Introduction

• dynamic trading environments:
  • agents are arriving and departing
  • agents have stochasticall changing values
• seller and buyer may benefit from dynamic mechanism that governs relationship

• two important constraints:

  1. participation constraints: determines if and when privately informed agent agrees to enter into mechanism
  2. incentive constraints: determines continued truth-telling regarding evolving private information
Incentive and Participation Constraint

• extensive analysis of the sequence of incentive constraints in the literature of dynamic mechanism design
• narrow analysis of participation constraint: single ex-ante participation constraint, agent either immediately accepts contract or is never offered a contract again
• implicit assumption: arrival time of buyer is public information
• explicit assumption today: arrival time of buyer is private information
• in consequence private information of buyer is two-dimensional:
  1. stochastic value over time
  2. stochastic arrival time
pursue the implications of private information of arrival time and value for a classic problem in revenue maximization

sell good or service to a buyer with unit demand repeatedly over time

a few motivating examples to keep in mind:
  - lease, sales, service contracts
  - mobile phone contracts, club memberships
  - display advertising contracts
Standing Offer

- seller makes standing (or open) offer for possibly long-term contract
- stationary offer: the same contract/mechanism is offered in every future period
- rather than a take-it-now-or-never offer

→ standing offer presents buyer with option
→ chooses to enter contract now or at some later time
Progressive Participation

- buyer is timing participation time to his current value
- sometimes participates immediately, sometimes at future time

→ progressive participation

→ probability of participation is increasing over time

- option of participation time is source of new information rent
Progressive Mechanism

- symmetry across constraints:
  1. interim (sequential) incentive constraints
  2. interim (sequential) participation constraints

- information rent due to:
  1. private value
  2. private arrival time
Contrast with Dynamic Mechanism

- standard models of sequential screening:
  1. unknown value
  2. but known arrival time

- agent either accepts contract immediately or receives zero outside option
- ex-ante participation constraint at $t = 0$
- less rent extraction in progressive mechanism compared to dynamic mechanism
Model
Random Arrival and Random Value

- time is continuous $t \in [0, \infty)$
- buyer arrives and departs (is replaced) with rate $\gamma > 0$
- arrival time $\alpha \in [0, \infty)$ is private information of buyer
- value (willingness-to-pay) $\theta_t$ is private information of buyer
- initial value $\theta_\alpha$ given by common prior $F$:
  \[
  \theta_\alpha \in [0, \bar{\theta}] = \Theta \subset \mathbb{R}_+
  \]
- value $\theta_t$ follows geometric Brownian motion:
  \[
  d\theta_t = \sigma \theta_t dW_t
  \]
Allocation and Contract

• buyer has flow unit demand for object

\[ \theta_t x_t - p_t \]

• at every \( t \) contract prescribes contingent on sequence of past and current reports:

1. a flow allocation \( x_t \in [0, 1] \),
2. a flow payment \( p_t \in \mathbb{R} \),

• contract can only start after arrival:

\[ p_t = x_t = 0, \text{ for all } t < \alpha \]

• principal can commit to any direct dynamic mechanism
Payoffs

• positive common discount rate \( r > 0 \)
• expected utility of agent is:

\[
\mathbb{E} \left[ \int_0^\infty e^{-(r+\gamma)(t-\alpha)} (\theta_t x_t - p_t) \, dt \right]
\]

• expected profit of principal is:

\[
\mathbb{E} \left[ \int_0^\infty e^{-r(t-\alpha)} (p_t - c(x_t)) \, dt \right]
\]

• constant marginal cost \( c(x) = cx \), normalization: \( c = 0 \)
Stationary Mechanism

- find revenue maximizing stationary mechanism
- stationary mechanism requires that the same menu of dynamic allocations is offered in every period
- each item of the menu of dynamic allocations defines a sequence of report-contingent allocations
- each sequence can be contingent in arbitrary way on past and present report
- in particular, none of the allocation sequence has to be stationary
- not today: when is the optimal stationary mechanism the optimal mechanism (allowing for time dependent mechanisms)
- stationary mechanism is optimal under some conditions
First Steps: Observable Arrival
Observable Arrival

