Notes

Introduction

1. Pp. 75 ff., Nelson Goodman, *Fact, Fiction, and Forecast*.
2. 'A Formulation of the Logic of Sense and Denotation' and 'Outline of a Revised Formulation of the Logic of Sense and Denotation'.
3. 'Intensional Isomorphism and Identity of Belief'. This conception of synonymy is assessed in §19 below; evidently it is an outgrowth of Church's effort to find a formally adequate resolution to the paradox of analysis.
4. Axioms 63–8, 'Outline', part two.
5. It is very important to realize that logical validity and epistemic justification also fall under conception 2.
6. See §14 for an illustration.
7. What I call Russell's theory is a synthesis of positions Russell took in the writings of his early period.
8. The generous grades on desiderata 8 and 9 are meant to imply only that the various resolutions of the paradoxes are successful in avoiding the paradoxes; it is doubtful that an ideal resolution has yet been found.
9. This argument is adapted from George Myro's important paper 'Aspects of Acceptability'.

Chapter 1

1. Whether this quantifier logic should be first-order or higher-order is not relevant at the moment. That question will be taken up in §10 and again in chapter 4.

   It will be noticed that in a higher-order language having sentential (i.e., propositional) variables (I) can be represented by treating 'is necessary', 'is true', and '*x* believes' as operators that take sentences into sentences. However, this is so only because in such higher-order settings there is no hard distinction between predicates and operators. Thus, the conclusion in the text stands. By the way, in chapter 4 I propose to treat sentences such as '*a* is red' as having the form '*a* is *b*' where the copula 'is' is a 2-place logical predicate expressing the predication relation. If this treatment is right, perhaps '*a* is necessary' and '*a* is true' should be treated analogously. If so, then strictly speaking 'is necessary' and 'is true' would not be predicates. Still, this does not affect the substance of my claim that 'is necessary' and 'is true' are predicates, for like ordinary predicates, 'is necessary' and 'is true' would still combine with a singular term to yield a sentence (open or closed). This is the only point that is needed for the succeeding steps in my argument in the text.
2. I take up the question of the definability of the bracket notation in §37.
3. Incidentally, the conclusion that 'that'-clauses should be treated as defined or undefined singular terms is compatible with all approaches considered on the chart in §4 except for the approaches of Carnap, Hintikka, and Davidson and the Quinean primitive-predicate approach.

4. The possibility of externally quantifiable occurrences of variables is not allowed in Quine's original bracket notation.
5. For example, it is intuitively valid that if it is true that *A*, then there is something that is true. It would be irrational to deny this and to hold instead that the antecedent could be true and the consequent false. This shows that 'is true' is a predicate satisfied by entities. Given this, the best theory of what makes 'It is true that *A*' true is that the constituent predicate 'is true' is satisfied by an appropriate entity. One could hold otherwise only by disunifying his treatment of truth, and what good reason could there be for that? Given the fact that what makes 'It is true that *A*' true is that the predicate 'is true' is satisfied by an appropriate entity, it would only be perverse to deny that the entity is other than one semantically correlated with the singular term 'that *A*'.

   Of course, one wants a semantical account, not just of 'that'-clause sentences concerning truth, but of an open-ended list of 'that'-clause sentences, sentences concerning validity, provability, evidence and epistemic justification, explanation, all the various psychological attitudes, meaning, assertion, the modalities, causation, probability, counterfactuality, moral prescription, etc. In view of the open-ended character of this list and in view of the fact that all forms of 'that'-clause sentences may be embedded in one another arbitrarily many finite number of times, a unified, general account demands the apparatus of quantification and cross reference in connection with 'that'-clauses.
6. The sentence (a) 'There is a language S' such that Seneca wrote as a sentence of S' words whose translation from S' into English is "Man is a rational animal"' is a typical nominalistic analysis of sentence (1) 'Seneca said that man is a rational animal', which contains a 'that'-clause. Church criticizes this analysis,

   > For it is not even possible to infer (1) as a consequence of [(a)], on logical grounds alone—but only by making use of the item of factual information, not contained in [(a)], that 'Man is a rational animal' means in English that man is a rational animal.
   >
   > Following a suggestion of Langford we may bring out more sharply the inadequacy of [(a)] as an analysis of (1) by translating into another language, say German, and observing that the two translated statements would obviously convey different meanings to a German (whom we may suppose to have no knowledge of English). The German translation of (1) is (1′) *Seneca hat gesagt, dass der Mensch ein vernünftiges Tier sei*. In translating [(a)], of course 'English' must be translated as 'Englisch' (not as 'Deutsch') and '"Man is a rational animal"' must be translated as '"Man is a rational animal"' (not as '"*Der Mensch ist ein vernünftiges Tier*"').

   (See Church, 'On Carnap's Analysis of Assertion and Belief'.) Incidentally, let us suppose with Quine that there are certain epistemological difficulties in determining which one in a class of candidate translations is correct. Still, *epistemological* difficulties in making a distinction do not in general entitle one to draw the *ontological* conclusion that the distinction does not exist. Quineans have been very hard put indeed to show why this generalization does not apply to the distinction between correct and incorrect translations.
7. For example, nominalistic approaches appear to provide no adequate treatment of prelinguistic intentional states such as those of infants and intelligent lower animals. Relatedly, they appear to provide no adequate treatment of complex intentional states of people who know no single language but only know fragments of several distinct languages—e.g., people who philosophize in Greek and make love in French. Carnapian and Quinean syntactical analyses are, in

addition, faced with a special difficulty of their own. They seem to require new *ad hoc* expressions such as 'believes-true A as a sentence of language L'. But what do such new expressions mean? It would seem that this question can be answered only if appeal is made to the theory of propositions.

The language-of-thought theory is a nominalistic account which avoids many of the above difficulties. But this theory is caught in the following dilemma. Either it must be construed as a form of representationalism (in the sense of §42); in this case the relation between thoughts and what they correspond to in the world remains mysterious. Or it must take the radical position that no one's beliefs ever have any real content; in this case the theory would seem to be logically self-defeating (since *it* would presumably be believed by its proponents). Either way, the theory has a deep problem not found in the theory of PRPs. Of course, a proponent of the theory might try to avoid this verdict by defining the relation holding between thoughts and what they correspond to in the world. However, these efforts all end in failure unless intensions are re-introduced: behavioristic definitions cannot be sufficiently discriminating, as Quine's indeterminacy argument shows in effect; physiologically oriented definitions must fail, for among other reasons, because of the open-endedness of the possible physiological bases of thought; and functional definitions run into the difficulties mentioned in note 27 of chapter 10, difficulties which can be surmounted only by appealing to intensions of the sort posited in the theory of qualities and concepts.

8. I emphasize again that this conclusion takes no stand on the question of whether this intensional abstraction operation is defined or undefined. Both positions will be considered later in the book.
9. Leon Henkin has proved a quasi-completeness result for higher-order quantifier logic: when the language of higher-order logic is interpreted with what Henkin calls general models (as opposed to the usual standard models), higher-order logic is complete. (Every standard model is a general model, but not conversely.) Advocates of the higher-order approach sometimes point to Henkin-style quasi-completeness results to try to show that on the issue of completeness the higher-order approach is not inferior to the first-order approach. However, it is a matter of considerable controversy whether these results warrant any such philosophical conclusion.
10. Transcendental predicates produce comparable difficulties for many of the syntactic theories that are categorial in style.
11. It should be recalled that in Quine's notation, unlike the notation I have proposed, externally quantifiable variables are not permitted to occur within bracketed expressions $[A]_\alpha$. (Strictly speaking, Quine uses $\alpha[A]$ where I use $[A]_\alpha$, but this notational variant is of no significance.) Also, even though Quine would be willing provisionally to interpret $[A(v_1, \ldots, v_i)]_{v_1 \ldots v_i}$, as a term that denotes an i-ary intensional entity, he in the end would want to interpret it as a term that denotes the formula $A(v_1, \ldots, v_i)$ itself. This nominalistic interpretation, however, need not concern us here. The issue I want to focus on is syntactic, not semantic.
12. I do not approve of this treatment. For, given the arguments of chapter 5, sets do not really exist, and sequences turn out to be just a special kind of *de re* property or *de re* relation-in-intension. (*De re* PRPs are those that we would naturally denote with intensional abstracts containing externally quantifiable variables.) The modified Quinean approach to the logic of *de re* PRPs is thus caught in a vicious regress, one fraught with internal technical inconsistencies.
13. This limitation is another count against the modified Quinean treatment itself.
14. In 'Intensional Logic in Extensional Language' Charles Parsons constructs a Fregean system which can represent quantifying-in (and which can satisfy

