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On being a student of Ken Arrow

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Looking back it is now clear that Kenneth Arrow was the most important economist of the twentieth century. He was able to distill complex ideas into lucid prose and into precise mathematical models of transcendent generality. He had the power to rigorously analyze those models, giving the three most famous proofs in economics, and finally he had the open-mindedness and humility to critique his own theories, pointing the way forward for the rest of the profession, and especially his students. No economics teacher, with the possible exception of Keynes, has ever been a better mentor (four of his students have won Nobel Prizes). I will try to describe what it felt like being one of his students, and then I shall make some remarks about general equilibrium, the area for which he won the Nobel Prize.

Arrow's two models of general equilibrium and social choice crystallized the mathematical-axiomatic approach that transformed economics from a field not much more mathematical than its sister social sciences like sociology and psychology into a discipline with the same mathematical rigor as physics and the other hard sciences.† As an undergraduate mathematics major at Yale University who loved to read political philosophy, I was intrigued and puzzled by Plato's famous dictum 'Let no man ignorant of geometry pass through these gates'. Only as a junior when I took a beautiful course in mathematical economics from Herbert Scarf did I fully appreciate the wisdom in those words. I resolved to go to graduate school at Harvard to study with Kenneth Arrow, the guru of mathematical economics. Two weeks before graduate students had to commit, the New York Times ran a curious front-page story about a visiting committee declaring that the once great Harvard economics department had devolved into a collection of fiefdoms that did not interact with each other or with students. Many decided to go elsewhere.‡ I went to Harvard with more enthusiasm, figuring I might have Arrow all to myself.

The first day I got to Harvard there was an orientation meeting hosted by one of the economics faculty at his home, at which the professors in first year courses were to give introductory talks. There were not enough chairs, so the students all sat on the living room rug. Arrow sat down on the floor next to me and introduced himself. He already knew who I was. He taught us first semester microeconomics. At the end there was a final exam, administered centrally in Memorial Hall with dozens of other courses. The exams in the other courses were distributed by the teaching assistants. Arrow came and handed out his own.

Arrow taught optimization and the economics of information, leaving general equilibrium and social choice to his younger colleagues Truman Bewley and Jerry Green. That led

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† Of course economics does not have the same empirical-experimental validity as say physics.

‡ Yale's graduate class ended up being twice as large as normal that year, and included several future superstars of our profession.

me to read his original papers on those subjects more closely. At the end of my first year he called me into his office and told me I would be his research assistant, paying me essentially to go and talk to him. (Between an NSF scholarship, Arrow's stipend, and free room and board as a resident tutor in Mather House, I lived better than any year until I got a raise for making tenure.) Arrow did not devote much time to preparing his lectures, or his public talks for that matter. He was invited to give the prestigious Tanner Lectures on moral philosophy at Harvard after he had gone to Stanford and I had gone back to teach at Yale, and he asked me to come hear them. I arrived the night before and soon realized who was typing in the next room until 2 a.m. that night, and again the next night. About 20 minutes before his second lecture ended he turned over the last page and spoke the rest without notes. In his classes he would begin sentences and then switch to another idea that suddenly occurred to him. But what was inspirational was that every idea was perfectly precise linguistically and mathematically, even if the order was jumbled. And every idea was connected to a literary reference, often Greek. When giving an example of the all too common situation where the best idea is ignored by the politicians, he cited Thersites, who was beaten for his suggestion to abandon the Trojan War. In one talk he described a countable number of oil fields with independent production by saying no man can step into the same river twice. His last public lecture at Harvard, before he moved back to his beloved Stanford, was titled 'Economics as Omphaloskepsis'. His unceasing analogizing across disciplines may be what Frank Hahn meant by Arrow's touch of genius.