- suppose arrival time is observable
- suppose seller can make a single, take-it-or-leave-it, offer
- classic dynamic revenue maximization problem
- Pavan, Segal and Toikka (2014) in discrete time
- Bergemann and Strack (2015) in continuous time
• revenue maximizing allocation in period $t$ is determined by dynamic version of virtual utility:

$$J(\theta_t) \triangleq \theta_t - \frac{1 - F(\theta_0)}{f(\theta_0)} \frac{d\theta_t}{d\theta_0}$$

• virtual utility in period $t$ is determined by:

1. information rent in period 0:

$$\frac{1 - F(\theta_0)}{f(\theta_0)}$$

2. stochastic flow in period $t$:

$$\frac{d\theta_t}{d\theta_0}$$

impact of initial state on current state (impulse response in discrete setting)
• we consider geometric Brownian motion:

\[ \theta_t = \theta_0 \exp \left( -\frac{\sigma^2}{2} t + \sigma W_t \right) \]

• stochastic flow is then only state dependent:

\[ \frac{d\theta_t}{d\theta_0} = \frac{\theta_t}{\theta_0} \]

• dynamic version of virtual utility

\[ J(\theta_t) \triangleq \theta_t - \frac{1 - F(\theta_0)}{f(\theta_0)} \frac{\theta_t}{\theta_0} = \theta_t \left( 1 - \frac{1 - F(\theta_0)}{f(\theta_0)} \frac{1}{\theta_0} \right) \]
Optimal Allocation with Observable Arrival

- revenue maximizing allocation gives object to agent at any time $t \geq 0$ iff virtual utility is positive:
  \[
  J(\theta_t) = \theta_t \left( 1 - \frac{1 - F(\theta_0)}{f(\theta_0)} \frac{1}{\theta_0} \right) \geq 0 \iff \theta_0 - \frac{1 - F(\theta_0)}{f(\theta_0)} \geq 0
  \]

- optimal allocation depends only on initial value $\theta_0$ for all $t$:
  \[
  x_t = \begin{cases} 
  1 & \text{if } \theta_0 \geq \hat{\theta}, \\
  0 & \text{otherwise}, 
  \end{cases}
  \]
  where $\hat{\theta}$ solves:
  \[
  J(\theta_0) \triangleq \theta_0 - \frac{1 - F(\theta_0)}{f(\theta_0)} = 0
  \]
• optimal allocation via a simple sales contract:
• object is sold irrevocably at $\hat{P}$:

$$\hat{P} = \frac{\hat{\theta}}{r + \gamma} \Leftrightarrow \hat{p} = \hat{\theta} \text{ (flow price)}$$

• either gets object at $t = 0$ or priced out of market forever ...
• ... independently of how his value evolves over time
• indirect utility of the agent at $t = 0$:

$$V(\theta_0) = \max \left\{ 0, \theta_0 - \hat{\theta} \right\} \frac{\theta_0 - \hat{\theta}}{r + \gamma}.$$
• value function with observable arrival:

Figure 1: Value Function with Observable Arrival
Next Step: Sales Contract with Unobservable Arrival
Unobservable Arrival

- suppose sales contract is now made as standing offer, stationary offer with:
  \[ \hat{P} = \frac{\hat{\theta}}{r + \gamma} \]
- buyer with initial value \( \theta_0 \) lower than \( \hat{\theta} \) never gets the object, his net utility is zero
- but can now improve by only reporting his arrival when \( \theta_t \) reaches value \( w > \hat{\theta} \):
  \[ \tau_w \triangleq \inf\{ t \geq 0 : \theta_t \geq w \} \]
  and get a utility of
  \[ \mathbb{E} \left[ e^{-(r+\gamma)\tau_w} V(\theta_{\tau_w}) \mid \theta_0 \right] = \mathbb{E} \left[ e^{-(r+\gamma)\tau_w} \frac{w - \hat{\theta}}{r} \mid \theta_0 \right] > 0. \]


- agent solves the stopping problem

$$\sup_{\tau} \mathbb{E} \left[ e^{-(r+\gamma)\tau} (\theta_t - \hat{\rho}) \right]$$