Davidson's finite-learnability requirement). However, Parsons accomplishes this by adjoining to the system an *ad hoc* device whose semantic force is to associate with each entity a special "rigid" concept, i.e., a special essential individuating concept of that entity. But such concepts, if they are credible at all, are just singular concepts, i.e., the sort of concepts that arise from singular predications. (E.g., the special individuating concept of me is surely just the concept of being identical to me, i.e., $[x = y]_x^y$ where $y =$ me.) Ideally, however, a logical theory should treat singular concepts and singular predication directly; only then will one be able to lay bare the logic for the special "rigid" concepts posited by Parsons. (This is what is done by means of my bracket notation and the semantics for it (see §13).) Singular concepts and singular predication, moreover, are not even countenanced by Frege's philosophy (though they are by Russell's). So once again quantifying-in would seem to be representable (and Davidson's learnability requirement would seem to be satisfiable) only by retreating from Frege's original view (and by taking up a neo-Russellian position instead).

Chapter 2

1. Nothing prevents us from adjoining primitive functional constants to L_ω, but that would require enriching the algebraic model structures (see §14) by adding operations for application of function to argument and relativized applications of function to argument.
2. These operations will be precisely defined in the next section. However, it might be helpful to describe provisionally the relativized predication operations, which are more difficult to understand than the others. An intensional abstract $[A]_\alpha$ binds those free variables in the embedded formula A that occur among the variables α. What is special about relativized predications is that some of the variables bound by the intensional abstract $[A]_\alpha$ occur free in an intensional abstract occurring within the embedded formula A. So, for example, the intensional abstract $[F[Gy]^y]_y$ binds the variable y that occurs free in the intensional abstract $[Gy]^y$ occurring within the embedded formula $F[Gy]^y$. And more generally, the abstract $[F[A]^{u_1 \ldots u_k}_{v_1 \ldots v_m}]_{u_1 \ldots u_k}$ binds the variables $u_1, \ldots, u_k$ in the embedded formula $F[A]^{u_1 \ldots u_k}_{v_1 \ldots v_m}$. This abstract is the k-ary relativized predication of $[Fx]_x$ of $[A]_{v_1 \ldots v_m u_1 \ldots u_k}$.

   There is an alternate strategy for dealing with relativized predications. Instead of denumerably many relativized predication operations having two arguments, one posits a single predication operation having three arguments, the additional argument serving to code the number of variables to be relativized in a particular application of the operation. This alternate strategy is sketched in my 'Theories of Properties, Relations, and Propositions'.
3. For more on the algebraic approach to extensional logic (without abstraction operations), see Henkin, Monk, and Tarski, *Cylindric Algebras*. Incidentally, the first seven operations also have a close relationship to the syntactic operations isolated in Quine's 'Variables Explained Away'.
4. Strictly speaking $\mathscr{P}$ is a prelinear order on $\mathscr{D}$.
5. Readers inclined to view $\mathscr{K}$ as a vestige of possible worlds should see p. 209 f. By the way, the truth values (T and F) may be defined in many ways; they may be identified respectively with $\mathscr{D}$ itself and with the null set, for example.

6. As things stand $\mathscr{D}$ is not closed under these operations. For example, Neg is not defined for elements of $\mathscr{D}_{-1}$. To close $\mathscr{D}$ under Neg, one could identify Neg(x), for $x \in \mathscr{D}_{-1}$, with some arbitrary element of $\mathscr{D}$. The same goes for the other operations. By the way, conservative Platonists might wish to modify clause (8.0) as follows: $\text{Pred}_0\colon \mathscr{D}_i \times (\mathscr{D} \sim \mathscr{D}_{-1}) \to \mathscr{D}_{i-1}$. This modification rules out the possibility of particulars being genuine subjects of predications.
7. In general, for $i \geq 1$ and $j \geq k \geq 1$, $\text{Pred}_k\colon \mathscr{D}_i \times \mathscr{D}_j \to \mathscr{D}_{j+k-1}$. In my informal remarks in the previous section Pred_0 is what I called absolute predication; Pred_1, unary relativized predication: Pred_2, binary relativized predication; ...; Pred_k, k-ary relativized predication.
8. In general,

$$8.k \quad \langle x_1, \ldots, x_{i-1}, y_1, \ldots, y_k\rangle \in H(\text{Pred}_k(u, v)) \equiv \langle x_1, \ldots, x_{i-1}, \text{Pred}_0(\ldots \text{Pred}_0(\text{Pred}_0(v, y_k), y_{k-1}), \ldots, y_1)\rangle \in H(u)$$

where $u \in \mathscr{D}_i$, $i \geq 1$, and $v \in \mathscr{D}_j, j \geq k \geq 1$. The following will help to illustrate the behavior of the predication operations Pred_0, $\text{Pred}_1, \ldots$:

Since $\text{Pred}_0([Fx]_x, [Guv]_{uv}) = [F[Guv]_{uv}]$, clause (8.0) insures that $H([F[Guv]_{uv}]) = \text{T} \equiv [Guv]_{uv} \in H([Fx]_x)$.

Since $\text{Pred}_0([Fxy]_{xy}, [Guv]_{uv}) = [Fx[Guv]_{uv}]_x$, clause (8.0) insures that $x \in H([Fx[Guv]_{uv}]_x) \equiv \langle x, [Guv]_{uv}\rangle \in H([Fxy]_{xy})$.

Since $\text{Pred}_1([Fx]_x, [Guv]_{uv}) = [F[Guv]^v_u]_v$, clause (8.1) insures that $v \in H([F[Guv]^v_u]_v) \equiv [Guv]^v_u \in H([Fx]_x)$.

Since $\text{Pred}_2([Fx]_x, [Guv]_{uv}) = [F[Guv]^{uv}]_{uv}$, clause (8.2) insures that $\langle u, v\rangle \in H([F[Guv]^{uv}]_{uv}) \equiv [Guv]^{uv} \in H([Fx]_x)$.

(Here I use, not mention, intensional abstracts from L_ω.)

9. Examples of type 1 and type 2 model structures are easily constructed. E.g., a type 1 model structure can be constructed relative to a model for first-order logic with identity and extensional abstraction, and a type 2 model structure can be constructed relative to a model for first-order logic with identity, extensional abstraction, and Quine's device of corner quotation.
10. $[A(u_1, \ldots, u_p)]_{u_1 \ldots u_p}$ and $[A(v_1, \ldots, v_p)]_{v_1 \ldots v_p}$ are *alphabetic variants iff*$_{\text{df}}$ their externally quantifiable variables are the same and, for each k, $1 \leq k \leq p$, u_k is free in A for v_k and conversely. A term t is said to be free for v_i in A if and only if, for all v_k, if v_k is free in t, then no free occurrence of v_i in A occurs either in a sub-context of the form $(\exists v_k)(\ldots)$ or in a sub-context of the form $[\ldots]_{\alpha v_k \gamma}$. (Recall that $(\forall v_k)(\ldots v_k \ldots)$ is an abbreviation for $\neg(\exists v_k)\neg(\ldots v_k \ldots)$.) Thus, if t is free for v_i in A, the result of substituting t for the free occurrences of v_i in A produces no "collision of variables". Let $A(v_1, \ldots, v_k)$ be any formula; $v_1, \ldots, v_k$ may or may not occur free in A. Then, I write $A(t_1, \ldots, t_k)$ to indicate the formula that results when, for each j, $1 \leq j \leq k$, the term t_j replaces each free occurrence of v_j in A.
11. The notion of t's being free for v_i in A is defined in note 10. Example: if $A(v)$ is $F[Gv]$ and t is $[Hw]$, then $A(t)$ is $F[G[Hw]]$. In this example t is free for v in $A(v)$; v is an externally quantifiable variable in $A(v)$, and w is an externally quantifiable variable in $A(t)$.
12. T1 is the simplest formulation of conception 1. In it the Barcan formula and its converse are derivable. This feature can be removed by slightly complicating the axioms and rules. Corresponding adjustments would then be made in the semantics. For simplicity of exposition these sophistications will not be pursued.

