Progressive Participation

Dirk Bergemann and Philipp Strack

December 2019

Introduction

- dynamic trading environments:
  - agents are arriving and departing
  - agents have stochasticall changing values
- seller and buyer may benefit from dynamic mechanism that governs relationship
- two important constraints:

1. participation constraints: determines if and when privately informed agent agrees to enter into mechanism
2. incentive constraints: determines continued truth-telling regarding evolving private information

Incentive and Participation Constraint

- extensive analysis of the sequence of incentive constraints in the literature of dynamic mechanism design
- narrow analysis of participation constraint:
  single ex-ante participation constraint,
  agent either immediately accepts contract or is never offered a contract again
- implicit assumption: arrival time of buyer is public information
- explicit assumption today: arrival time of buyer is private information
- in consequence private information of buyer is two-dimensional:

1. stochastic value over time
2. stochastic arrival time

Classic Setting of Revenue Maximization

- pursue the implications of private information of arrival time and value for a classic problem in revenue maximization
- sell good or service to a buyer with unit demand repeatedly over time
- a few motivating examples to keep in mind:
    - lease, sales, service contracts
    - mobile phone contracts, club memberships
    - display advertising contracts

Standing Offer

- seller makes standing (or open) offer for possibly long-term contract
- stationary offer: the same contract/mechanism is offered in every future period
- rather than a take-it-now-or-never offer

$\rightarrow$ standing offer presents buyer with option

$\rightarrow$ chooses to enter contract now or at some later time

Progressive Participation

- buyer is timing participation time to his current value
- sometimes participates immediately, sometimes at future time

$\rightarrow$ progressive participation

$\rightarrow$ probability of participation is increasing over time

- option of participation time is source of new information rent

Progressive Mechanism

- symmetry across constraints:

1. interim (sequential) incentive constraints
2. interim (sequential) participation constraints

- information rent due to:

1. private value
2. private arrival time

Contrast with Dynamic Mechanism

- standard models of sequential screening:

1. unknown value
2. but known arrival time

- agent either accepts contract immediately or receives zero outside option
- ex-ante participation constraint at $t = 0$
- less rent extraction in progressive mechanism compared to dynamic mechanism

Model

Random Arrival and Random Value

- time is continuous $t \in [0, \infty)$
- buyer arrives and departs (is replaced) with rate $\gamma > 0$
- arrival time $\alpha \in [0, \infty)$ is private information of buyer
- value (willingness-to-pay) θ_t is private information of buyer
- initial value θ_α given by common prior F:

$$\theta_\alpha \in [0, \overline{\theta}] = \Theta \subset \mathbb{R}_+$$

- value θ_t follows geometric Brownian motion:

$$\mathrm{d}\theta_t = \sigma\, \theta_t \mathrm{d}W_t$$

Allocation and Contract

- buyer has flow unit demand for object

$$\theta_t x_t - p_t$$

- at every t contract prescribes contingent on sequence of past and current reports:

1. a flow allocation $x_t \in [0, 1]$,
2. a flow payment $p_t \in \mathbb{R}$,

- contract can only start after arrival:

$$p_t = x_t = 0, \text{ for all } t < \alpha$$

- principal can commit to any direct dynamic mechanism

Payoffs

- positive common discount rate $r > 0$
- expected utility of agent is:

$$\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)(t-\alpha)}(\theta_t x_t - p_t)\,\mathrm{d}t\right]$$

- expected profit of principal is:

$$\mathbb{E}\left[\int_0^\infty e^{-r(t-\alpha)}(p_t - c(x_t))\,\mathrm{d}t\right]$$

- constant marginal cost $c(x) = cx$, normalization: $c = 0$

Stationary Mechanism

- find revenue maximizing stationary mechanism
- stationary mechanism requires that the same menu of dynamic allocations is offered in every period
- each item of the menu of dynamic allocations defines a sequence of report-contigent allocations
- each sequence can be contingent in arbitrary way on past and present report
- in particular, none of the allocation sequence has to be stationary
- not today: when is the optimal stationary mechanism the optimal mechanism (allowing for time dependent mechanisms)
- stationary mechanism is optimal under some conditions

First Steps: Observable Arrival

Observable Arrival

- suppose arrival time is observable
- suppose seller can make a single, take-it-or-leave-it, offer
- classic dynamic revenue maximization problem
- Pavan, Segal and Toikka (2014) in discrete time
- Bergemann and Strack (2015) in continuous time