- stopping problem as irreversible investment problem

- buyer claims object if value exceeds stationary threshold $\theta^*$

$$\theta^* \triangleq \frac{\beta}{\beta - 1} \hat{\rho}$$

where

$$\beta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r + \gamma)}{\sigma^2}} > 1$$
ability to delay his purchasing decision is “option value”

\[ E \left[ e^{-(r+\gamma)\tau_{\theta^*}} (\theta^* - p) \right] - \max \{ (\theta - p), 0 \} \]

sales price turns into expected future quantity

**Lemma**

*The expected discounted time \( \tau_w = \inf \{ t : \theta_t \geq w \} \) until the valuation reaches a value \( w \) given initial value \( \theta_0 \) is:*

\[ E \left[ e^{-(r+\gamma)\tau_w} \mid \theta_0 \right] = \min \left\{ \left( \frac{\theta_0}{w} \right)^{\beta}, 1 \right\}. \]
Price and Probability

- flow price $p$:
  \[
  \theta^* \triangleq \frac{\beta}{\beta - 1} p
  \]
  - controls discounted probability of flow sale:
  \[
  \min \left\{ \left( \frac{\theta_0}{w} \right)^\beta, 1 \right\} \Rightarrow \min \left\{ \left( \frac{\beta - 1}{\beta} \frac{\theta}{p} \right)^\beta, 1 \right\}
  \]
  - a higher price $p$ reduces the probability of sale

Proposition (Revenue in Sales Contract)

The revenue in a sales contract with price $p$ is given by:

\[
R_{sales}(p) = p \int_0^\infty \min \left\{ \left( \frac{\beta - 1}{\beta} \frac{\theta}{p} \right)^\beta, 1 \right\} f(\theta) \, d\theta.
\]
• value function with unobservable arrival:

Figure 2: Value Function with Observable vs Unobservable Arrival
• understand behavior of buyer faced with sales contract
• what is the optimal mechanism?
• is a sales contract still optimal with progressive participation?
Revenue Equivalence
An Auxiliary Static Problem

- in related static problem, buyer reports only initial valuation \( \theta_0 \) and seller chooses discounted expected quantity

\[ q : \Theta \rightarrow \mathbb{R}_+ \]

- in any incentive compatible mechanism, value of buyer and seller are only function of \( q \)

- define “expected aggregate quantity” by:

\[
q(\theta_0) \triangleq \mathbb{E} \left[ \int_0^\infty e^{-(r+\gamma)t} x_t \exp \left( -\frac{\sigma^2}{2} t + \sigma W_t \right) dt \mid \theta_0 \right]
\]

- first term is discounted quantity in period \( t \):

\[ e^{-(r+\gamma)t} x_t \]

- second term is stochastic flow:

\[
\frac{d\theta_t}{d\theta_0} = \exp \left( -\frac{\sigma^2}{2} t + \sigma W_t \right)
\]
Revenue Equivalence

- expected quantity $q$ and virtual utility $J$ allow us to completely summarize the objective function of buyer and seller:

**Proposition (Revenue Equivalence)**

*In any incentive compatible mechanism the value of the buyer is*

$$V(\theta) = \int_0^\theta q(z)dz + V(0)$$

*and the revenue of the seller is:*

$$\mathbb{E} \left[ \int_0^\infty e^{-(r+\gamma)t} p_t dt \right] = \int_0^\theta J(\theta)q(\theta)dF(\theta) - V(0).$$

- in particular, information rent:

$$V'(\theta) = q(\theta)$$
Necessary Condition for Optimal Mechanism

- monotonicity of allocation

**Proposition (Monotonicity of Discounted Quantity)**

In any incentive compatible mechanism, the function $q(\theta_0)$ is increasing in $\theta_0$.

- buyer must find it optimally to report his arrival immediately
- implies that there cannot be kinks in the value function as this would imply a first order gain for the agent waiting

In any incentive compatible mechanism, the function $q(\theta_0)$ is continuously differentiable and increasing.
Strengthen Necessary Conditions

- start with necessary conditions for truthful reporting of arrival:
  - considering a specific, small class of deviations
- find optimal mechanisms using tools from optimization theory
- verify that in candidate mechanism arrival is reported truthfully
Small Class of Deviations