Chapter 3

1. Such a contradiction can be derived using either T1 or T2. Of course, only T2, which is our logic for conception 2, is relevant here, for the paradox of analysis is a puzzle in the logic for intentional matters, to which conception 2 is tailored.
2. See p. 215, Benson Mates, 'Synonymity' (p. 125 in Linsky). I use the term 'Mates' puzzle' to apply to all *prima facie* substitutivity failures involving synonymous predicates (or formulas) such that the substitutivity failure can be traced to some form of ignorance about linguistic, historical, or social matters. For further discussion of Mates' puzzle, see §39.
3. The instance of Mates' puzzle generated by the formula 'x does not know that whatever chews [pronounced *chōōz*] chews [pronounced *chôz*]' dramatizes the linguistic character of the problem.
4. I do not count Carnap's analysis of assertion and belief in *Meaning and Necessity* as a serious formal attempt to resolve the paradox of analysis because it is so fraught with problems, including its well-known violation of the Langford-Church translation test. When I speak of Church's resolution of the paradox I refer to a synthesis of the views found in his papers. I do not include the remark in his review of Max Black and Morton White; with tinkering perhaps that remark can be made to mesh with the resolution I offer in this chapter.
5. Church uses this theory of synonymy as the intuitive motivation for the Alternative (0) theory of concepts in his 'Outline', part two. Church holds the principle that the formulas $A(v)$ and $B(v)$ are synonymous if and only if the concept of being a thing v such that $A(v)$ = the concept of being a thing v such that $B(v)$. In this way, his theory of synonymy doubles as theory of concept identity.
6. This is the Leibnizian definition of identity. Frege's and Russell's definitions are substantially the same.
7. In a Churchian language '$y = z$' cannot be converted into '$(\forall f)(fy \equiv fz)$' by rules (1)–(7). The most that can be achieved is the conversion of '$y = z$' into '$(\lambda yz)(\forall f)(fy \equiv fz)(y, z)$' by use of rule (5). And this is not enough to obtain the above instance of the paradox.
8. Pp. 68–71, 'Intensional Isomorphism'.
9. Axiom 66, 'Outline', part two.
10. See p. 69, 'Intensional Isomorphism'.
11. In a Churchian language the rules (1)–(7) at most allow 'Outweighs(y, z)' to be converted into '(λyz)(the weight of y is greater than the weight of z)(y, z)'; they do not permit the production of 'the weight of y is greater than the weight of z'.
12. The ancestor relation is an historically interesting example: x is an ancestor of y *iff* every relation closed under the parent relation holds between x and y; however, 'x is an ancestor of y' and 'every relation closed under the parent relation holds between x and y' are not synonymous isomorphic.
13. The concept of logical consequence for propositions can be defined on analogy with the definition of logical validity for propositions found in §47.
14. And of course what is learned is not a variety of operational facts involving, e.g., ruler and compass; such facts, while not irrelevant, are only incidental.
15. Scope ambiguities provide a helpful analogy: treating them as semantic is inferior to treating them as structural.
16. The following example illustrates the purpose of clause (5). The unanalysed identify concept $[\underline{y = z}]_{yz}$ has two converses, the first one being a concept possessed by people who are sensitive to the fact that it is a converse of $[\underline{y = z}]_{yz}$ and the second one being a concept that can be possessed by people who are not sensitive to this fact. To mark this distinction, I use $[\underline{y = z}]_{zy}$ to denote the

former concept and $\underline{[y = z]_{zy}}$ to denote the latter concept. Generalizing on this, I allow that, for any formula A and any sequence of variables α, the following three intensional abstracts are well-formed: $[A]_\alpha$, $[\underline{A}]_\alpha$, $\underline{[A]_\alpha}$. In the limiting case where $[A]_\alpha$ is a normalized term, $[\underline{A}]_\alpha = \underline{[A]_\alpha}$ is valid; similarly, $[\underline{A}] = \underline{[A]}$ is valid.

17. In this semantics the denotation of the complex term, e.g., $[Gx \mathbin{\&} Hx]_x$ will be the defined concept that is the conjunction of G-ness and H-ness, and the denotation of the complex term $[\underline{Gx \mathbin{\&} Hx}]_x$ will be the undefined concept that is the conjunction of G-ness and H-ness. This suggests an easy solution to the problem of specifying appropriate model structures. Simply build these model structures so that they include two corresponding sorts of conjunction, one that yields defined concepts and one that yields undefined concepts. The same thing goes for each of the other fundamental logical operations, negation, existential generalization, etc. Now type 3 model structures, which are designed to model qualities and complex concepts (see §43), already contain two sorts of conjunction, negation, etc. However, in order to develop the second method mentioned in the text, one would have to construct still another type of model structure (called type $\underline{2}$). A type $\underline{2}$ model structure $\mathscr{M}$ is any structure

$$\langle \mathscr{D}, \mathscr{P}, \mathscr{K}, \mathscr{G}, \mathrm{Id}, \\ \mathrm{Conj}^{\mathrm{u}}, \mathrm{Neg}^{\mathrm{u}}, \mathrm{Exist}^{\mathrm{u}}, \mathrm{Exp}^{\mathrm{u}}, \mathrm{Inv}^{\mathrm{u}}, \mathrm{Conv}^{\mathrm{u}}, \mathrm{Ref}^{\mathrm{u}}, \mathrm{Pred}_0^{\mathrm{u}}, \mathrm{Pred}_1^{\mathrm{u}}, \ldots, \\ \mathrm{Conj}^{\mathrm{d}}, \mathrm{Neg}^{\mathrm{d}}, \mathrm{Exist}^{\mathrm{d}}, \mathrm{Exp}^{\mathrm{d}}, \mathrm{Inv}^{\mathrm{d}}, \mathrm{Conv}^{\mathrm{d}}, \mathrm{Ref}^{\mathrm{d}}, \mathrm{Pred}_0^{\mathrm{d}}, \mathrm{Pred}_1^{\mathrm{d}}, \ldots \rangle$$

that simultaneously satisfies the following three conditions. First, the elements of $\mathscr{M}$ are such that the following two diminished structures $\mathscr{M}_{\mathrm{u}}$ and $\mathscr{M}_{\mathrm{a}}$, respectively, are themselves type 2 model structures:

$$\langle \mathscr{D}, \mathscr{P}, \mathscr{K}, \mathscr{G}, \mathrm{Id}, \\ \mathrm{Conj}^{\mathrm{u}}, \mathrm{Neg}^{\mathrm{u}}, \mathrm{Exist}^{\mathrm{u}}, \mathrm{Exp}^{\mathrm{u}}, \mathrm{Inv}^{\mathrm{u}}, \mathrm{Conv}^{\mathrm{u}}, \mathrm{Ref}^{\mathrm{u}}, \mathrm{Pred}_0^{\mathrm{u}}, \mathrm{Pred}_1^{\mathrm{u}}, \ldots \rangle$$

$$\langle \mathscr{D}, \mathscr{P}, \mathscr{K}, \mathscr{G}, \mathrm{Id}, \\ \mathrm{Conj}^{\mathrm{d}}, \mathrm{Neg}^{\mathrm{d}}, \mathrm{Exist}^{\mathrm{d}}, \mathrm{Exp}^{\mathrm{d}}, \mathrm{Inv}^{\mathrm{d}}, \mathrm{Conv}^{\mathrm{d}}, \mathrm{Ref}^{\mathrm{d}}, \mathrm{Pred}_0^{\mathrm{d}}, \mathrm{Pred}_1^{\mathrm{d}}, \ldots \rangle.$$

Secondly, the ranges of the nine sort-u logical operations are disjoint from the ranges of the nine sort-d logical operations. The sort-d operations are to be thought of as those whose values are *defined* concepts, and the sort-u operations are to be thought of as those whose values are undefined concepts. The third condition is a bit cumbersome to state. Put roughly, it is that, for every undefined concept in $\mathscr{D}$, there is an associated concept in $\mathscr{D}$ that is fully defined; and conversely, for every defined concept in $\mathscr{D}$, there is an associated concept in $\mathscr{D}$ that is fully undefined.

18. This general style of resolution can be extended to cover difficult instances of the paradox of analysis such as those produced by analyses of extensional abstraction (§27), number (§32), intensional abstraction (§37), definite descriptions (§38), and so on.

