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Class

> The notion of class is so fundamental to thought that we cannot hope to define it in more fundamental terms.
>
> W.V.O. Quine
> *Set Theory and Its Logic*

The above passage typifies the attitude most set theorists take towards their subject. From the point of view of the theory of properties, relations, and propositions, however, this attitude has two flaws. First, the notion of class is not fundamental to thought. And secondly, insofar as the notion of class is useful in mathematics and empirical science, it can be defined in more fundamental terms, namely, in terms of the predication relation. These are striking criticisms not likely to be accepted without support. The purpose of the present chapter is to provide that support.

What justifies the ontology of sets? In chapter 1, I argued that the theory of PRPs is part of logic. Since logic always belongs to the best comprehensive theory of the world, the ontology of PRPs is justified. In view of the youthfulness of set theory, however, it would be unwise to assume that the same is true for sets. We should entertain the hypothesis that set theory is the result of conflating certain constructions that, although they do play a role in the logic of natural language, do not play the role that set theory presumes. I am inclined to this hypothesis and, indeed, to the proposition that there is simply no sound justification for the ontology of sets. How might the set theorist attempt to justify his ontology? There are three strategies open to him. The first is to show that sets are included in what might be called our *naturalistic ontology*. If they are, then we may assume that whatever justifies our naturalistic ontology also justifies the ontology of sets. The second strategy is to show that, like the theory of PRPs, set theory is part of logic. In this case, the ontology of sets would be justified in the same way as the ontology of PRPs. And the third strategy is to show that set theory plays some unique role in mathematics or in empirical science. If it does, then its ontology would be justified pragmatically. None of these strategies is successful, however, as I will now explain.

27. The Unnaturalness of Sets

Paul Halmos begins his popular book *Naive Set Theory* with this observation:

> A pack of wolves, a bunch of grapes, or a flock of pigeons are all examples of sets of things.

Perhaps it is true that the idea of a set is somehow "genetically" related to ideas of such naturalistic objects as packs, bunches, and flocks.[1] Nevertheless, it is certain that sets are not the same sort of thing as packs, bunches, flocks, etc. Here are a few of the many reasons. First, packs, bunches, flocks, tribes, and so on, displace volumes, have mass, and come into and pass out of existence. Sets, by contrast, are non-physical and eternal. Secondly, sets cannot change their members; packs, bunches, flocks, etc. can. If a wolf in* a given pack dies (or gives birth), the pack is still the same pack. But the set of wolves-before-the-death (birth) is not the same set as the set of wolves-after-the-death (birth). Thus, a set of wolves and a pack of wolves are different. Thirdly, packs, bunches, flocks, etc. do not exist if nothing is in them; this is not so for sets. If there were no wolves, there would be no packs of wolves. But the set of wolves would exist nonetheless, for it would just be the null set. Indeed, if sets exist, the null set is a set that exists necessarily.[2]

If sets are not the same sort of thing as packs, bunches, flocks, etc., what are they? It is now commonplace to say that sets are collections or classes. What is meant by this? Art collections, social classes, sets of dishes: is it true that these are cases of the kind of sets posited in set theory? No, definitely not. They are no more the kind of sets posited in set theory than are packs, bunches, and flocks, etc., and for much the same reasons. First, art collections and sets of dishes can displace volumes, have mass, and come into and pass out of existence. And social classes, although they seem not to displace volumes or have mass, can come into and pass out of existence. Secondly, art collections, social classes, sets of dishes,

* Note that in order not to bias the discussion I will use the natural and, I hope, neutral locution 'is in' (and its cognates) rather than the technical locution '$\in$'. A moment's reflection will show that in adopting this practice I do not commit any fallacies of equivocation.

etc. can change their members.[3] Thirdly, ordinary collections, social classes, and ordinary sets do not exist if nothing is in them. (If China has no aristocrats, it has no aristocracy.)

These three differences suffice to show that ordinary collections, social classes, and ordinary sets are different from set-theoretical sets. However, there might be a fourth difference, having special philosophical interest. This difference concerns a transitivity property. Consider a billionaire who collects art collections in the style in which Howard Hughes used to collect companies. This man purchases outright entire art collections. Now if his, say, ten art collections contain one Cezanne each, then we would say that there are ten Cezannes in his collection of art collections. And in general, if a painting is in an art collection that is itself in a collection of art collections, then we would say that the painting itself is in the collection of art collections. The sets of set theory are not like this at all. No individual paintings are in the set of art collections; only art collections are.[4] Thus, the set of art collections and the collection of art collections are different. This sort of difference also seems to hold between set-theoretical sets, on the one hand, and social classes and ordinary sets, on the other. For example, if Jones is in the intelligentsia and the intelligentsia is in the upper class, then we would say that Jones is in the upper class. Or if a saucer is in a matched cup-and-saucer set that is itself in a set of eight matched cup-and-saucer sets, then we would say that the saucer is in the set of cup-and-saucer sets. And we say that there are four socks in a pair of pairs of socks. None of these things hold for the set-theoretical counterparts—the set of upper classes, the set of sets containing a cup and a matching saucer, the set consisting of $\{\text{sock}_1, \text{sock}_2\}$ and $\{\text{sock}_3, \text{sock}_4\}$. Put formally, the difference here is that ordinary collections, social classes, and ordinary sets seem transitive whereas the sets of set theory typically are not. That is, the following transitivity principle seems to be valid for ordinary collections, social classes, and ordinary sets whereas it is not valid for the sets posited in set theory:

$$x \text{ is in } y \supset (\forall z)(z \text{ is in } x \supset z \text{ is in } y).$$