- restricting buyer to small class of deviations in reporting arrival
- report arrival at first time value crosses stationary cut-off $w$:
  \[ \tau_w = \inf \{ t \geq 0 : \theta_t \geq w \} . \]
- payoff of using this deviation is given by:
  \[ \mathbb{E} \left[ e^{-r \tau_w} V(\theta_{\tau_w}) \mid \theta_0 \right] = V(w) \left( \frac{\theta_0}{w} \right)^\beta \]
- report of arrival times generates qualitatively different incentive constraints
- it is not just claiming to be a different type, it is actually being (becoming) a different type
Truthtelling of Arrival Time

- thus reporting arrival time truthfully requires:
  \[ V(\theta_0) \geq V(w) \left( \frac{\theta_0}{w} \right)^\beta \iff V(w)w^{-\beta} \leq V(\theta_0)\theta_0^{-\beta} \]

- thus the product \( V(w)w^{-\beta} \) has to be decreasing everywhere:
  \[ V'(\theta_0) \leq \beta \frac{V(\theta_0)}{\theta_0}, \]
by revenue equivalence theorem, indirect utility:

\[ V'(\theta_0) = q(\theta_0) \]

Proposition (Bound on Derivative)

The derivative of the agent’s value function is bounded above by

\[ q(\theta_0) = V'(\theta_0) \leq \beta \frac{V(\theta_0)}{\theta_0}, \]

in any mechanism where it is optimal to report arrivals truthfully.

- information rent cannot grow too fast
- can always be guaranteed by raising \( V(\theta_0) \) uniformly over \( \theta_0 \)
Relaxed Program: Lower Bound on Utility

- rewrite condition as lower bound on value of lowest type since
  \[ V(\theta) = V(0) + \int_0^\theta q(z)dz \]

**Proposition (Lower Bound on Utility)**

*In any incentive compatible mechanism we have:*

\[ V(0) \geq \max_\theta \left\{ \frac{\theta q(\theta)}{\beta} - \int_0^\theta q(z)dz \right\} \]

- participation constraint is determined by class of global deviations rather than local deviations
- information rent does not stem/refer from stochastic flow/virtual utility
Relaxed Program: Upper Bound on Revenue

- using revenue equivalence theorem
- turn lower bound on value bound into upper bound on revenue

Corollary (Upper Bound on Revenue)

An upper bound on revenue in any incentive compatible mechanism:

$$\int_{\Theta} q(z) J(z) dF(z) - \max_\theta \left\{ \frac{\theta q(\theta)}{\beta} - \int_0^\theta q(z) dz \right\}.$$

- additive but non-local optimality condition
- rewrite in value rather than allocation terms
Optimal Mechanism
Attaining Upper Bound

- find indirect utility which maximizes upper bound of revenue
- restate as an optimal control problem:

\[
\max_V \int_{\theta}^{\bar{\theta}} V'(z) J(z) f(z) \, dz - V(0)
\]

subject to

\[V'(\theta) \in \left[0, \frac{1}{r + \gamma}\right] \text{ for all } \theta,\]

\[V \text{ is convex,}\]

\[V'(\theta) \leq \beta \frac{V(\theta)}{\theta} \text{ for all } \theta.\]

- \(V'(\theta)\) is control variable, \(V(\theta)\) is state variable
- \(J\) is weakly increasing, but \(V'(\theta)\) cannot grow too fast
• we shall ignore convexity (monotonicity) constraint $V''(\theta)$
• focus on limit of growth of information rent:

$$q(\theta) = V'(\theta) \leq \beta \frac{V(\theta)}{\theta}$$

• opposite to ironing procedure where growth is bounded below (weakly increasing)
• it represents a mixed control-state constraint
• special features we use: (i) objective depends only on control but not state variable; (ii) control enters multiplicatively
Comparison Principle

- assert specific property of solution of differential equation if auxiliary equation has a certain property
- an central comparison result is Gronwall’s inequality:

**Lemma (Gronwall’s Inequality)**

Let \( u \) and \( \beta \) be continuous functions. If \( u \) is differentiable and satisfies differential inequality: 
\[
u'(t) \leq \beta(t) u(t),
\]
then \( u \) is bounded by solution of corresponding differential equation: 
\[
v'(t) = \beta(t) v(t),
\]
thus:
\[
u(t) \leq u(a) \exp\left(\int_a^t \beta(s) \, ds\right).
\]