I might mention that there is an entirely different technique for representing the ambiguity in intensional abstracts. For example, suppose that 'the proposition that, for all y, if Fy then Fy' and 'the proposition that, for all y, if Fy then $(Gy \mathbin{\&} Hy)$' denote the same type 2 entity on one reading and different type 2 entities on another. (The second reading is that which pertains to ignorance of definitions.) Then the second reading of 'the proposition that, for all y, if Fy then $(Gy \mathbin{\&} Hy)$' might be represented as follows: $[(\forall y)(Fy \supset y$ has the property that is the conjunction of $[Hz]_z$ and $[Gz]_z)]$. The type 2 intension denoted by this intensional abstract is clearly different from that which is denoted by $[(\forall y)(Fy \supset Fy)]$. Now, using this technique, one could contextually define the intensional abstracts of $\underline{L}_{\omega}$. However, in the text I adopt a primitive underlining technique, rather than this one, since it is readily axiomatizable and since it stears well clear of the logical paradoxes.

Chapter 4

1. I should mention that my remarks on higher-order logic will be aimed at higher-order logic in general. I will not discuss the special case of those second-order logics wherein strings as '$f = g$', '$p = q$', and 'xBp' are ill-formed. Given their inability to express identity (and non-identity) among the very entities over which they quantify, I find these second-order logics both unnatural and non-general. In addition, given their restrictions on the use of propositional variables these second-order logics are of little use in the treatment of modal and intentional matters.
2. The historical reason for selecting Δ to play this role is that, when type subscripts are supplied, Δ is the symbol that plays a somewhat related role in Church's formulation of the logic of sense and denotation. For heuristic purposes it might be helpful to think of the predication relation (the Δ-relation) as the property-theoretic analogue of the ϵ-relation from first-order set theory. However, unlike the ϵ-relation, which includes sets in its range, the predication relation includes properties and relations-in-intension in its range. I reserve judgement here on the Aristotelian question of whether the range of the predication relation includes, in addition to properties and relations-in-intension, objects from other metaphysical categories, e.g., individual particulars, species, quantities, actions, positions, locations, times, stuffs, etc. I do, however, envisage a global logical theory that takes the affirmative on this question. (See my 'Predication and Matter' for a partial defense of this position.) All that is important for the present purposes is that the range of the predication relation should include at least properties and relations. In this connection I also reserve judgement on the Aristotelian question of whether the copula in natural language expresses more than one relation. And, further, I reserve judgement on the Platonic question of whether the copula in natural language is satisfiable in varying degrees, from more perfect to less perfect. What matters for our purposes is that the relation expressed by Δ should include in its extension the relations having-as-a-property and standing-in-a-relation.
3. An attractive alternative first-order representation of this argument can be obtained by adapting Richard Grandy's theory of *anadic logic*. On Grandy's theory predicates need not have any fixed degree. Using this idea, one may treat Δ as a special kind of anadic predicate such that $v_1 \ldots v_n \, \Delta \, w_1 \ldots w_m$ is a well-formed formula, for $n \geq 1$ and $m \geq 0$. Accordingly, the argument in the text could be represented as follows:

$$\frac{xy \, \Delta \, [H^2(x, y)]_{xy} \;\&\; uv \, \not\Delta \, [H^2(x, y)]_{xy}}{\therefore \; (\exists w)(xy \, \Delta \, w \;\&\; uv \, \not\Delta \, w)}$$

   Thus, the notation for ordered pair is no longer needed. Although I find this approach to the treatment of predication quite attractive, I will for simplicity pursue the treatment given in the text.
4. For the first-order generalization of ramified type theory, see §26. Incidentally,

not only Russell's but also Church's higher-order logic can be constructed within a first-order theory of PRPs with Δ. Note here the relationship to David Kaplan's 'How to Russell a Frege-Church'. By the way, it is uncertain whether Kaplan's "Russellization" can be made to work for Church's Alternative (0), which is designed for intentional matters.

5. To see this, note that, e.g., $(\forall f)(f = f)$ and $(\forall f)(\forall g)(f = g \vee f \neq g)$ are typical laws of higher-order logic. In these sentences, linguistic predicates occur as linguistic subjects. By the way, I use 'linguistic subject' and 'linguistic predicate' to contrast with 'ontological subject' and 'ontological predicate'. My use comes close to Strawson's use of 'logical subject' and 'logical predicate'; see chapter 8 of his *Individuals*, for example.
6. The subject/predicate distinction plays much the same role in the syntax for first-order intensional languages.
7. Indeed, given the algebraic semantics for L_ω the following principle holds: if x is a k-ary intensional entity expressed by the linguistic predicate F^k_h, then x is denoted by the complex term $[F^k_h(v_1, \ldots, v_k)]_{v_1 \ldots v_k}$.
8. Frege is faced with a dilemma: either he cannot express the sample argument, or he too is committed to holding that red and blue are functions.
9. Some might expect that, since this first-order theory for the predication relation contains existence assertions, it is not really logic. (See pp. 195–7, Parsons, 'Frege's Theory of Number'.) In §36 I will argue against this point of view. For the present suffice it to say that even in the case of language L_ω without Δ all standard models are infinite and, indeed, they include denotable properties whose extensions include ω-sequences.
10. Whether there is a complete axiomatization for the valid$_1$ (valid$_2$) sentences of the first-order extensional language L (see §37) is an open question. If there is not, the chart—and, hence, the thesis of the present section—would need to be complicated accordingly.
11. See pp. 195–7, Parsons, 'Frege's Theory of Number' for a defense of this view.
12. Leśniewski stands as one of the few exceptions.
13. Henkin's quasi-completeness result for higher-order quantification theory provides another point of view on the question of the origin of incompleteness in higher-order theories. The relationship between these two points of view is a topic for further reflection.
14. When 'fx' is read aloud in English, we say, 'x is f'!
15. This rule and the Ref-rule are from the definition of what a model structure is; see p. 51 f.
16. The methodological discussion in §37 is relevant to the issue of whether classical logic should be tampered with here.
17. The same thing can easily be done for Quine's resolutions and for the more recent Fitch-Gilmore-Feferman resolution. (An idea analogous to Fitch's original insight lies behind Kripke's recent resolution of the Epimenides paradox in 'Outline of a Theory of Truth'.) For adaptation to the logic for L_ω with predication, Gilmore's lucid paper 'The Consistency of Partial Set Theory Without Extensionality' is ideal.
18. Strictly speaking this principle of predication is fashioned after the class-building principle in the Kelly-Morse set theory (appendix, Kelly, *General Topology*) rather than the von Neumann-Gödel-Bernays set theory. A strict von Neumann-style principle of predication is obtained by restricting the range of all quantified variables in the formula A to safe objects. See note 26.
19. In the setting of T2′ special care is required in formulating the power axioms. See note 27. By the way, a nice feature of the ZF-style theory is that in it $=$ can be defined in terms of Δ: $x = y$ *iff*$_{\text{df}}$ $(\forall z)(x \, \Delta \, z \equiv y \, \Delta \, z)$.

20. The ZF-style theory in the setting of T1 is consistent if ZF in the setting of first-order logic with extensional abstraction (see §15) is consistent. And the GB-style theory in the setting of T1 is consistent if Kelly-Morse set theory in the setting of first-order logic with extensional abstraction is consistent. But the relative consistency of the ZF and GB-style theories in the setting of T2′ remains to be studied.

By the way, it is widely believed that Frege's law V, i.e., $\{x: f(x)\} = \{x: g(x)\} \equiv (\forall x)(f(x) \equiv g(x))$, is responsible for the logical paradoxes in Frege's logic. True enough, the system does appear to be free of these paradoxes if law V is dropped. However, the sytem is also free of them if law V is retained and, instead, the higher-order variables and quantifiers are dropped, thereby converting the system into a first-order logic. The resulting system is virtually the same as first-order logic with extensional abstraction, which we know to be sound and complete. In that logic the first-order counterpart of law V, i.e., $\{x: A\} = \{x: B\} \equiv (\forall x)(A \equiv B)$, is just one of the axioms. This goes to show that Frege's higher-order syntax is as responsible for the logical paradoxes in his logic as is law V. Law V generates no such paradox if, as I have urged in the present chapter, first-order syntax is taken as canonical.