This transitivity principle is equivalent to the following: $(\exists x)(z \text{ is in } x \,\&\, x \text{ is in } y) \supset z \text{ is in } y$. Ordinary collections, social classes, and ordinary sets would thus seem to be closed under a union operation.

There might be yet another difference between the sets of set theory and ordinary collections, social classes, and ordinary sets, one which concerns a corresponding power operation. Consider an example. If the individual cup and individual saucer in a matched cup-and-saucer set are themselves in a full set of dishes, then we say that the matched cup-and-saucer set itself is in the set of dishes. The sets posited by set theory are not like this. The set-theoretical set of dishes contains only individual dishes, not cup-and-saucer sets. For another example, suppose that I have a collection of famous rare stamps known as the First Issue Collection. This collection contains rare stamps from a wide variety of countries. Suppose further that I have a particularly valuable collection of rare Dutch stamps. Now if every stamp in my collection of Dutch stamps is in the First Issue Collection, we would say that my Dutch stamp collection is in the First Issue Collection. But the set-theoretical set of stamps that I own contains only stamps; it does not contain, e.g., the set of Dutch stamps that I own. To put this formally, the following power principle might be valid for ordinary collections, social classes, and ordinary sets whereas it is not valid for the kind of sets posited in set theory:

$$(\forall z)(z \text{ is in } x \supset z \text{ is in } y) \supset x \text{ is in } y$$

for all x and y, where $x \neq y$.

It is commonplace among historians of logic and mathematics to remark that it was not until well into the nineteenth century that people became clear about the significant difference between membership and inclusion. However, given the above principles of transitivity and power, it follows that for ordinary collections, social classes, and ordinary sets these relations are virtually equivalent; i.e., for ordinary collections, social classes, and ordinary sets x and y:

$$x \text{ is in } y \equiv (\forall z)(z \text{ is in } x \supset z \text{ is in } y)$$

where $x \neq y$. In addition, this principle would seem to hold for at least certain ordinary collections and ordinary sets x and y, where $x = y$. In view of this, it might be more accurate to say that it was not until well into the 19th century that people became confused about the nature of membership and inclusion relations. For it was not until the set theorists' distinction was thought up that the commitment to the new, extraordinary kind of collection was made official. For that matter, it was not until the set theorists' distinction was thought up that it became possible to generate the paradoxes of naive set theory. Without the set-theoretical distinction between membership and inclusion there would be no set-theoretical paradoxes. What I mean by this will become clearer below.

By abstracting from the intuitive notions of ordinary collections, social classes, and ordinary sets as characterized in the foregoing discussion, one arrives at the general notion of what I will call an *aggregate*. Aggregates are like sets in that whenever a thing w satisfies a formula A, w is in the set of As, and w is in the aggregate of As. That is, the following schemas hold for sets and aggregates, respectively:

(a) $A(w) \supset w$ is in the set of things y such that $A(y)$
(a′) $A(w) \supset w$ is in the aggregate of things y such that $A(y)$.

Furthermore, whenever something is in the set of As it also satisfies the formula A. That is, the following converse of (a) holds for sets:

(b) w is in the set of things y such that $A(y) \supset A(w)$.

Here aggregates part company with sets, however. Recall that membership and inclusion are virtually equivalent for aggregates. Thus, if a thing w is in the aggregate of As it does not follow that w satisfies A; w may instead be in something else that satisfies A, *or* something else that satisfies A could be in w, *or* something else that is in w could be in some third thing that satisfies A—any of these alternatives would do equally well. So, the schema for aggregates that corresponds to (b) offers several alternatives:

(b′) w is in the aggregate of things y such that $A(y) \supset (A(w)$ or
$(\exists u)(w$ is in u & $A(u))$ or
$(\exists u)(u$ is in w & $A(u))$ or
$(\exists u, v)(u$ is in w & u is in v & $A(v)))$.

Membership and inclusion are quite distinct in set theory, of course. So the consequent of (b) contains none of the supplementary alternatives that we had to add in (b′); whatever is in the set of As must satisfy A—there is no alternative. This dissimilarity of schemas (b′) and (b) is, of course, one more difference between ordinary aggregates and sets. But what is more important is that this feature of naive set theory is the very feature that renders it inconsistent. Schema (b) requires that all things in the set of things satisfying a given formula (e.g., the formula 'y is not in y') must themselves satisfy the formula, and this is what plunges naive set theory into contradictions. Because (b′) does not likewise restrict the identity of the things that are in ordinary aggregates, the theory of ordinary aggregates avoids the fate of naive set theory.[5]

The theory of aggregates is from a formal point of view rather like Leśniewski's mereology (i.e., the part/whole calculus);[6] each of the principles for aggregates also holds for mereological sums. In this, ordinary collections, social classes, and ordinary sets are far closer to mereological sums than to abstract sets. Set theory just does not get its motivation from the naturalistic ontology of ordinary collections, social classes, and ordinary sets.