Arrow was at his best in seminars other people gave. He twirled his pen in the air, catching it with different fingers at the same time he performed mental gymnastics with the speaker. He would always ask the most penetrating questions, tell the funniest stories, and most importantly, make crucial clarifying comments. I remember one speaker presenting a model in which some Arrow securities were traded but not the others. Hahn commented that all the troubles in the model came from those pesky Arrow securities. Arrow said the trouble was the absence of (the other) Arrow securities. Another speaker pleaded that one mistake shouldn't spoil his whole talk; Arrow told him he was more than right, that a roomful of darkness couldn't put out a single candle. Yet another speaker, a physicist, was explaining the butterfly effect in chaos theory, and wondered if there were any actual historical examples of such a thing (and not just in economic models). Arrow referred him to Pascal's view that if Cleopatra's nose had been a millimeter longer, Roman history would be quite different. There was a time, while I was a graduate student, when he would fall asleep during the seminar, but wake up just in time to again ask the best questions. I asked him how he did it, and he said the trick was to wake up before he opened his eyes.†

Virtually every individual conversation I had with Arrow was memorable. One day I ran into his office terrified because I had discovered that an obscure (to me) German economist named Bertram Schefold had anticipated some of the key concepts in my paper on Sraffa. Arrow advised me that one should always be grateful to find knowledge in the universe. During another period I had to call him every two or three days asking for another letter of recommendation for a fellowship or job. One day when I called he picked up the phone from his secretary and, in the most friendly, accommodating voice, said 'John, what can you do for me?' Another time he called me into his office and told me that when he was a graduate student he knew that he was smart; he just doubted he would ever think of anything truly original to say. He wanted me to know he thought of me as a comer. When I excitedly turned in my dissertation of over 200 pages of dense mathematics he said 'Such a big book Mr. Gibbon', which I learned later was how David Hume responded when receiving the first copy of the Decline and Fall of the Roman Empire from its (apparently) overly proud and long-winded author.

Arrow opened up many worlds for me. He introduced me to his fascinating friends, including the economists Frank Hahn, Mordecai Kurz, Bob Aumann, Brian Arthur, Graciela Chichilnisky, and Roy Radner, and his students Eric Maskin and Roger Myerson, and also non-economists like John Rawls and Hilary Putnam and Murray Gell-mann. He brought me every summer to IMSSS, the great summer symposium he ran with Kurz. Afterward he invited me to lecture at the first and last (19th) Jerusalem summer school that he organized.‡ And finally he introduced me to the Santa Fe Institute, appointing me the second director of the economics program (jointly with David Lane) after Brian Arthur.

Like Murray Gell-mann, Arrow had a universal curiosity, and command of many different fields, including theater,§ history,¶ and literature.‖ But he wanted to apply rigorous mathematics to do something beautiful and general and practically important. He recounted to me, as I am sure he

† I also asked him how he managed to travel so much and yet still write so many papers. He said he worked very well on airplanes. A little later David Herlihey, the head of Mather House and an economic historian, told me had had just flown back from Japan, intending to write up his notes on the plane, but my adviser was snoring so loudly in the row behind that Herlihy couldn't write. I reported this conversation to Arrow and asked how it squared with his claims about working on planes. He said good sleep is very productive work.

‡ At the 19th and last summer Arrow was director, there was a celebratory concert given by Israel's most famous trio. At intermission everyone kept their seats, and many toasts were given to honor Arrow. The last speaker Eric Maskin, who was taking over the next year as director of the economics summer school, lamented that there was nothing left to say about Arrow. With that he asked the clarinetist to give him his seat, and after intermission the concert continued without missing a beat but with the new clarinetist.

§ Arrow was a devotee of Gilbert and Sullivan, memorizing the lyrics and performing various roles on stage in Harvard student productions.

¶ Waiting in the lunch line at the Santa Fe Institute in 1995 I remarked that it was the 550th anniversary of the Council of Florence, at which the Byzantines had agreed to end the Schism and join the Catholic Church in exchange for a Papal army to protect Constantinople against the Turks. I knew this because my father was a Professor of Byzantine and Renaissance History, and I shamelessly wanted to show that I knew something I figured the two polymaths didn't. Unfortunately for me, they responded by competing over who could name more delegates to the Council (the Byzantine Patriarch had just died and was not in attendance).

‖ Among other things, Arrow had almost a complete mastery of the Shakespearian classics.

did to many others, how he decided to become an economist. As a young man he told Tjalling Koopmans, the Director of the Cowles Foundation, that he intended to become an actuary, because it required mathematical skills and paid a lot of money. Koopmans said 'But it has no music'.