21. As indicated in §22, the following is an example of a definition of truth tailored to T2′:

$$Tx \; \textit{iff}_{\mathrm{df}} \; (\exists y)(x \approx_{\mathrm{N}} [(\exists z)z \,\Delta\, y]^{y} \;\&\; (\exists z)z \,\Delta\, y).$$

And the following is an example of a definition of truth tailored to T1:

$$Tx \; \textit{iff}_{\mathrm{df}} \; (\exists y)(x = [(\exists z)z \,\Delta\, y]^{y} \;\&\; (\exists z)z \,\Delta\, y).$$

The proofs that $T[A] \equiv A$ are straightforward proofs using pairing, abstraction, and null axioms, and $\mathscr{A}$17 or A8. It should be stressed, however, that neither of the above definitions is the philosophically motivated definition based on the correspondence theory of truth given in §45.

22. Corner quotes can be eliminated in favor of Gödel numbers, which can be defined in terms of Δ.

23. See footnote 25, p. 758, Church, 'Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski'.

24. So if T is defined as in the third line of note 21, then this occurrence of v_i cannot be bound by $(\exists v_i)$ in a context of the form

$$(\exists v_i)(v_k \approx_{\mathrm{N}} [(\exists v_j)v_j \,\Delta\, v_i]^{v_i} \;\&\; (\exists v_j)v_j \,\Delta\, v_i)$$

or

$$(\exists v_j)(v_k \approx_{\mathrm{N}} [(\exists v_i)v_i \,\Delta\, v_j]^{v_j} \;\&\; (\exists v_i)v_i \,\Delta\, v_j).$$

25. For remarks relevant to this distinction between predicative and impredicative principles of predication, see pp. 52–4, Chihara, *Ontology and the Vicious-Circle Principle*, and pp. 758–60, Church, 'Comparison'.

26. This predicative principle of predication is the intensional analogue of the von Neumann, as opposed to the Kelly-Morse, class-building principle. See note 18.

27. Special adjustments are needed in the counterparts of the ZF and GB power principles:

(ZF-Style Power Principle)
$$(\exists y)(\forall x)((\exists z)(z \doteqdot x \;\&\; z \,\Delta\, y) \equiv x \subseteq w)$$

(GB-Style Power Principle)
$$(\exists b)(\forall a)((\exists c)(c \doteqdot a \;\&\; c \,\Delta\, b) \equiv a \subseteq d)$$

where $\alpha \doteqdot \beta$ *iff*$_{df}$ $(\forall\delta)(\delta \,\Delta\, \alpha \equiv \delta \,\Delta\, \beta)$ and $\alpha \subseteq \beta$ *iff*$_{df}$ $(\forall\delta)(\delta \,\Delta\, \alpha \supset \delta \,\Delta\, \beta)$. Special adjustments are also needed in the counterparts of the ZF replacement axiom as follows:

(ZF-Style Replacement Principle)

$$(\forall x, y, z)((A(x, y) \,\&\, A(x, z)) \supset y = z) \supset$$
$$(\exists w)(\forall y)((\exists z)(z \doteqdot y \,\&\, z \,\Delta\, w) \equiv (\exists x)(x \,\Delta\, u \,\&\, A(x, y)))$$

where w is distinct from y and is not free in A. (This principle could, if necessary, be restricted further by permitting no predicates beyond Δ and $=$ to occur in A.) The GB-style replacement principle can be formulated on analogy with the ZF-style replacement principle.

28. On the semantical theory defended in §38 there is only one fundamental kind of semantical relation—namely, meaning—and all other semantical relations are derivative, being definable in terms of this one kind of meaning. If this theory is right, there is no reason to treat explicitly any semantical paradoxes beyond those generated by this one kind of meaning.

There is also no need to impose the indicated restrictions concerning grounded formulas in the setting of conception 1. The reason is that both meaning and intentionality are conception 2 phenomena. (The thesis that meaning is a conception 2 phenomenon is defended in §38. This thesis, coupled with the thesis that there is just one underlying kind of meaning, frees us to construct unrestricted ZF and GB-style theories of conception 1 PRPs. These conception 1 theories make it possible to give an especially simple no-class construction of pure ZF and GB set theory (see §31), and this in turn simplifies the argument of §§30–2. But the conclusion of that argument can also be won in the setting of conception 2; see note 16 in the next chapter for an indication of how to do this by means of an attractive informal argument.)

In comparison with Tarski-style and Russell-style resolutions of the semantical paradoxes, the proposed resolution permits a language to have a single univocal meaning predicate and single univocal truth predicate rather than an infinite hierarchy of meaning and truth predicates. Such infinite hierarchies of meaning and truth predicates lead to violations of Davidson's finite learnability requirement (see desideratum 13 in §4). And in comparison with a resolution of the semantical paradoxes that is fashioned after Kripke's technique in 'Outline', the resolution proposed here permits a meaning predicate M^2 that suffers from no "gaps": $M^2(\ulcorner A(\alpha)\urcorner, [A(\alpha)]_\alpha)$ holds for all L_ω-formulas $A(\alpha)$ even where M^2 is itself a predicate in $A(\alpha)$. A further advantage this resolution has over those given in the style of Tarski or Kripke is that it does not regard the semantical and intentional paradoxes as unrelated phenomena; rather it resolves them both by one and the same account.

If certain arguable modal principles about meaning (e.g., x means $y \supset \Box x$ means y) were adjoined to the ZF and GB-style theories described in the text, then certain new semantical paradoxes could be derived. But this threat evidently can be shortcircuited by restricting T2′ axiom $\mathscr{A}17$ to grounded formulas A_u and B_u: $\Box(A_u \equiv_\alpha B_u) \equiv [A_u]_\alpha \approx_N [B_u]_\alpha$. (Analogous restrictions could, if necessary, be imposed elsewhere in T2′.) Even with this restriction, the schema $T[A_u] \equiv A_u$ can still be derived for the univocal truth predicate T which is definable in the theory.

29. A somewhat related pragmatic resolution of the semantical paradoxes is suggested in Charles Parsons' 'The Liar Paradox'. Tyler Burge also argues for a pragmatic resolution in 'Semantical Paradox'; however, he locates the pragmatic element in an indexical slot in a truth predicate 'true$_i$' for sentences. In my tentative resolution the truth predicate for propositions is treated as a 1-place

predicate definable in terms of Δ and $\approx_N$; the pragmatic element is instead located in the contextually determined implicit universe of discourse u. Although Burge criticizes (footnote 13, p. 176) Parsons' shift-in-the-domain-of-discourse resolution, Burge's resolution may be viewed as a derivative form of the version of the shift-in-the-domain-of-discourse resolution that I have described. To see this, notice that a truth predicate 'true$_u$' for sentences, where u is an indexical slot, is definable as follows: true$_u(x)$ *iff*$_{df}$ $(\exists y)(y \, \Delta \, u \,\&\, M^2(x, y) \,\&\, Ty)$. Here u limits not the universe of *sentences* but rather the universe of *propositions*.

30. For a discussion of an analogous relationship between ordinary ZF and simple type theory, see pp. 266–86, Quine, *Set Theory and Its Logic*, revised edition.
31. Let '$A(x, y)$' abbreviate:

$$\text{formula } x \text{ expresses set } y \,\&\, \langle x, y\rangle \in \{xy: x \text{ is a formula } \&\ y = \emptyset\}.$$

And let '"$A(x, y)$" expresses the set $\{xy: A(x, y)\}$' be adopted as an axiom. (Assume that 'is a formula' has been defined in terms of Gödel numbers, which in turn are defined in terms of ϵ; assume also that $\{xy: \ldots\}$ is contextually defined in terms of ϵ. Then one can easily derive contradictions in ZF and GB set theories even when their comprehension schemas $(\exists y)(\forall x)(x \in y \equiv (x \in u \,\&\, B(x)))$ and $(\exists y)(\forall a)(a \in y \equiv B(a))$ are restricted to "grounded" set-theoretical formulas B (i.e., set-theoretical formulas B whose quantified variables are all restricted in their ranges to antecedently given sets). No analogous contradiction is derivable in T2′ from the proposed ZF-style or GB-style axioms for Δ. The reason is that there is no principle of extensionality in these logics for intensional entities.