The moral is that sets are not in evidence in any of the above naturalistic ontologies. Those who persist in the attempt to motivate the concept of class along such naturalistic lines sooner or later find themselves offering the "invisible-plastic-bag" conception. But this, I think, only confirms the point that sets do not fall within our naturalistic ontology.

28. No Basis in Logic

The second candidate strategy for justifying the ontology of sets is to attempt to show that set theory is grounded in logic. The most promising line is to look for evidence that set theory is embedded in the logical syntax of natural language. I can think of only one syntactic construction in natural language that might fill the bill, namely, pluralization.[7] Let us see how the set theorist might try to show that set theory has a special role to play in the treatment of plurals.

Consider the following sentences:

(1) The walnuts outweigh the pecans.
(2) The counties outnumber the states.

These sentences are not transformed universal conditionals:

(1′) $(\forall x, y)((\text{Walnut}(x) \ \& \ \text{Pecan}(y)) \supset \text{Outweigh}(x, y))$
(2′) $(\forall x, y)((\text{County}(x) \ \& \ \text{State}(y)) \supset \text{Outnumber}(x, y))$.

For whereas (1′) and (2′) are false, (1) and (2) are true.

Provisionally, then, let us represent (1) and (2) as 2-place relational sentences:

(1″) Outweigh (the walnuts, the pecans)
(2″) Outnumber (the counties, the states).

Here the plurals are provisionally treated as (defined or undefined) singular terms. It becomes appropriate, then, to ask what the primary semantical correlates of these provisional singular terms are. A natural hypothesis is that in (1) the primary semantical correlates of 'the walnuts' and 'the pecans' are aggregates of the ordinary sort characterized earlier (specifically, the aggregate of all walnuts and the aggregate of all pecans). On the face of it, this hypothesis seems successful. This gives rise to the presumption that the plurals in (2) should be treated analogously; i.e., this suggests that the primary semantical correlates of 'the counties' and 'the states' in (2) are also aggregates. But what kind of aggregate? Not ordinary aggregates, certainly. Since the ordinary aggregate of the counties is identical to the ordinary aggregate of the states, (2) would be false. Yet on its primary reading (2) is true. Therefore, if one continues to be swayed by the presumption that the primary semantical correlates of 'the counties' and 'the states' are aggregates, then a new, extraordinary kind of aggregate must be hypostasized. These new, extraordinary aggregates should differ from ordinary aggregates in at least the following respect: the things in the extraordinary aggregate of *F*s must be exactly those things that satisfy the predicate *F*. But this is precisely what is required of sets according to the abstraction principle of naive set theory (recall schemas (a) and (b) in §27). This gives rise to the further presumption that the extraordinary aggregate that is the primary semantical correlate of the plural 'the *F*s' in sentences akin to (2) is a set, specifically the set of *F*s.

Although the above line of reasoning has a certain appeal, it leads immediately to a fatal dilemma. Consider the following problematical sentences:

The walnuts *both* outweigh and outnumber the pecans.

Although the counties occupy exactly the same territory as the states, *they* outnumber the states, and, in addition, *they* resent federal intervention more than the states do.

These French stamps were once in the First Issue Collection;

however, after a while *they* outnumbered the Dutch stamps and, for that reason, *they* were moved to another collection.

The whales once outnumbered the human beings; now, however, *they* are nearly extinct.

In view of the earlier discussion about the nature of set-theoretical sets, if the plurals in these problematical sentences are treated in the same kind of naive surface-syntactical way adopted above in connection with sentence (2), then their primary semantical correlates clearly cannot be sets. (For example, the set of walnuts cannot outweigh the set of pecans since no set weighs anything.) These primary semantical correlates would have to be some further kind of entity. But in this case, uniformity requires us also to identify the primary semantical correlates of the plurals in (2) not with sets but with this further kind of entity. So if the plurals in the above problematical sentences get the naive surface-syntactical treatment that we provisionally gave to (2), then what initially seemed to be a justification for set theory in the natural logic of (2) evaporates. On the other hand, suppose the plurals in the above problematical sentences are treated in a sophisticated deep-structural way.[8] In this case, we nullify the original presumption that the plurals in (2) ought to be treated on analogy with the plurals in (1) (i.e., the presumption that the plurals in (2) are singular terms whose primary semantical correlates are some sort of aggregates). This makes (2) fair game for alternate sophisticated treatments; the various treatments of (2) must compete on their own terms. But if the contest is to take place in this stark arena, then, as I show next, set theory cannot win for itself a place in natural logic. Thus, either way, set theory fails to find motivation in the treatment of plurals in sentences like (2) in natural language.