Revenue Maximization with Observable Arrival

- revenue maximizing allocation in period t is determined by dynamic version of virtual utility:

$$J(\theta_t) \triangleq \theta_t - \frac{1 - F(\theta_0)}{f(\theta_0)} \frac{d\theta_t}{d\theta_0}$$

- virtual utility in period t is determined by:

1. information rent in period 0:

$$\frac{1 - F(\theta_0)}{f(\theta_0)}$$

2. stochastic flow in period t:

$$\frac{d\theta_t}{d\theta_0}$$

impact of initial state on current state
(impulse response in discrete setting)

Stochastic Flow

- we consider geometric Brownian motion:

$$\theta_t = \theta_0 \exp\left(-\frac{\sigma^2}{2} t + \sigma W_t\right)$$

- stochastic flow is then only state dependent:

$$\frac{d\theta_t}{d\theta_0} = \frac{\theta_t}{\theta_0}$$

- dynamic version of virtual utility

$$J(\theta_t) \triangleq \theta_t - \frac{1 - F(\theta_0)}{f(\theta_0)} \frac{\theta_t}{\theta_0} = \theta_t \left(1 - \frac{1 - F(\theta_0)}{f(\theta_0)} \frac{1}{\theta_0}\right)$$

Optimal Allocation with Observable Arrival

- revenue maximizing allocation gives object to agent at any time $t \geq 0$ iff virtual utility is positive:

$$J(\theta_t) = \theta_t \left(1 - \frac{1 - F(\theta_0)}{f(\theta_0)} \frac{1}{\theta_0}\right) \geq 0 \Leftrightarrow \theta_0 - \frac{1 - F(\theta_0)}{f(\theta_0)} \geq 0$$

- optimal allocation depends only on initial value θ_0 for all t :

$$x_t = \begin{cases} 1 & \text{if } \theta_0 \geq \widehat{\theta}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\widehat{\theta}$ solves:

$$J(\theta_0) \triangleq \theta_0 - \frac{1 - F(\theta_0)}{f(\theta_0)} = 0$$

Sales Contract

- optimal allocation via a simple sales contract:
- object is sold irrevocably at $\widehat{P}$:

$$\widehat{P} = \frac{\widehat{\theta}}{r + \gamma} \Leftrightarrow \widehat{p} = \widehat{\theta} \text{ (flow price)}$$

- either gets object at $t = 0$ or priced out of market forever ...
- ... independently of how his value evolves over time
- indirect utility of the agent at $t = 0$:

$$V(\theta_0) = \max \left\{ 0, \frac{\theta_0 - \widehat{\theta}}{r + \gamma} \right\}.$$

Value Function with Observable Arrival

- value function with observable arrival:

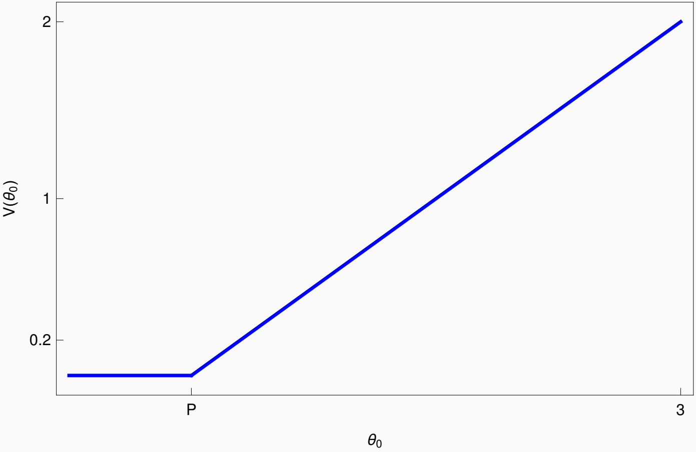

Figure 1: Value Function with Observable Arrival

Next Step: Sales Contract with Unobservable Arrival

Unobservable Arrival

- suppose sales contract is now made as standing offer, stationary offer with:
$$\widehat{P} = \frac{\widehat{\theta}}{r+\gamma}$$
- buyer with initial value θ_0 lower than $\widehat{\theta}$ never gets the object, his net utility is zero
- but can now improve by only reporting his arrival when θ_t reaches value $w > \widehat{\theta}$:
$$\tau_w \triangleq \inf\{t \geq 0 \colon \theta_t \geq w\}$$
and get a utility of
$$\mathbb{E}\left[e^{-(r+\gamma)\,\tau_w} V(\theta_{\tau_w}) \mid \theta_0\right] = \mathbb{E}\left[e^{-(r+\gamma)\,\tau_w} \frac{w-\widehat{\theta}}{r} \mid \theta_0\right] > 0.$$