Chapter 5

1. Later in this section I give an example of how the idea of set might be "genetically" related to ideas of certain naturalistic objects. Incidentally, in the present discussion I do not assume that the naturalistic ontology of packs, bunches, flocks, etc. is justified. The point rather is that, if this ontology is justified, that would confer no justification on the ontology of sets since sets are quite unlike packs, bunches, flocks, etc. I should also mention that since writing this section I have learned that Ruth Barcan Marcus makes many similar points in her 'Classes, Collections, and Individuals'.
2. For another sort of problem, suppose that by time t some given bunch of grapes has dwindled down to a single grape. Does the bunch still exist? If so, is the bunch = the grape? I'm not sure. But notice that in set theory the answers are already prescribed: the singleton of the grape exists, and it is not the same thing as the grape. How bizarre singleton sets and null sets are.
3. This difference gives rise to another: the time invariant principle of extensionality, which is supposed to be valid for the sets of set theory, is not valid for ordinary collections, social classes, and ordinary sets. Consider art collections. It is in principle possible that the Tate collection should contain at t exactly those art works that the Guggenheim collection contains at $t'(t \neq t')$ and yet that the Tate collection and the Guggenheim collection should always remain distinct.
4. It might be objected that what I say in the text in this and in the following paragraph results from a confusion between membership ($\in$) and inclusion ($\subseteq$). However, this objection begs the question. The alleged membership/inclusion distinction is a set-theoretical distinction, yet what is presently at issue is whether set-theoretical concepts can be justified by appealing to the ontology of

ordinary collections, social classes, and ordinary sets. In controversies such as this, one has no choice but to fall back on naive intuition, and naive intuition concerning the relation of being in ordinary collections, social classes, and ordinary sets provides *prima facie* evidence for my conclusions.

By the way, doubts about the $\in/\in$-inclusion distinction in set theory do not carry over to the Δ/Δ-inclusion distinction in the theory of PRPs.

5. What is the aggregate of things that are not in themselves?
6. See, e.g., Tarski, 'On the Geometry of Solids'.
7. Not unrelatedly, Gödel attempts to justify set theory on grounds of intuitions about what he calls "pluralities", pp. 137 ff., 'Russell's Mathematical Logic' (pp. 220 ff. in Benacerraf and Putnam).
8. I am inclined to treat plurals in this way. For example, the second problematical sentence cited a moment earlier in the text might be provisionally thought of as follows:

   $(\exists x, y)(x =$ (being a) county & $y =$ (being a) state & the-x-aggregate occupies the same territory as the-y-aggregate & the-x-extension outnumbers the-y-extension & the-typical-x resents federal intervention more than the-typical-y).

   The expressions ⌜the-α-aggregate⌝, ⌜the-α-extension⌝, and ⌜the-typical-α⌝ can then be treated as contextually defined operators on α. For example, if ⌜the-α-extension⌝ is represented by ⌜$\{\beta: \beta \, \Delta \, \alpha\}$⌝, it can be contextually defined in the way suggested in the text a bit later. Incidentally, it might be crucial that α ranges over properties rather than sets, given the extensionality of sets. Is it not true that the typical policeman $\neq$ the typical short-order cook (or at least that the ideal policeman $\neq$ the ideal short-order cook) and that this would be so even if, because of widespread moonlighting practices, all and only policemen coincidentally turned out to be short-order cooks?
9. Note, this is just the first-order version of Frege's law V. See note 20, chapter 4.
10. This is of course part of Russell's theory of meaning. Russell's theory is unlike the Fregean theory according to which, not only do predicates and formulas express something, but also they name something. I defend Russell's theory over Frege's in §38.
11. This is just the first-order analogue of Russell's higher-order "no-class" definition of extensional abstracts. So it is clear that the Russellian theory of meaning and the no-class theory are of a piece. Incidentally, even Quine acknowledges, 'Classes may be thought of as properties in abstraction from any differences which are not reflected in differences of instances' (pp. 120–1, *Mathematical Logic*, revised edition).
12. Question: on this account what is the primary semantical correlate of the extensional abstract $\{v_i: Av_i\}$? Answer: nothing in particular because on this account the extensional abstract $\{v_i: Av_i\}$ is only an *indefinite description* of a property (specifically, a property that is co-extensive with the property expressed by the formula Av_i).
13. I speak of extensional semantics in the sense of Tarski, Carnap, and their followers.
14. Or *ideal*, e.g., the typical county, the typical state, the ideal policeman, etc. See note 8.
15. These no-class constructions entail that ZF (GB) with extensionality is consistent if ZF (GB) without extensionality is consistent. These relative consistency results differ from those obtained by other authors. For more on this topic, see notes 23 and 26. By the way, the no-class construction of GB can be adapted to obtain a no-class construction of—and a relative consistency result for—Kelly-Morse set theory.

16. There is a more direct, though less rigorous, way to win this conclusion. (See my 'Foundations Without Sets' for elaboration of this line of argument.) One need only show that pure and applied set theory can be interpreted informally as theories of properties. This can be done for the pure set theories ZF and GB by informally interpreting them as theories of an appropriate kind of property, for example, pure L-determinate type 1 properties. (Property x is L-determinate *iff* $\Box(\forall y)(y \,\Delta\, x \supset \Box y \,\Delta\, x)$, and x is pure L-determinate *iff* x is L-determinate, the instances of x are L-determinate, their instances are L-determinate, and so on all the way down.) Since type 1 properties are identical if necessarily equivalent, pure L-determinate properties will be identical if they have the same pure L-determinate instances. But this is just what the axiom of extensionality says when pure set theory is interpreted as a theory of pure L-determinate properties, so a universe of pure L-determinate properties validates this axiom. (Pure set theory can also be interpreted as a theory of a special kind of conception 2 property; see note 18 for an example.) Applied set theory, on the other hand, can be interpreted as a theory of properties in which identity for empirical sets is read simply as equivalence for empirical properties. When the principle of extensionality for empirical sets is interpreted this way, it is a trivial tautology. It will become clear that the no-class constructions in the text are simply formalizations of these informal interpretations derived within ZF and GB-style logics for the predication relation.
17. See George Boolos, 'The Iterative Conception of Set'.
18. Someone might doubt that the property $[y$ is an L-determinate property whose instances are instances of a property formed at a stage prior to $\alpha]_y$ has "enough" instances to validate the relevant Zermelo-style axioms. This doubt is unfounded, however, for this property is necessary equivalent to:

    $[(\exists u)(u$ is an aggregate of properties that are instances of a property formed at a stage prior to α & $y = [v$ is a property in $u]_v)]_y$.

    Here I use the notion of *aggregate* which was characterized in the previous section. (The notion of *sum* from an unrestricted part/whole logic would serve our purposes equally well.) By using the notion of aggregate (sum), one obtains a property that clearly has "enough" instances to validate the relevant Zermelo-style axioms. At the same time, one avoids the sort of circularity found in the set-theoretical motivation that Russell gave for the axioms of reducibility in the setting of his no-class theory (see pp. 80–3, 'Mathematical Logic as Based on the Theory of Types', *Logic and Knowledge*).

    By the way, if one were to use the formulation given in this note, one could also construct an iterative hierarchy of type 2 properties that validates a ZF-style theory for conception 2. Such a theory would be especially congenial with the picture of concepts given in chapter 8.

    Since Platonists object to the idea that PRPs, type 1 or type 2, are really "formed", one might better think of these hierarchies as stage-by-stage certifications of sub-portions of the extension of the predication relation over the field of PRPs.
19. Here $\subseteq$ is defined in terms of Δ: $u \subseteq v$ *iff*$_{\mathrm{df}}$ $(\forall w)(w \,\Delta\, u \supset w \,\Delta\, v)$. Recall from §22 that in T1 x is a property *iff*$_{\mathrm{df}}$ $(\exists y)x = [z \,\Delta\, y]^y_z$ and from §29 that x is L-determinate *iff*$_{\mathrm{df}}$ $\Box(\forall y)(y \,\Delta\, x \supset \Box y \,\Delta\, x)$.
20. Sentence A may be taken from ZF or GB. If it is taken from GB, then I will assume that its special set variables $a, b, c, \ldots$ are contextually defined in terms of $\in$.