To complete the above argument I must show that, if there is no presumption in favor of a set-theoretical treatment of sentences such as (2), then the set-theoretical treatment succumbs to superior competitors. So as not to bias the argument, let us agree to represent (2) provisionally along the following lines:

(2‴) $\text{Outnumber}(\{x: Cx\}, \{x: Sx\})$.

Here $\{x: Cx\}$ and $\{x: Sx\}$ are extensional abstracts; that is, they are (defined or undefined) abstract singular terms for which the following general law holds:

(3) $\{x: Ax\} = \{x: Bx\} \equiv (\forall x)(Ax \equiv Bx)$.[9]

Further, let us allow that for all non-paradox-producing formulas Ax in which y is free for x: (4) $y \in \{x: Ax\} \equiv Ay$. And finally, let us allow that (2‴) is true if and only if there is no 1-1 function from $\{x: Sx\}$ onto $\{x: Cx\}$ though there is a 1-1 function from $\{x: Sx\}$ into $\{x: Cx\}$. In this case (2‴) comes out true, as desired. Next consider briefly what seems to me to be the intuitive picture of the semantics for natural language. According to this picture, predicates and formulas do not refer to anything; they simply express.[10] A formula A, for example, expresses the property, relation, or proposition denoted by a certain associated gerundive phrase, infinitive phrase, or 'that'-clause formed from A. Specifically, it expresses the property, relation, or proposition denoted in L_ω by the normalized singular term $[A]_\alpha$. Now for all non-paradox-producing formulas A, the following law holds: (5) $\alpha \,\Delta\, [A]_\alpha \equiv A$. In view of this, the extensional abstract $\{v_i: A\}$ can be contextually defined in terms of the predication relation:

(6) $\ldots\{v_i: A\}\ldots$ *iff*$_{\text{df}}$ $(\exists v_j)((v_i \,\Delta\, v_j \equiv_{v_i} A) \,\&\, \ldots v_j \ldots)$

where v_j is a new distinct variable.[11] And $\in$ may be contextually defined as follows: (7) $u \in v$ *iff*$_{\text{df}}$ $u \,\Delta\, v$. To be convinced of the adequacy of these contextual definitions, notice that, for all non-paradox-producing formulas A and B, the above law (3) follows directly from (5) and (6), and law (4) follows directly from (5), (6), and (7). However, these laws are all that are needed for an adequate treatment of sentences such as (2‴).[12] Thus, extensional abstracts, and sentences such as (2‴), can be adequately treated within the logic for the predication relation, a theory already part of natural logic. And, this is accomplished without having to hypostasize the extraordinary aggregates of set theory. So if there is no presumption in favour of the set-theoretical treatment of sentences such as (2), then as far as natural logic is concerned the outlined alternative treatment wins hands down.

It might be objected that no economy follows from adopting this contextual treatment of extensional abstracts since sets have already entered the picture through an independent pathway, namely, through extensional semantics.[13] According to Frege's semantical theory, all meaningful expressions have two kinds of meaning: sense and nominatum. Frege identified the nominata of predicates (and open sentences) with what he called functions. But since at least the time of Tarski's work in extensional semantics, it has been common instead to view the nominatum of a predicate (open sentence) as a set, namely, the set of things that satisfy the predicate (open sentence). That is, on this view the nominatum of the predicate F is the set of Fs. Since extensional semantics already makes use of sets here, no economy of theory is gained (so someone might argue) by giving extensional abstracts such as $\{x: Fx\}$ the alternative treatment. In fact, for those persuaded by this set-theoretical semantical theory, it is only natural to identify the primary semantical correlate of the extensional abstract $\{x: Fx\}$—and thus that of the plural 'the Fs'—with the nominatum of the predicate F, i.e., with the set of Fs.

This objection, it seems to me, has gotten the proper order of the argument turned around. What good reason is there for accepting the extensional semantical theory? After all, the natural, intuitive picture of the semantics for predicates and formulas is Russell's, not Frege's. According to this picture predicates and formulas do not name anything; they simply express. The primary semantical correlate of predicates and formulas are just the properties, relations, and propositions expressed by them. What point, then, is there in having a Fregean two-kinds-of-meaning semantics rather than the simpler, more natural Russellian one-kind-of-meaning semantics? Surely something is gained at least theoretically? No, in fact, as I will show in §38, a Fregean theory provides no more semantical information than its simpler Russellian counterpart. So one can hardly justify a set-theoretical treatment of extensional abstracts and plurals by appealing to the set-theoretical content in an unnatural and informationally superfluous semantical theory.

