathcal{T}}\left(s_i\left(\theta_i\right)|t_i\right) = \begin{cases} 1, \text{ if } \widehat{\theta}_i\left(t_i\right) = \theta_i, \\ 0, \text{ if } \widehat{\theta}_i\left(t_i\right) \neq \theta_i. \end{cases}$$

Theorem

The following are equivalent:

1. *s is an ex post equilibrium*
2. *$\sigma^{s,\mathcal{T}}$ is an interim equilibrium of $(u, \mathcal{T})$ for all type space $\mathcal{T}$*
3. *$\sigma^{s,\mathcal{T}}$ is an interim equilibrium of $(u, \mathcal{T})$ for all full support common prior payoff type spaces $\mathcal{T}$*

- earlier result in "Robust Mechanism Design" (2005)

Conclusion

- belief free analysis of game/mechanism design
- moderate interdependence yields strong predictions
- epistemic foundations of belief free solution concepts
- given prior over payoff types what is the set of equilibrium predictions