Option Value

- agent solves the stopping problem

$$\sup_{\tau} \mathbb{E}\left[e^{-(r+\gamma)\tau}\,(\theta_t - \widehat{p})\right]$$

- stopping problem as irreversible investment problem
- buyer claims object if value exceeds stationary threshold $\theta^\star$

$$\theta^\star \triangleq \frac{\beta}{\beta - 1}\widehat{p}$$

where

$$\beta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\gamma)}{\sigma^2}} > 1$$

Option Value and Waiting Time

- ability to delay his purchasing decision is "option value"

$$\mathbb{E}\left[e^{-(r+\gamma)\tau_{\theta^\star}}(\theta^\star - p)\right] - \max\{(\theta - p), 0\}$$

- sales price turns into expected future quantity

Lemma

The expected discounted time $\tau_w = \inf\{t\colon \theta_t \geq w\}$ until the valuation reaches a value w given initial value θ_0 is:

$$\mathbb{E}\left[e^{-(r+\gamma)\tau_w} \mid \theta_0\right] = \min\left\{\left(\frac{\theta_0}{w}\right)^\beta, 1\right\}.$$

Price and Probability

- flow price p:

$$\theta^\star \triangleq \frac{\beta}{\beta - 1} p$$

- controls discounted probability of flow sale:

$$\min\left\{\left(\frac{\theta_0}{w}\right)^\beta, 1\right\} \Rightarrow \min\left\{\left(\frac{\beta - 1}{\beta}\,\frac{\theta}{p}\right)^\beta, 1\right\}$$

- a higher price p reduces the probability of sale

Proposition (Revenue in Sales Contract)

The revenue in a sales contract with price p is given by:

$$R_{sales}(p) = p \int_0^\infty \min\left\{\left(\frac{\beta - 1}{\beta}\,\frac{\theta}{p}\right)^\beta, 1\right\} f(\theta)\, \mathrm{d}\theta\,.$$

Unobservable Arrival

- value function with unobservable arrival:

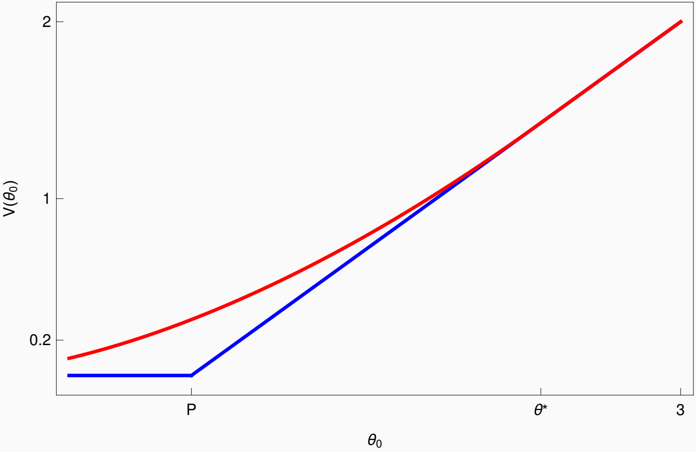

Figure 2: Value Function with Observable vs Unobservable Arrival

Open Questions

- understand behavior of buyer faced with sales contract
- what is the optimal mechanism?
- is a sales contract still optimal with progressive participation?