One wonders, then, why set theory and set-theoretical semantics have caught on. Sociology of knowledge aside, one might give a "genetic" account something like the following. We have seen that there is *prima facie* evidence that plurals behave rather like singular terms. If they are singular terms, though, what are their primary semantical correlates? Some plurals seem to have ordinary concrete aggregates as their primary semantical correlates. (Recall (1) above.) This fosters the presumption that the primary semantical correlates of all plurals are aggregates. If this were so, however, then for some uses of plurals a new kind of abstract aggregate would have to be hypostasized—or so the set-theorist reasons—a kind of aggregate whose members are exactly the things satisfying the predicate from which the plural is generated. (Recall (2) above.) Thus, unlike ordinary concrete aggregates, this new abstract aggregate must wear on its sleeve the satisfaction conditions of the generating predicate. Since the new aggregate, the set of *F*s, appears to bear such a simple and direct semantical relation to the generating predicate *F*, there is a further tendency to identify the set of *F*s as the primary semantical correlate of the predicate *F* itself, as well as of the plural 'the *F*s'. And so one might arrive at the full logico-semantical belief that the set of *F*s is the primary semantical correlate of the predicate *F*.

On this account set theory and set-theoretical semantics appear to be fostered by a compulsion to concretize; that is, they appear to spring from a compulsion to think of the primary semantical correlates of predicates on analogy with ordinary concrete aggregates. Yet, as I have said, intuitively the primary semantical correlates of predicates are simply the properties or relations they express. All other semantical correlates of predicates are derivative and, furthermore, are typically natural or social:[14] packs, bunches, flocks, tribes, races, species, kinds, etc., or ordinary collections, social classes, ordinary sets, etc. Against this background set theory is seen to be an artifice out of place in the natural logical world. How ironic that a compulsion to concretize gives birth to the most abstract artifice ever produced by the human mind.

What then is the overall conclusion so far? On the basis of the foregoing critical survey it appears that the ontology of sets does not fall within our naturalistic ontology and also that set theory is not embedded in natural logic. Thus, contrary to received opinion, the notion of class does not appear to be fundamental to thought.

There remains one more strategy by which one might try to justify the ontology of sets: perhaps set theory, while not natural, is at least uniquely useful in mathematics or in empirical science. To this pragmatic issue I turn next.

29. The Dispensability of Sets

If the concepts of set theory are grounded neither in our naturalistic ontology nor in natural logic and if concepts of set theory arise from a conflation of the concepts of ordinary aggregate and property, why should one take set theory seriously? Evidently the only remaining reason is that set theory might nevertheless play some unique role in pure mathematics or in the empirical sciences.

Consider pure mathematics first. Here set theory is used in an entirely abstract way to aid and to unify the study of such matters as cardinality, order, mapping, etc. Let x be an arbitrary non-empty set. I will say that x_0 is an *ultimate element* of x if and only if $x_0 \in x_1 \in x_2 \in \ldots \in x$ and nothing is in x_0 itself. Now, it is a matter of complete indifference what the ultimate elements are of any set that might be contemplated in pure mathematics. Hence, as far as pure mathematics is concerned, the study of sets can be limited to those sets whose only ultimate element is the null set. The theory of such sets is called *pure set theory*. We may conclude, therefore, that if set theory should turn out to have a unique role to play in pure mathematics, that role can be filled by pure set theory.

Now consider the empirical sciences. Let us suppose that in the service of classification and measurement there are occasions when it is useful to consider collectively (as well as individually) the individuals with which a given empirical science deals. Let us call the sets postulated for these purposes *empirical sets*. In connection with measurement there might, in addition, be a call for certain key relations, such as equinumerosity, that hold between pure and empirical sets. The theory that characterizes empirical sets and those key relations holding between pure and empirical sets may be called *applied set theory*. If set theory has any role to play in the empirical sciences, that role can be filled by applied set theory.

So, to repeat our earlier question, why should set theory be taken seriously? The answer stated more precisely is this: pure set theory might have a unique role to play in pure mathematics, or applied set theory might have a unique role to play in the empirical sciences.

I think that pure and applied set theory have no such unique roles. In fact, this claim can be proved. Specifically, it can be proved that first-order pure and applied set theory can be modeled within the first-order logic for the predication relation. (Since first-order pure set theory countenances sets of sets, sets of sets of sets, etc., this result goes well beyond Russell's no-class construction, which works only for sets of non-sets. In what follows I will give no-class constructions for both of the leading first-order pure set theories, Zermelo-Fraenkel (ZF) and von Neumann-Gödel-Bernays (GB).[15]) This result shows that any theoretical tasks that pure and applied set theory perform can be accomplished equally well by a theory that, unlike set theory, has a legitimate origin in natural logic.[16]

Thus, my larger conclusion is that neither naturalistic ontology, natural logic, pure mathematics, nor the empirical sciences provide any ground for believing that sets exist: there is neither naturalistic, logical, nor pragmatic warrant for set theory. Set theory does not belong in a rational view of reality.