- bound a function that is known to satisfy a certain differential inequality by solution of corresponding differential equation
Using Comparison Principle

Proposition (A Specific Control Problem)

Let $\Phi : \mathbb{R} \times [0, \overline{\theta}] \rightarrow \mathbb{R}_+$ be increasing and uniformly Lipschitz continuous. Let $\mathcal{J} : [0, \overline{\theta}] \rightarrow \mathbb{R}$ be continuous, satisfy $\mathcal{J}(\theta) = -1$ and $z \mapsto \min\{\mathcal{J}(z), 0\}$ be non-decreasing. Consider:

$$\max_w \left\{ \int_0^{\overline{\theta}} \mathcal{J}(\theta)w'(\theta)\,d\theta - w(0) \right\}.$$

over all absolutely continuous functions $w : [0, \overline{\theta}] \rightarrow \mathbb{R}_+$ that satisfy $w'(\theta) \leq \Phi(w(\theta), \theta)$. There exists $\hat{\theta} \in [0, \overline{\theta}]$ and

$$w(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \hat{\theta}], \\ \Phi(w(\theta), \theta), & \text{if } \theta \in (\hat{\theta}, \overline{\theta}]. \end{cases}$$
• offer a characterization of relaxed optimal mechanism

Proposition (Optimal Control)

There exists $\theta'$ such that a solution to the control problem is:

$$V(\theta) = \begin{cases} 
(\frac{\theta}{\theta'})^\beta \frac{\theta'/\beta}{\gamma+r}, & \text{for } \theta \leq \theta', \\
\frac{\theta'/\beta}{\gamma+r} + \frac{\theta-\theta'}{\gamma+r}, & \text{for } \theta' \leq \theta,
\end{cases}$$

and satisfies for all $\theta \in [0, \bar{\theta}]$:

$$V'(\theta) = \frac{1}{r + \gamma} \min \left\{ \left( \frac{\theta}{\theta'} \right)^{\beta-1}, 1 \right\}.$$
Proposition (Optimal Control)

There exists $\theta^* \geq 0$ such that the quantity $q^*$ that maximizes revenue is given by

$$q^*(\theta) = \min \left\{ \beta \left( \frac{\theta}{\theta^*} \right)^{\beta - 1}, 1 \right\}.$$ 

- threshold $\theta^*$ depends on prior distribution, but shape of $q^*(\theta)$ is fixed by geometric Brownian and patience alone
Implementation and Welfare
Indirect Implementation

• simple indirect implementation of optimal mechanism

Proposition (Sales Price)

The allocation \( q^* \) is implemented by selling the agent the object forever at a flow price of

\[
p^* = \theta^* \left[ \frac{\beta - 1}{\beta} \right].
\]

• consider uniform distribution of initial values, then

\[
p^* = \frac{\beta - 1}{\beta} \theta^* < \frac{1}{2} < \frac{1}{2} \frac{1 + \beta}{\beta} = \theta^*
\]

• by contrast–with observable arrival:

\[
p = \frac{1}{2} = \theta
\]
Impact of Unobservable Arrival

- how does unobservability of arrival impact optimal prices?

**Figure 3**: Progressive threshold (red), dynamic threshold and price (black), and progressive price (blue)
Progressive Participation

- how does the probability of consumption change over time?

**Figure 4:** Consumption probability over time, progressive (orange), dynamic (blue)
Progressive vs Dynamic Mechanism

- aggregate discounted quantity
- steeper curves with larger $\beta = (3/2, 3, 6)$

**Figure 5:** Quantities assigned in progressive and dynamic mechanism
Welfare Implications

- comparing dynamic and progressive mechanism

**Proposition (Welfare Implications)**

1. *The sales prices is uniformly lower in the progressive mechanism.*
2. *The consumer surplus is uniformly larger in the progressive mechanism.*
3. *The social welfare is uniformly larger in the progressive mechanism than in short-term contracting.*

- social welfare is not necessarily larger in progressive mechanism than in dynamic mechanism when initial private information is negligible
Conclusion

- stationary contracts and progressive mechanism design
- argument used geometric Brownian motion for relaxed program
- many comparative static results to be explored
- open question: when is stationary contract optimal allowing for time dependent mechanisms