The most beautiful, most important, and most general model in economics is the Arrow Debreu model of General Equilibrium. It was simple enough to be understood immediately by mathematicians with no training in economics, yet general enough, given ever subtler interpretations of the notion of commodity, to encompass a large fraction of economics known up until that time, as special cases. Moreover, many subsequent developments in economics could be cast as elementary relaxations of the Arrow-Debreu framework. Today general equilibrium plays an absolutely central role in fields as diverse as international trade, public finance, development, finance, and macroeconomics.

In 1951 and 1954 Arrow and Debreu sought to give a formal mathematical answer to two questions discussed by Adam Smith in 1776. Can free markets alone coordinate the diverse desires and talents of millions of consumers and producers, all pursuing their own selfish interests without any regard for or knowledge of the others, in a way that promotes the common good? Axioms were provided for the commodity space, for endowments and preferences of each consumer, and for the production possibilities of each firm. Equilibrium was then defined via prices and budget sets, embodying the neoclassical methodological premises of individual rationality, market clearing, perfect competition, and rational expectations.

The first fruit of the more precise formulation of equilibrium was the transparent demonstration that Arrow and Debreu independently gave in 1951, that every equilibrium is Pareto optimal, that is, that no other feasible allocation is preferred by every consumer. So simple and illuminating is this proof that it is no exaggeration to call it the most important argument in all of neoclassical economic theory. The old proofs, which still linger in many intermediate textbooks, rested on three irrelevant assumptions: differentiable and concave utilities, and strictly positive consumption of every good by every individual. Arrow's proof (like Debreu's simultaneously written proof) drops all three assumptions, contains no calculus, and is only a few lines long.

Arrow and Debreu together, and McKenzie separately, gave the touchstone proof in economics, of the existence of competitive equilibrium. Leon Walras, the inventor of general equilibrium, had informally suggested that equilibrium would always exist, because for every market there is a corresponding price, and by increasing the price in markets for which there is too much demand and decreasing the other prices, the economy would grope its way to equilibrium in which demand equals supply for all markets. Arrow and Hurwicz realized early on that proving the convergence of Walras' tatonnement dynamics was problematic in general, though they did prove convergence for several important kinds of economies beginning in 1958.

Arrow and Debreu independently became convinced, on reading the general proof of existence of Nash equilibrium for games by John Nash in 1950, that a general existence proof for competitive economies must be attainable via Brouwer's fixed point theorem, even though it had eluded von Neumann and Wald. Von Neumann and Wald knew Brouwer's fixed point theorem very well, and had used it in other proofs, but great mathematicians as they were, they were unable to prove the existence of equilibrium except in very special cases.

By humbly paying careful attention to the precise formulation of the details of the model, Arrow and Debreu succeeded where von Neumann and Wald failed. I give three of these details. Arrow and Debreu noticed that if income becomes zero when a price simultaneously hits zero, then demand can jump discontinuously. They recognized the need to make an assumption ruling this out (namely that endowments are strictly positive in every good, so that income is never zero), but still preserving the possibility that each individual might consume zero of many goods. They noticed that if it is assumed that consumption is bounded below (say by zero), an unobjectionable hypothesis, then the assumption that consumption is also bounded above can be made for free, because it will never be binding at the fixed point. Brouwer's fixed point theorem cannot naturally be used to show that demand equals supply, but only that demand is less than or equal to supply. Arrow observed that when a price is zero, and the good is free like air, it is natural for there to be excess supply. So they modified the definition of equilibrium to allow supply to exceed demand, but only for goods with zero price.

Part of the astonishing generality of the Arrow Debreu model is due to the simultaneous and equal treatment it gives to producers and consumers. The classical economists, including Smith, Ricardo, Marx, and Cassel, had more or less assumed that relative prices are determined by fixed coefficients of production. The marginalists, like Jevons and Menger, claimed that price is determined by marginal utility. The Arrow-Debreu model recognized that each price was determined by its own supply and demand (as in the traditional picture of the cross from elementary economics) and also by prices of other commodities in a general equilibrium. This gave full scope to the possibility that indirect consequences can reverse the intentions of economic agents, as when producers striving to increase their profits compete so hard that they drive profits to zero. Arrow called this phenomenon of unintended consequences the most important idea in the social sciences.