An example will help clarify what I meant when I said that first-order pure and applied set theories can be modeled within the first-order logic for the predication relation and that this shows that any of the theoretical tasks that these set theories perform can be accomplished equally well by the logic for the predication relation. Consider the miniature theory for ordered pairs:

$$\langle u, v\rangle = \langle x, y\rangle \equiv (u = x \;\&\; v = y).$$

It is widely known that this theory can be modeled by set theory. What this means is that from a syntactical point of view the notation used in the ordered-pair theory can be introduced into the language of set theory as an abbreviation for longer set-theoretic locutions and that this can be done in such a way that the miniature theory can then be derived as a theorem using just the original axioms and rules of set theory. Thus, on those occasions in pure mathematics and empirical science when previously one spoke of ordered pairs one may now merely speak of certain sets, namely, those sets singled out by the abbreviation scheme with which the ordered-pair notation is introduced. Thus, any mathematical or scientific jobs that ordered-pair theory can do can be done by set theory equally well.

In an analogous manner, then, I will show that from a syntactic point of view the notations used in first-order pure and applied set theory can be introduced into the logic for the predication relation as abbreviations for longer property-theoretic locutions and that this can be done in such a way that the set-theoretical axioms and rules can be derived as theorems using just the logic for the predication relation. This permits one to speak merely of properties on those occasions in pure mathematics and in empirical science when one previously spoke of sets. Thus the logic for the predication relation may fill perfectly well any of set theory's mathematical or empirical scientific roles.

There are two opposing philosophical purposes one might have in modeling one theory within another. One purpose is *reduction*. In the ordered-pair case, e.g., one's aim might be to show that no mathematical or scientific utility is lost if ordered pairs are *identified* with a certain kind of set. The other purpose is *elimination*. Thus, in the ordered-pair case one's motive might be to show that no mathematical or scientific utility is lost if ordered pairs are held *not to exist*.

Which of these purposes should one have, reduction or elimination? The possibility of modeling one theory within another shows that the modeled theory has no mathematical or scientific utility not possessed by the modeling theory. Suppose that the motivation offered for particular axioms in the modeling theory is at least as strong as the motivation offered for those in the modeled theory. And suppose that we already have good philosophical or logical reasons for accepting the ontological framework of the modeling theory. In this case the decision whether to reduce or eliminate the entities of the modeled theory should be based on whether there is any independent philosophical or logical reason to think those entities exist. Now, in a moment we shall see that the motivation offered on behalf of the axioms for the predication relation is at least as strong as the motivation offered on behalf of the axioms for the ϵ-relation. Further, we have already seen that there are good philosophical (§5) and logical (§§6–9) reasons for accepting an ontology of PRPs. And we have seen (in the previous two sections) that there are no independent philosophical or logical reasons for accepting an ontology of sets. Therefore, the fact that set theory can be modeled within the logic for the predication relation supports the decision to eliminate sets from our ontology.

This brings us to the motivation for the axioms in formulations of the logic for the predication relation. The point that needs to be made here is that, for any credible motivation that can be given for a particular formulation of set theory, an analogous motivation, which is at least as satisfactory, can be given for the axioms in a corresponding formulation of the logic for the predication relation. To see how this goes for a simple example, consider the usual motivation offered in support of Zermelo's axioms for pure set theory, namely, the motivation provided by the iterative conception of set. On this conception, sets are thought of as being "formed" in stages from the null set $\emptyset$ by means of repeated applications of a power operation:

Stages	1	2	...	α	...
Pure Sets	$\{\emptyset\}$	$\{\emptyset, \{\emptyset\}\}$	...	$\{y$: y is a set and every element of y belongs to a set formed prior to $\alpha\}$	...

If for every stage there is a later stage immediately following no stage, then the union of these sets is a model for Zermelo's axioms.[17] However, on analogy with the iterative conception of set, there are also iterative conceptions of PRPs. The easiest to describe is the iterative conception of pure L-determinate type 1 properties. (x is L-determinate *iff*$_{\text{df}}$ $\Box(\forall y)(y \,\Delta\, x \supset \Box\, y \,\Delta\, x)$.) On this conception such properties may be thought of as being "formed" in stages from the necessarily null type 1 property Λ by means of repeated applications of a power operation:

Stages	1	2	...	α	...
Pure Properties	$[\Lambda]$	$[\Lambda, [\Lambda]]$	...	$[y$ is an L-determinate property whose instances are instances of a property formed prior to $\alpha]_y$	...[18]

If for every stage there is a later stage immediately following no stage, then the union of these properties is a model for the axioms of a Zermelo-style theory for pure L-determinate type 1 properties. Moreover, the same sort of thing can be done for other iterative conceptions of PRPs.

Now the general point is this. Whenever motivation is offered for the axioms in a given set theory, it is never stronger than an analogous motivation for the axioms in an associated property theory. For this reason, general philosophical and logical considerations (such as those given earlier in this chapter) should every time guide us to choose the property theory—with its no-class construction—over the set theory.

Before I proceed to the no-class constructions for ZF and GB a general point of clarification is in order. Many of the formal metatheoretic constructions in this book are given within a set-theoretic framework, and one might wonder whether it is consistent to conduct metatheoretic constructions within a set-theoretic framework while denying that sets really exist. It is. For each of these metatheoretic constructions may be viewed as only a convenient shorthand for a metatheoretic construction within a property-theoretic framework. That is, in the last analysis my underlying metatheoretic framework is really property theory, not set theory.