Revenue Equivalence

An Auxiliary Static Problem

- in related static problem, buyer reports only initial valuation θ_0 and seller chooses discounted expected quantity

$$q : \Theta \to \mathbb{R}_+$$

- in any incentive compatible mechanism, value of buyer and seller are only function of q
- define "expected aggregate quantity" by:

$$q(\theta_0) \triangleq \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} x_t \exp\left(-\frac{\sigma^2}{2} t + \sigma\, W_t\right) \mathrm{d}t \mid \theta_0\right]$$

- first term is discounted quantity in period t:

$$e^{-(r+\gamma)\,t} x_t$$

- second term is stochastic flow:

$$\frac{d\theta_t}{d\theta_0} = \exp\left(-\frac{\sigma^2}{2} t + \sigma\, W_t\right)$$

Revenue Equivalence

- expected quantity q and virtual utility J allow us to completely summarize the objective function of buyer and seller:

Proposition (Revenue Equivalence)

In any incentive compatible mechanism the value of the buyer is

$$V(\theta) = \int_0^\theta q(z)\mathrm{d}z + V(0)$$

and the revenue of the seller is:

$$\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} p_t \mathrm{d}t\right] = \int_0^{\overline{\theta}} J(\theta)q(\theta)\mathrm{d}F(\theta) - V(0).$$

- in particular, information rent:

$$V'(\theta) = q(\theta)$$

Necessary Condition for Optimal Mechanism

- monotonicity of allocation

Proposition (Monotonicity of Discounted Quantity)

In any incentive compatible mechanism, the function $q(\theta_0)$ is increasing in θ_0.

- *buyer must find it optimally to report his arrival immediately*
- *implies that there cannot be kinks in the value function as this would imply a first order gain for the agent waiting*

In any incentive compatible mechanism, the function $q(\theta_0)$ is continuously differentiable and increasing.

A Bound for Optimal Revenue

Strengthen Necessary Conditions

- start with necessary conditions for truthful reporting of arrival:

$\rightarrow$ considering a specific, small class of deviations

- find optimal mechanisms using tools from optimization theory
- verify that in candidate mechanism arrival is reported truthfully

Small Class of Deviations

- restricting buyer to small class of deviations in reporting arrival
- report arrival at first time value crosses stationary cut-off w :

$$\tau_w = \inf\{t \geq 0 \colon \theta_t \geq w\}.$$

- payoff of using this deviation is given by:

$$\mathbb{E}\left[e^{-r\,\tau_w} V(\theta_{\tau_w}) \mid \theta_0\right] = V(w)\left(\frac{\theta_0}{w}\right)^{\beta}$$

- report of arrival times generates qualitatively different incentive constraints
- it is not just claiming to be a different type, it is actually being (becoming) a different type

Truthtelling of Arrival Time

- thus reporting arrival time truthfully requires:

$$V(\theta_0) \geq V(w) \left(\frac{\theta_0}{w}\right)^{\beta} \Leftrightarrow V(w)w^{-\beta} \leq V(\theta_0)\theta_0^{-\beta}$$

- thus the product $V(w)w^{-\beta}$ has to be decreasing everywhere:

$$V'(\theta_0) \leq \beta \frac{V(\theta_0)}{\theta_0},$$

Relaxed Program: Bound on Derivative

- by revenue equivalence theorem, indirect utility:

$$V'(\theta_0) = q(\theta_0)$$

Proposition (Bound on Derivative)

The derivative of the agent's value function is bounded above by

$$q(\theta_0) = V'(\theta_0) \leq \beta \frac{V(\theta_0)}{\theta_0},$$

in any mechanism where it is optimal to report arrivals truthfully.

- information rent cannot grow too fast
- can always be guaranteed by raising $V(\theta_0)$ uniformly over θ_0

Relaxed Program: Lower Bound on Utility

- rewrite condition as lower bound on value of lowest type since

$$V(\theta) = V(0) + \int_0^\theta q(z)\mathrm{d}z$$

Proposition (Lower Bound on Utility)

In any incentive compatible mechanism we have:

$$V(0) \geq \max_\theta \left\{ \theta q(\theta)/\beta - \int_0^\theta q(z)\mathrm{d}z \right\}$$

- participation constraint is determined by class of global deviations rather than local deviations
- information rent does not stem/refer from stochastic flow/virtual utility

Relaxed Program: Upper Bound on Revenue

- using revenue equivalence theorem
- turn lower bound on value bound into upper bound on revenue

Corollary (Upper Bound on Revenue)

An upper bound on revenue in any incentive compatible mechanism:

$$\int_{\Theta} q(z)J(z)\mathrm{d}F(z) - \max_{\theta}\left\{\theta q(\theta)/\beta - \int_{0}^{\theta} q(z)\mathrm{d}z\right\}.$$

- additive but non-local optimality condition
- rewrite in value rather than allocation terms

