Econ 121b: Intermediate Microeconomics

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1 Lecture 4: Utility Maximization

1.1 Multivariate Function Maximization

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ be a consumption bundle and $f \colon \mathbb{R}^n_+ \to \mathbb{R}$ be a multivariate function. The multivariate function that we are interested in here is the utility function $u \colon \mathbb{R}^n_+ \to \mathbb{R}$ where $u(x)$ is the utility of the consumption bundle x.

The F.O.C. for maximization of f is given by:

$$\frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_i} = 0 \quad \forall\, i = 1, 2, \ldots, n$$

This is a direct extension of the F.O.C. for univariate functions as explained in Lecture 3. The S.O.C. however is a little different from the single variable case. Let's look at a bivariate function $f \colon \mathbb{R}^2 \to \mathbb{R}$. Let's first define the following notations:

$$f_i(x) \triangleq \frac{df(x)}{dx_i}, \ f_{ii}(x) \triangleq \frac{d^2 f(x)}{dx_2^2}, \ i = 1, 2,$$

$$f_{ij}(x) \triangleq \frac{d^2 f(x)}{dx_i dx_j}, \ i \neq j$$

The S.O.C. for the maximization of f is then given by,

$$(i) \qquad f_{11} < 0$$

$$(ii) \quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$$

The first of the S.O.C.s is analogous to the S.O.C. for the univariate case. If we write out the second one we get,

$$f_{11}f_{22} - f_{12}f_{21} > 0$$

But we know that $f_{12} = f_{21}$. So,

$$f_{11}f_{22} > f_{12}^2 > 0$$

$$\Rightarrow f_{22} < 0 \quad (\text{since } f_{11} < 0)$$

Therefore the S.O.C. for the bivariate case is stronger than the analogous conditions from the univariate case. This is because for the bivariate case to make sure that we are at the peak of a function it is not enough to check if the function is concave in the directions of x_1 and x_2, as it could not be concave along the diagonal and therefore the need to introduce cross derivatives in to the condition. For the purposes of this class we'd assume that the S.O.C. is satisfied for the utility function being given, unless it is asked specifically to check for it.

1.2 Budget Constraint

A budget constraint is a constraint on how much money (income, wealth) an agent can spend on goods. We denote the amount of available income by $I \geq 0$. $x_1, \ldots, x_N$ are the quantities of the goods purchased and $p_1, \ldots, p_N$ are the according prices. Then the budget constraint is

$$\sum_{i=1}^{N} p_i x_i \leq I.$$

As an example, we consider the case with two goods. In that case we get that $p_1 x_1 + p_2 x_2 \leq I$, i.e., the agent spends her entire income on the two goods. The points where the budget line intersects with the axes are $x_1 = I/p_1$ and $x_2 = I/p_2$ since these are the points where the agent spends her income on only one good. Solving for x_2, we can express the budget line as a function of x_1:

$$x_2(x_1) = \frac{I}{p_2} - \frac{p_1}{p_2}x_1,$$

where the slope of the budget line is given by,

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$

The **budget line** here is defined as the equation involving x_1 and x_2 such that the decision maker exhausts all her income. The set of consumption bundles (x_1, x_2) which are feasible given the income, i.e. (x_1, x_2) for which $p_1 x_1 + p_2 x_2 \leq I$ holds is defined as the **budget set**.

1.3 Indifference Curve

Indifference Curve (IC) is defined as the locus of consumption bundles (x_1, x_2) such that the utility is held fixed at some level. Therefore the equation of the IC is given by,

$$u(x_1, x_2) = \bar{u}$$

To get the slope of the IC we differentiate the equation w.r.t. x_1:

$$\frac{\partial u(x)}{\partial x_1} + \frac{\partial u(x)}{\partial x_2}\frac{dx_2}{dx_1} = 0$$

$$\Rightarrow \frac{dx_2}{dx_1} = -\frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}} = -\frac{MU_1}{MU_2}$$

where MU_i refers to the marginal utility of good i. So the slope of IC is the (negative of) ratio of marginal utilities of good 1 and 2. This ratio is referred to as the Marginal Rate of Substitution or MRS. This tells us the rate at which the consumer is ready to substitute between good 1 and 2 to remain at the same utility level.

1.4 Constrained Optimization

Consumers are typically endowed with money I, which determines which consumption bundles are affordable. The budget set consists of all consumption bundles such that $\sum_{i=1}^{N} p_i x_i \leq I$. The consumer's problem is then to find the point on the highest indifference curve that is in the budget set. At this point the indifference curve must be tangent to the budget line. The slope of the budget line is given by,

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$

which defines how much x_2 must decrease if the amount of consumption of good 1 is increased by dx_1 for the bundle to still be affordable. It reflects the opportunity cost, as money spent on good 1 cannot be used to purchase good 2 (see Figure 1).