The main engine for generalizing the Arrow Debreu model lies in the various interpretations of commodity that Arrow was able to conjure. Hicks, perhaps anticipated by Fisher, was the first to suggest an elaborate notion of commodity, in which two identical apples, that were deliverable at different time periods, were regarded as different commodities. Thus saving, or the lending of money, can be thought of as the trade today of an apple in exchange for a future dated apple, exactly like the trade of an apple for an orange; the ratio of the apple prices can then be interpreted as the gross real rate of interest. The mundane general equilibrium model of trading apples for oranges can be used without substantive change to explain the real rate of interest, which is the starting point of macroeconomics and finance. Similarly the same commodity or the same labor at different geographical locations can be treated like different commodities, and again the general equilibrium model can encompass international trade.

Arrow took a more imaginative tack in extending commodities. In 1969 he added a firm technology that transformed

each (public) good into many copies, each indexed by a different consumer name. This enabled the Arrow-Debreu model to include the theory of public goods and externalities, for example making clear why the efficient production of private goods equates marginal cost with each consumer's marginal utility, whereas the efficient production of public goods equates marginal cost with the sum of consumers' marginal utilities, as Samuelson had pointed out. It also made clear that the efficiency losses from externalities could be ascribed to missing markets.†

Arrow's boldest stroke by far was in imagining in 1953 that a New York apple is a different commodity depending on how much it is snowing in Paris. By distinguishing physical objects depending on the state of nature (which includes a complete description of all uncertainty, even those apparently unrelated to the object), Arrow was able to analyze the optimal allocation of risk with exactly the same general equilibrium apparatus used to analyze the exchange of apples and oranges, and thereby to usher in the field of modern finance.

Before Arrow, uncertainty was represented in financial theory by joint normal *distributions*, which he replaced with arbitrary *random variables*. As Arrow explained, this was a natural step for somebody steeped in statistics. But it immediately implied that the output of a firm, or the payoff of a bond, could be thought of exactly the same way as a basket of commodities or fruit, where the bond payoff in each state corresponds to the quantity of a different kind of fruit. It meant that the payoff of an insurance contract could be modeled by the cash flows it provides in each state, without worrying that it is not normally distributed. This immediately led Arrow to describe what are now called Arrow securities, namely contingent securities that pay one dollar (or one apple) in exactly one state. The price of a bond or an insurance contract is then just the sum of the prices of its constituent Arrow securities, just as the price of a basket of fruit is the sum of its individual fruit prices. Following Arrow, modern financial theorists soon introduced the fundamental vocabulary of the field, talking of *state prices* for evaluating securities, and the payoffs of some benchmark securities *spanning* the payoffs of other target securities, and the *no-arbitrage* relationship that must hold between the target security price and the spanning benchmark securities prices. The celebrated Black–Scholes model of option pricing, especially in its binomial tree formulation, fits perfectly into this Arrow framework.

One of Ken Arrow's enduring legacies is the criticism he made of his own work, and the implied research agenda it opened up for his students. He created the modern definition of equilibrium, yet he embraced the non-equilibrium approach to economics of the Santa Fe Institute. He gave the most illuminating and general proof of the 'Pareto efficiency' of equilibrium with complete markets, yet he sometimes regarded the proof as important because it made clear why equilibrium would not be efficient without complete markets. As he said, one should always work both sides of the street.

Much of my own work has been devoted to investigating the consequences of missing markets (the absence of some Arrow securities), especially in environments where lenders must worry about default. Two years ago I gave the 'Arrow lecture' at Stanford University, and last year I gave the last 'Arrow lecture' while Ken was alive at Columbia University, Ken's alma mater. At 95 years of age, three months before he was to die, Ken traveled all the way from Palo Alto to New York to be a discussant for the talk, no doubt working the whole time on the plane. Soon after he got back, a heart valve condition worsened and he entered the hospital. Absolutely committed scholar that he was, he asked the doctor whether not having an operation would give him the higher odds of finishing his last paper, and chose accordingly.

At his 70th birthday celebration, when he was in perfect health, I ended my dinner toast by quoting Phaedo's speech about his mentor Socrates from a Platonic dialogue 'Of all the men I have known, he was the wisest and the most just'. Ken responded 'Ah, please pass the hemlock'.

† Arrow seemed unaware in 1969 of the related work of Liindahl in 1919.