Optimal Mechanism

Attaining Upper Bound

- find indirect utility which maximizes upper bound of revenue
- restate as an optimal control problem:

$$\max_V \int_{\underline{\theta}}^{\overline{\theta}} V'(z)J(z)\,f(z)\,\mathrm{d}z - V(0)$$

subject to

$$V'(\theta) \in \left[0, \frac{1}{r+\gamma}\right] \text{ for all } \theta,$$

$$V \text{ is convex},$$

$$V'(\theta) \leq \beta \frac{V(\theta)}{\theta} \text{ for all } \theta\,.$$

- $V'(\theta)$ is control variable, $V(\theta)$ is state variable
- J is weakly increasing, but $V'(\theta)$ cannot grow to fast

Second Relaxation

- we shall ignore convexity (monotonicity) constraint $V''(\theta)$
- focus on limit of growth of information rent:

$$q(\theta) = V'(\theta) \leq \beta \frac{V(\theta)}{\theta}$$

- opposite to ironing procedure where growth is bounded below (weakly increasing)
- it represents a mixed control-state constraint
- special features we use: (i) objective depends only on control but not state variable; (ii) control enters multiplicatively

Comparison Principle

- assert specific property of solution of differential equation if auxiliary equation has a certain property
- an central comparison result is Gronwall's inequality:

Lemma (Gronwall's Inequality)

Let u and β be continuous functions. If u is differentiable and satisfies differential inequality: $u'(t) \leq \beta(t)\, u(t)$, then u is bounded by solution of corresponding differential equation: $v'(t) = \beta(t)\, v(t)$, thus:

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)\, \mathrm{d}s\right).$$

- bound a function that is known to satisfy a certain differential inequality by solution of corresponding differential equation

Using Comparison Principle

Proposition (A Specific Control Problem)

Let $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}_+$ *be increasing and uniformly Lipschitz continuous. Let* $\mathcal{J} : [0, \overline{\theta}] \to \mathbb{R}$ *be continuous, satisfy* $\mathcal{J}(\underline{\theta}) = -1$ *and* $z \mapsto \min\{\mathcal{J}(z), 0\}$ *be non-decreasing. Consider:*

$$\max_{w} \left\{ \int_0^{\overline{\theta}} \mathcal{J}(\theta) w'(\theta) \mathrm{d}\theta - w(0) \right\} .$$

over all absolutely continuous functions $w : [0, \overline{\theta}] \to \mathbb{R}_+$ *that satisfy* $w'(\theta) \leq \Phi(w(\theta), \theta)$. *There exists* $\hat{\theta} \in [0, \overline{\theta}]$*and*

$$w(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \hat{\theta}], \\ \Phi(w(\theta), \theta), & \text{if } \theta \in (\hat{\theta}, \overline{\theta}]. \end{cases}$$

Optimal Utility

- offer a characterization of relaxed optimal mechanism

Proposition (Optimal Control)

There exists θ' such that a solution to the control problem is:

$$V(\theta) = \begin{cases} \left(\frac{\theta}{\theta'}\right)^{\beta} \frac{\theta'/\beta}{\gamma+r}, & \text{for } \theta \leq \theta', \\ \frac{\theta'/\beta}{\gamma+r} + \frac{\theta-\theta'}{\gamma+r}, & \text{for } \theta' \leq \theta, \end{cases}.$$

and satisfies for all $\theta \in [0, \overline{\theta}]$:

$$V'(\theta) = \frac{1}{r+\gamma} \min\left\{ \left(\frac{\theta}{\theta'}\right)^{\beta-1}, 1 \right\}.$$

Optimal Allocation

Proposition (Optimal Control)

There exists $\theta^\star \geq 0$ such that the quantity $q^\star$ that maximizes revenue is given by

$$q^\star(\theta) = \min\left\{\beta\left(\frac{\theta}{\theta^\star}\right)^{\beta-1}, 1\right\}.$$

- threshold $\theta^\star$ depends on prior distribution, but shape of $q^\star(\theta)$ is fixed by geometric Brownian and patience alone

Implementation and Welfare

Indirect Implementation

- simple indirect implementation of optimal mechanism

Proposition (Sales Price)

The allocation $q^\star$ is implemented by selling the agent the object forever at a flow price of

$$p^\star = \theta^\star \left[\frac{\beta - 1}{\beta}\right].$$

- consider uniform distribution of initial values, then

$$p^\star = \frac{\beta - 1}{\beta}\theta^\star < \frac{1}{2} < \frac{1}{2}\frac{1+\beta}{\beta} = \theta^\star$$

- by contrast–with observable arrival:

$$p = \frac{1}{2} = \theta$$

Impact of Unobservable Arrival

- how does unobservability of arrival impact optimal prices?