21. (Comprehension) $x \Delta [x \Delta u \,\&\, A]^u_x \equiv (x \Delta u \,\&\, A)$
(Null) $x \Delta [x \neq x]_x \equiv x \neq x$
(Pairing) $x \Delta [x = u \vee x = v]^{uv}_x \equiv (x = u \vee x = v)$
(Union) $x \Delta [(\exists z)(x \Delta z \,\&\, z \Delta u)]^u_x \equiv (\exists z)(x \Delta z \,\&\, z \Delta u)$
(Power) $x \Delta [(\forall z)(z \Delta x \supset z \Delta u)]^u_x \equiv (\forall z)(z \Delta x \supset z \Delta u)$
(Infinity) $(\exists v)([x \neq x]_x \Delta v \,\&\, (\forall z)(z \Delta v \supset [w \Delta z \vee w = z]^z_w \Delta v))$
(Replacement) $(\forall xyz)((A(x, y) \,\&\, A(x, z)) \supset y = z) \supset$
$(y \Delta [(\exists x)(x \Delta u \,\&\, A(x, y))]^u_y \equiv_y (\exists x)(x \Delta u \,\&\, A(x, y)))$
(Regularity) $(\forall x)((\exists y) y \Delta x \supset (\exists y)(y \Delta x \,\&\, (\forall z)(z \Delta x \supset z \not\Delta y)))$.
These axioms do not conflict with the resolution of the intentional and semantical paradoxes given in §26. For I am presently working in the setting of conception 1 whereas the intentional and semantical paradoxes—and their resolution—fall within conception 2. If conception 2 entities were brought in directly or indirectly, qualifications would be in order.

22. T1 rules R1–R3 may be applied directly to any axiom in TZF^-. Note that TZF^- is consistent if ZF in the setting of first-order logic with extensional abstraction (see §15) is consistent.

23. A common way to give a consistency proof for one theory relative to another is to model the first theory within the second. This technique is frequently used to prove that set theory is consistent if a given sub-theory with fewer axioms is consistent. We have seen that the relevant difference between set theory and the logic for the predication relation lies in the presence or absence of the axiom of extensionality. So the logic for the predication relation will model set theory if set theory with the axiom of extensionality can be proved in this way to be consistent relative to set theory without that axiom. In 'More on the Axiom of Extensionality' Dana Scott shows that ZF with extensionality cannot be proved consistent relative to ZF without extensionality, at least when no abstraction operation is taken as primitive. The present result has the force of proving the desired relative consistency for ZF when an abstraction operation—either intensional or extensional—is taken as primitive. By the way, if modifications in the statement of the original ZF axioms are permitted, ZF with extensionality can be proved consistent relative to ZF without extensionality. For example, Scott reports this for a modified (but equivalent) formulation of ZF in which $(\forall v)(v \in y \equiv v \in z)$ replaces $y = z$ in the antecedent of the replacement axiom. Indeed, ZF can be proved consistent relative to a certain modified ZF-style intuitionistic set theory that lacks the axiom of extensionality (Harvey Friedman, 'The Consistency of Classical Set Theory Relative to a Set Theory with Intuitionistic Logic'). All these relative-consistency results may double as no-class constructions of ZF.

24. (Comprehension) $a \Delta [A(v)]_v \equiv A(a)$
(Null) $(\exists a)(\forall b)(b \Delta a \equiv b \neq b)$
(Pairing) $(\exists a)(\forall b)(b \Delta a \equiv (b = c \vee b = d))$
(Union) $(\exists a)(\forall b)(b \Delta a \equiv (\exists c)(b \Delta c \,\&\, c \Delta d))$
(Power) $(\exists a)(\forall x)(x \Delta a \equiv x \subseteq b)$
(Infinity) $(\exists a)([x \neq x]_x \Delta a \,\&\, (\forall b)(b \Delta a \supset [y \Delta b \vee y = b]^b_y \Delta a))$
(Replacement) $(\forall bcd)((\langle b, c\rangle \Delta x \,\&\, \langle b, d\rangle \Delta x) \supset c = d) \supset$
$(\exists a)(\forall c)(c \Delta a \equiv (\exists b)(b \Delta e \,\&\, \langle b, c\rangle \Delta x))$
(Regularity) $(\forall x)((\exists a) a \Delta x \supset (\exists a)(a \Delta x \,\&\, (\forall b)(b \Delta x \supset b \not\Delta a)))$.
Here $a, b, c, \ldots$ are contextually defined variables ranging over safe entities, i.e., entities that have properties; all quantified variables in the formula A are restricted in their range to safe entities, and v is free for a in A and conversely. The supplementary remarks in note 21 also apply *mutatis mutandis* to these GB-style axioms.

25. T1 rules R1–R3 may be applied directly to any axiom in TGB^-. Note that TGB^- is consistent if GB in the setting of first-order logic with extensional abstraction (see §15) is consistent.
26. One might think that a no-class construction of GB is already at hand due to a relative consistency result of R. O. Gandy ('On the Axiom of Extensionality', part two). Gandy showed that a certain modified version of von Neumann-Gödel-Bernays class theory with extensionality is consistent relative to that theory without extensionality. This result does not provide what we need, though. Gandy's modified version of GB contains a primitive extensional abstraction operator (λv_i) for which $(\lambda v_i)A = (\lambda v_i)B \equiv (\forall v_i)(A \equiv B)$ holds whenever $(\lambda v_i)A$ and $(\lambda v_i)B$ are well-formed. Thus, (λv_i) behaves like the primitive class-abstraction operator $\{v_i: \ldots\}$. By adapting Dana Scott's result (see note 23), one evidently can show that Gandy's proof fails if the abstraction operator (λv_i) is not taken as primitive but instead is contextually defined in terms of ϵ on analogy with the contextual definition of $\{x: \ldots\}$ that I suggested in §28. But what we need is a theory lacking extensionality which can model class theory with extensionality and in which all extensional abstraction operators can be contextually defined. So Gandy's result seems not to do the job. The intensional abstraction operation of L_ω is what is wanted in the no-class construction of GB.
27. A similar no-class construction is also possible for a GB-style applied set theory. And no-class constructions for pure and applied set theory are possible in the setting of conception 2.

Chapter 6

1. In place of (5) he might say, equivalently, that for all properties z of natural numbers, if 0 has z and the successor of each natural number having z itself has z, then every natural number has z.
2. Recall that the first-order intensional logic T2—unlike, say, first-order quantifier logic—is committed to an infinite ontology of PRPs. By the way, the neo-Fregean definitions work in the setting of the logic T1. However, in order to derive Peano's postulates in T1, logical principles for Δ must be adjoined. I do not discuss the T1 approach in the text only because the T2 approach is so neat.
3. This is not true for Russell, according to whom sets do not really exist; Russell's entities are propositional functions. This is not just a scholarly point, for with a little fiddling Russell can easily avoid the criticism.
4. Note that 'are' is the plural form of the copula 'is'.
5. Recall that since the extensional abstract $\{v: Av\}$ is contextually defined in terms of Δ, its use carries ontological commitment to properties, not to sets.

   I emphasize, however, that this treatment of extensional abstracts is only tentative. One attractive alternative is to treat extensional abstracts as denoting ordinary aggregates. Someone who adopts this alternative would be led naturally to the sort of conclusion reached in Glenn Kessler's 'Frege, Mill, and the Foundations of Arithmetic', namely, that numbers are relations-in-intension holding between ordinary aggregates and properties (where the role of these properties is to provide a principle by which to identify—and hence, to count—things in the aggregates). This, though, is only a slight variation on the logicist position I am defending. Numbers still would be intensional entities, and if the "aggregate slot" in a relation that is a number is treated as a certain kind of parameter, then number theory still can be construed as part of pure intensional logic with Δ.

6. Constructions involving 'the number of *F*s', 'as many as', 'more than', 'less than', and other verbal forms from the idiom of cardinality can be easily defined by neo-Fregean means in L_ω using contextually defined definite descriptions, extensional abstracts, and Δ. Implicit in these definitions is Frege's well-argued, rather uncontroversial thesis that the natural numbers are cardinal numbers as opposed to ordinal numbers or quantities in the sense of amounts. To my knowledge there are no good arguments for either of these opposing views.
7. See pp. 57–8, Paul Benacerraf, 'Numbers'.
8. Near the end of the paper Benacerraf makes a positive proposal about how to analyse number-theoretic language, not in terms of particular objects called numbers, but *indefinitely* in terms of whole structures that behave in the appropriate way. But if the argument just summarized in the text were valid, then evidently a variant of it would apply against this positive proposal:

   There are many different ways—e.g., Frege's original way, the neo-Fregean way, Benacerraf's way, etc.—in which, for all we know, number-theoretic language could be correctly analysed.