30. Pure Set Theory Without Sets

There are many attractive no-class constructions of pure first-order set theory, some better suited to one philosophical view than another. The construction I will give, however, is the simplest I can find. I begin with the following definitions:

> x ultimately comprehends y *iff*$_{\text{df}}$
> $(\forall z)((x \subseteq z \ \& \ (\forall w)(w \,\Delta\, z \supset w \subseteq z)) \supset y \,\Delta\, z)$

> x is a pure L-determinate property *iff*$_{\text{df}}$
> x is an L-determinate property & whatever x ultimately comprehends is an L-determinate property.[19]

Thus, x ultimately comprehends y if and only if y is an instance of x or y is an instance of an instance of x or y is an instance of an instance of an instance of x or.... And x is a pure L-determinate property if and only if x is an L-determinate property whose instances are L-determinate properties and whose instances have as instances only L-determinate properties and so on. Now consider any sentence A in the standard language of pure first-order set theory,[20] and let A' be the sentence that arises from A by replacing all occurrences of $\in$ with Δ and by relativizing all quantifiers to pure L-determinate properties. Then I contextually define A in L_ω with Δ as follows: A *iff*$_{\text{df}}$ A'. Next take the standard axioms of Zermelo-Fraenkel set theory; drop the axiom of extensionality, and rewrite the remaining axioms using Δ, $=$, and intensional abstraction.[21] TZF^- is the intensional logic obtained when these axioms are adjoined to T1.[22]

> *Metatheorem:* Every sentence that is a theorem of Zermelo-Fraenkel set theory is, given its contextual definition in terms of Δ, $=$, and intensional abstraction, a theorem of the intensional logic TZF^- (i.e., for every set-theoretical sentence A, if $\vdash_{ZF} A$, then $\vdash_{TZF^-} A$).[23]

Proof. First, we prove in TZF^- that each property is included ($\subseteq$)

in some Δ-transitive property. Take the union of the original property, the union of the result, the union of that union, ... The union of all these unions is a Δ-transitive property that includes the original property. To prove that this new property exists, we form an appropriate function on an intensional ω by means of a recursion principle (whose proof in TZF$^-$ does not require extensionality). This function's range, which we obtain by means of the replacement axiom, is the desired Δ-transitive property. Using this theorem, we can then prove in TZF$^-$ that each L-determinate property whose instances are all pure L-determinate properties is itself a pure L-determinate property. After this, we show by induction that $\vdash_{TZF^-} A \equiv \Box A$ for formulas A whose constituent predicates are Δ or = (or both Δ and =) and all of whose constituent terms are variables whose ranges are restricted to pure L-determinate properties. With these facts at hand, the derivation of each ZF set-existence axiom is straightforward; we simply use the associated TZF$^-$ property-existence axiom plus appropriate instances of the TZF$^-$ comprehension schema. To derive the ZF extensionality axiom, we first prove in TZF$^-$ that $u = [z \, \Delta \, u]^u_z$ for all properties u. From this we may derive in T1 that, for all properties x and y, $\Box(\forall z)(z \, \Delta \, x \equiv z \, \Delta \, y) \supset x = y$. And from this plus the theorem that every instance of a pure L-determinate property is a pure L-determinate property, we derive

$$(\forall x, y)(x \text{ and } y \text{ are pure L-determinate properties} \supset ((\forall z)(z \text{ is a pure L-determinate property} \supset (z \, \Delta \, x \equiv z \, \Delta \, y)) \supset x = y)).$$

But given the contextual definition of the sentences of ZF in terms of Δ, =, and intensional abstraction, this sentence is just the expanded form of the ZF principle of extensionality:

$$(\forall x, y)((\forall z)(z \in x \equiv z \in y) \supset x = y).$$ *End of proof.*

Next take the axioms of von Neumann-Gödel-Bernays class theory, drop the axiom of extensionality, and rewrite the remaining axioms using Δ, =, and intensional abstraction.[24] TGB$^-$ is the intensional logic obtained when these axioms are adjoined to T1.[25]

Metatheorem: Every sentence that is a theorem of von Neumann-Gödel-Bernays class theory is, given its contextual definition in terms of Δ, =, and intensional abstraction, a theorem of the intensional logic TGB$^-$ (i.e., for every set-theoretical sentence A, if $\vdash_{GB} A$, then $\vdash_{TGB^-} A$).[26]

The proof of this is analogous to the previous proof.

The intuitive content of the results is easy to state: Zermelo-Fraenkel set theory and von Neumann-Gödel-Bernays class theory have an alternate interpretation according to which they are just theories of pure L-determinate properties, a kind of property that forms a sub-universe within which there are no intensional distinctions.