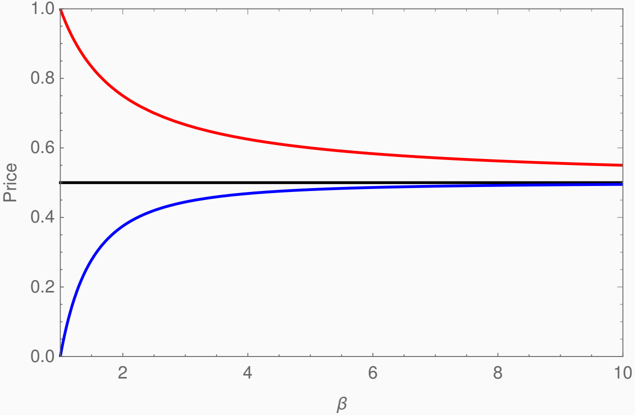

Figure 3: Progressive threshold (red), dynamic threshold and price (black), and progressive price (blue)

Progressive Participation

- how does the probability of consumption change over time?

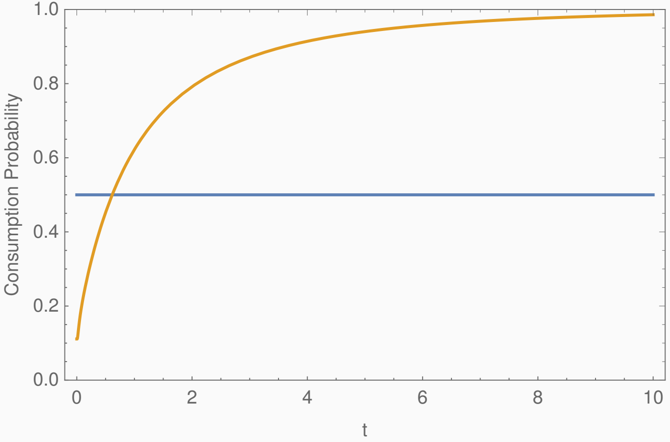

Figure 4: Consumption probabibility over time, progressive (orange), dynamic (blue)

Progressive vs Dynamic Mechanism

- aggregate discounted quantity
- steeper curves with larger $\beta = (3/2, 3, 6)$

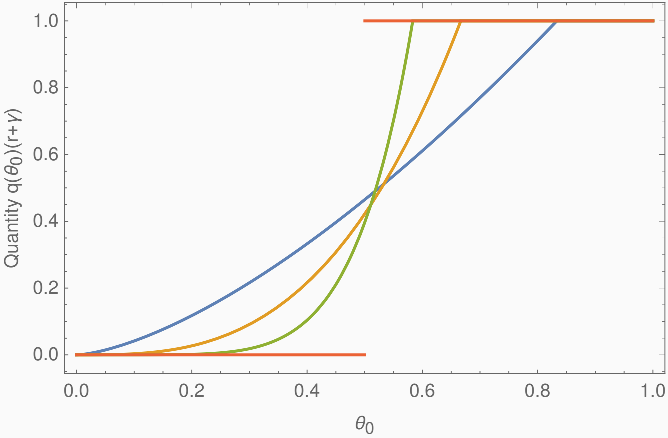

Figure 5: Quantities assigned in progressive and dynamic mechanism

Welfare Implications

- comparing dynamic and progressive mechanism

Proposition (Welfare Implications)

1. *The sales prices is uniformly lower in the progressive mechanism.*
2. *The consumer surplus is uniformly larger in the progressive mechanism.*
3. *The social welfare is uniformly larger in the progressive mechanism than in short-term contracting.*

- social welfare is not necessarily larger in progressive mechanism than in dynamic mechanism when initial private information is negligible

Conclusion

- stationary contracts and progressive mechanism design
- argument used geometric Brownian motion for relaxed program
- many comparative static results to be explored
- open question: when is stationary contract optimal allowing for time dependent mechanisms