   ∴ Number-theoretic language could not be correctly analysed in any of these ways.

9. This fact is what guided Frege's original research into the analysis of the natural numbers. And it is evidently one that Paul Benacerraf would accept, for he endorses '. . . the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallels the semantics for the rest of language'. (p. 661, Benacerraf, 'Mathematical Truth')
10. I should mention that Benacerraf does consider sentences of the form 'The *F*s are *n*' on pp. 58–60, where he says that, e.g., 'The lions in the zoo are seventeen' probably comes into the language by deletion from 'The lions in the zoo are seventeen in number', which in turn probably derives from something like 'Seventeen lions are in the zoo'. However, in view of the fact that the problem raised by Benacerraf's criticism is a species of the indeterminacy problem in logico-linguistic theory, one wonders whether it is consistent to use these assertions in an argument against logicism. (See note 8 above.) Waiving this reservation, however, one would like to know what is the logical form of 'Seventeen lions are in the zoo' and what is the analysis of 'seventeen' as it occurs in this sentence. The answer should permit an account of the inference from 'There are seventeen lions in the zoo' and 'Fifteen plus two is seventeen' to 'There are fifteen plus two lions in the zoo' and also an account of the equinumerosity principle stated earlier in this section. This indicates that, even if numerical adjectives in natural language were operators, as Benacerraf suggests (p. 60), they must nonetheless have semantical correlates that behave with respect to each other in exactly the same way that the natural numbers do. This seems to be reason enough for identifying the semantical correlates of numerical adjectives with the natural numbers themselves. In any event, does not the "arithmetic" for the semantical correlates of numerical adjectives fall squarely within the province of natural logic, and is this not all the logicist needs to make good his philosophy? If so, why not eschew the operator approach to numerical adjectives and return to the essentially simpler neo-Fregean theory?

    Incidentally, if one thinks a bit about the interesting grammatical phenomena cited by Benacerraf on p. 60 (center), one sees that they can be nicely predicted by the neo-Fregean theory.
11. A complete set of axioms for $\langle NN, =, 0, ' \rangle$ consists of (1′)–(4′) plus the following: (where n stands for n consecutive occurrences of $'$)

(6) $(\forall x)(NNx \supset (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y')))$
(7.1) $(\forall x)x \neq x'$
(7.2) $(\forall x)x \neq x''$
⋮
(7.n) $(\forall x)x \neq x^n$
⋮

See, e.g., §3.1 'Natural Numbers with Successor' in Enderton, *A Mathematical Introduction to Logic* and Quine's comments on axioms (7.*n*), p. 99 in *From a Logical Point of View*. Axiom (6) follows from (1′), (2′), (5′), plus the validity under discussion in the text. Axioms (7.*n*) follow from axiom $\mathscr{A}12$.

12. See, e.g., Landau, *Foundations of Analysis*, for a set-theoretic construction of number theory with + and · and of real and complex analysis beginning with just Peano's postulates.
13. Indeed, identity and necessary equivalence are definable in terms of predication. For the definition of =, see note 19, chapter 4. For the definition of $\approx_N$, see §46.
14. On this point, the logicism I am defending is possibly closer to Russell's than Frege's, for Russell makes it clear that his logicism requires only that the truths of mathematics be logical validities.
15. Question: in a Zermelo-style theory how many singleton properties are Δ-instances of the property with which the number 1 has been identified on my neo-Fregean analysis? Answer: as many as you like up to unsafe points.
16. Strictly speaking, the Kelly-Morse style theory (see appendix, Kelly, *General Topology*).
17. Allusions to this view can be found in Gödel's paper 'Russell's Mathematical Logic', pp. 137 ff. (pp. 220 ff. in Benacerraf and Putnam).
18. On Hilbert's view, like Frege's view before it, proof is the key to the account of our knowledge of complex logical truths. I am doubtful that the role of proof is as great as Hilbert and Frege thought. But such doubts do not affect the argument I give in the text.
19. On the justification of the laws of arithmetic (as opposed to analysis and set theory), Hilbert himself would give a reply that falls back on intuitions.
20. For more on this see p. 75 ff., Chihara, *Ontology*, and p. 674, Benacerraf, 'Mathematical Truth'.

Chapter 7

1. In *Meaning and Necessity* Carnap does not take this attitude toward all intensionality in language. In particular, he gives a fully extensional ("formal-mode") account of 'belief'-sentences. This account is a descendant of the account given in his *The Logical Syntax of Language*, which in turn appears to have been derived from the account given in Wittgenstein's *Tractatus*. The fact that in *Meaning and Necessity* Carnap offers no unified account of all intensionality in language would seem to be a count against his theory.
2. Russell had a rather complicated theory of extensionality and intensionality in language. For Russell there did exist certain *prima facie* cases of intensionality and extensionality that were not at all what they seemed to be. For example, all *prima facie* violations of Leibniz's law were deemed only apparent and were explained away by means of the theory of descriptions. In a similar fashion, Russell also explained away the *prima facie* extensionality generated by exten-

sional abstracts. This he did by means of his no-class analysis. But Russell held that there existed some genuine intensionality in language, for he took at face value all the usual *prima facie* violations of the principle of the substitutivity of equivalents. At the same time, he held that there is some genuine extensionality in language. Specifically, he held that Leibniz's law is universally valid. And, in the same vein, he held that the logical connectives are extensional in the sense that they are truth-functional and, hence, that the principle of the substitutivity of equivalents holds for all contexts that are exclusively built up by means of them. Thus, in the final analysis Russell subscribed to the view that language does bifurcate into two ultimate kinds, intensional and extensional. In this, his view is on a par with the views of C. I. Lewis, the Carnap of *Meaning and Necessity*, Hintikka, Montague, Kripke, *et al.*

3. In linguisitc theory the conflict between interpretive and generative semantics is somewhat similar in flavor.
4. Strictly speaking, definite descriptions are formed by applying an unanalysed term-forming operator to what Frege calls concept-names. However, this fine point is immaterial here. Incidentally, in the text I confine my comments to Frege's theory of definite descriptions in natural language. His treatment of definite descriptions in the formal language of the *Grundgesetze* is different; a special variant of the "favorite-object" approach is taken there on the matter of vacuous descriptions.
5. Analogously, it appears that in 'Quantifying In' David Kaplan praises Frege's theory of intensionality from the conservative point of view while in 'What is Russell's Theory of Descriptions?' he criticizes Russell's theory of descriptions from the liberal point of view. At many points in contemporary work in modal logic there also appear to be methodological vacillations over these issues.
6. See also note 14, chapter 1.
7. Apparent violations of Leibniz's law produced by extensional abstracts and functional constants can be explained away by analogous means. By the way, Russell's theory of descriptions is not essential to the program in the text. It would be possible, though more complex, to treat definite descriptions much as Frege does. However, that would force me to enrich my algebraic model structures with appropriate new logical operations to handle definite-description concepts (and, then, to make adjustments for certain new propositions that would lack a truth value).
8. The intensional language $\underline{\mathrm{L}}_{\omega}$, introduced in §20, can be translated into a finite-based first-order extensional language in much the same way as L_{ω}.
9. There are only heuristic reasons for choosing the symbols $\bar{F}_i^j$ for the names I need; any arbitrary symbol would suffice as long as one gives L the prescribed interpretation. Incidentally, when one interprets L as intended, no instances of Frege's '$a = a$'/'$a = b$' puzzle arise. (Frege's puzzle is discussed in the following section.)

   In what follows I will show how contextually to define intensional abstracts that contain predicates selected from $F_1^1, \ldots, F_p^q$. One could extend this method to intensional abstracts that contain, in addition, predicates selected from $\mathscr{Conj}^3, \ldots, \mathscr{Pred}^3$. One would adjoin $\overline{\mathscr{Conj}}, \ldots, \overline{\mathscr{Pred}}$ to the primitive names of L. This step calls for complications in the specification of the semantics of L, for in this region one treads close to certain logical paradoxes. I should mention also, that there exist simplifications of the construction given in the text. For example, all predicates beyond Δ might prove to be definable in terms of appropriate primitive names plus Δ. The construction, therefore, should not be thought of as definitive; I give it in order to show that the thesis of extensionality is defensible.