31. Applied Set Theory Without Sets

I will now show how to give a no-class construction for a fairly elementary applied set theory which countenances empirical sets of particulars and of type 1 PRPs. The intuitive idea is that notation that previously had been interpreted as being about such empirical sets will now be introduced as an abbreviation for a longer property-theoretic locution that concerns the properties common to the elements of these empirical sets. (This way of treating set-theoretical notation is reminiscent of Russell's no-class construction of type-stratified set theory.) The construction, however, can be extended by analogy to more sophisticated applied set theories, including ones that countenance empirical sets of sets, etc. and that are fitted out with intensional and extensional abstraction operations.

$\mathscr{S}$ is a first-order language for elementary applied set theory. The primitive symbols of $\mathscr{S}$ are:

Logical operators:	$\&, \neg, \exists$
Predicates:	$=, \in, \Delta, F_1^1, \ldots, F_p^q$
Variables:	$x, y, z, \ldots$
	$\alpha_1, \alpha_2, \alpha_3, \ldots$
	$\beta_1, \beta_2, \beta_3, \ldots$
Punctuation:	$($, $)$.

Atomic formulas: $v_i = v_j$, $\alpha_i = \alpha_j$, $\beta_i = \beta_j$, $\alpha_i \in \alpha_j$, $v_i \in \beta_j$, $v_i \,\Delta\, v_j$, $F_i^j(v_1, \ldots, v_j)$. Let complex formulas be built up from these in the usual way. The variables $x, y, z, \ldots$ are to be thought of as ranging over particulars and type 1 PRPs; $\alpha_1, \alpha_2, \alpha_3, \ldots$, over pure sets; $\beta_1, \beta_2, \beta_3, \ldots$, over empirical sets of particulars and type 1 PRPs. And $F_1^1, \ldots, F_p^q$ are non-set-theoretic predicates. Now every sentence A of $\mathscr{S}$ can be contextually defined in L_ω with Δ as follows: substitute Δ for all occurrences of $\in$; replace all atomic formulas $\beta_i = \beta_j$ with $(\forall x)(x \,\Delta\, \beta_i \equiv x \,\Delta\, \beta_j)$; restrict all quantifers on pure set variables to pure L-determinate properties; replace all pure and empirical set variables with new distinct non-set variables. The result A^* is a sentence in L_ω with Δ. Then adopt the following contextual definition: A *iff*$_{\text{df}}$ A^*. To see that this definition does the job, consider the theory TZF_a^- which is just like TZF^- except that now $F_1^1, \ldots, F_p^q$ may occur in the axiom schemas. In TZF_a^- we can derive, not only all the closures of the axioms of pure Zermelo-Fraenkel set theory, but also all closures of the following two axioms for the applied set theory:

(Extensionality)

$$(\forall x)(x \in \beta_i \equiv x \in \beta_j) \supset \beta_i = \beta_j$$

(Comprehension)

$$(\exists \beta_i)(\forall v)(v \in \beta_i \equiv (v \,\Delta\, u \,\&\, A))$$

where A is any formula of $\mathscr{S}$ in which β_i does not occur free. Hence, we have a no-class construction for not only pure first-order set theory but the applied first-order set theory as well.[27]

Summing up, we have seen that both pure and applied first-order set theories can be modeled within the first-order logic for the predication relation. Therefore, in view of the conclusion that the ontology of sets does not fall within our naturalistic ontology and the conclusion that set theory is not part of logic, there is simply no justification for positing the extraordinary abstract aggregates of set theory over and above PRPs and ordinary aggregates.

With this conclusion in hand I want to back up a bit. In the last three sections I have been operating under the assumption that set theory has at least a provisional role to play in mathematical matters. But now I want to challenge even that assumption, at least as it pertains to the analysis of numbers. For in the next chapter I will defend the thesis that in a proper construction of classical mathematics numbers should not even provisionally be identified with sets. Numbers should boldly be identified with properties.

6

Number

> It was Frege who first forced both philosophers and mathematicians to acknowledge the lack of any philosphical account of the nature and epistemological basis of mathematics. He himself constructed a complete system of philosophy of mathematics . . . . [T]he philosophical system, considered as a unitary theory, collapsed when . . . shown to be incapable of fulfillment . . . by Russell's discovery of the set-theoretic paradoxes. . . . [M]uch as we now owe to Frege . . ., it would now be impossible for anyone to consider himself a whole-hearted follower. . . .
>
> Michael Dummett
> *Elements of Intuitionism*

These excerpts express what appears to be the prevalent attitude toward logicism among leading contemporary philosophers of mathematics. Despite this, I am still inclined to hold a logicist position. In what follows I will employ the theory of PRPs to defend it. Along the way I will reply to the standard criticisms of logicism, none of which hits its mark in my opinion. I begin by considering logicism in the context of arithmetic. This after all was what Frege himself was concerned with, and it is here that the doctrine is most defensible.

32. A Neo-Fregean Analysis

Ask a practicing mathematician what the Peano postulates for number theory are. If he does not have a philosophical or historical ax to grind, in the majority of cases he will state the following:

(1) 0 is a natural number.
(2) Natural numbers have unique successors.
(3) 0 is not the successor of anything.
(4) If the successor of $x =$ the successor of y, then $x = y$.