The marginal rate of substitution, on the other hand, reflects the relative benefit from consuming different goods. The slope of the indifference curve is $-MRS$. So the relevant optimality condition, where the slope of the indifference curve equals the slope of the budget line, is

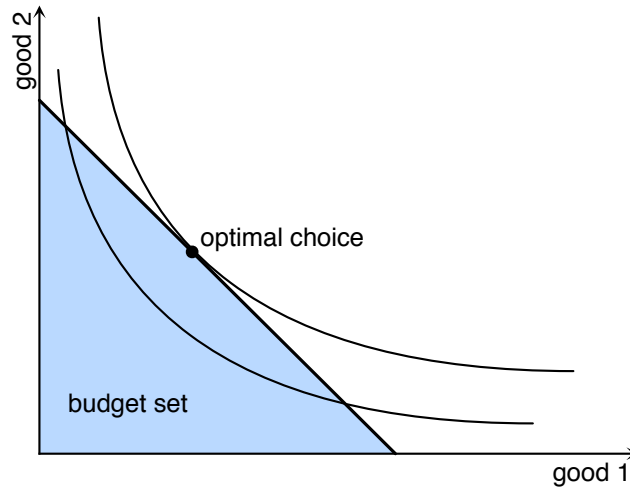$$\frac{p_1}{p_2} = \frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}}.$$

Figure 1: Indifference curve and budget set

We could equivalently talk about equating marginal utility per dollar. If

$$\frac{\frac{\partial u(x)}{\partial x_2}}{p_2} > \frac{\frac{\partial u(x)}{\partial x_1}}{p_1}$$

then one dollar spent on good 2 generates more utility then one dollar spent on good 1. So shifting consumption from good 1 to good 2 would result in higher utility. So, to be at an optimum we must have the marginal utility per dollar equated across goods.

Does this mean then that we must have $\frac{\partial u(x)}{\partial x_i} = p_i$ at the optimum? No. Such a condition wouldn't make sense since we could rescale the utility function. We could instead rescale the equation by a factor $\lambda \geq 0$ that converts "money" into "utility." We could then write $\frac{\partial u(x)}{\partial x_i} = \lambda p_i$. Here, λ reflects the marginal utility of money. More on this in the subsection on Optimization using Lagrange approach.

1.4.1   Optimization by Substitution

The consumer's problem is to maximize utility subject to a budget constraint. There are two ways to approach this problem. The first approach involves writing the last good as a function of the previous goods, and then proceeding with an unconstrained maximization. Consider the two good case. The budget set consists of the constraint that $p_1 x_1 + p_2 x_2 \leq I$. So the problem is

$$\max_{x_1, x_2} \ u(x_1, x_2) \ \text{ subject to } \ p_1 x_1 + p_2 x_2 \leq I$$

But notice that whenever u is (locally) non-satiated then the budget constraint holds with equality since there in no reason to hold money that could have been

used for additional valued consumption. So, $p_1 x_1 + p_2 x_2 = I$, and so we can write x_2 as a function of x_1 from the budget equation in the following way

$$x_2 = \frac{I - p_1 x_1}{p_2}$$

Now we can treat the maximization of $u\left(x_1, \frac{I - p_1 x_1}{p_2}\right)$ as the standard single variable maximization problem. Therefore now the maximization problem becomes,

$$\max_{x_1} u\left(x_1, \frac{I - p_1 x_1}{p_2}\right)$$

The F.O.C. is then given by,

$$\frac{du}{dx_1} + \frac{du}{dx_2}\frac{dx_2(x_1)}{dx_1} = 0$$

$$\Rightarrow \frac{du}{dx_1} - \frac{p_1}{p_2}\frac{du}{dx_2} = 0$$

The second equation substitutes $\frac{dx_2(x_1)}{dx_1}$ by $-\frac{p_1}{p_2}$ from the budget line equation. We can further rearrange terms to get,

$$\frac{\frac{du}{dx_1}}{p_1} = \frac{\frac{du}{dx_2}}{p_2}$$

$$\Rightarrow \frac{\frac{du}{dx_1}}{\frac{du}{dx_2}} = \frac{p_1}{p_2}$$

This exactly the condition we got by arguing in terms of budget line and indifference curves. In the following lecture we shall look at a specific example where we would maximize a particular utility function using this substitution method and then move over to the Lagrange approach.