2

Intensional Logic

Intensional logic has never been completely and adequately formulated. To be convinced of this, consider two representative arguments:

Whatever x believes y believes.
x believes that A.
∴ y believes that A.

Being a bachelor is the same thing as being an unmarried man.
It is necessary that all and only bachelors are bachelors.
∴ It is necessary that all and only bachelors are unmarried men.

Neither of these intuitively valid arguments is even expressible in standard first-order predicate logic, even when epistemic and modal operators are adjoined. And while it is true that both of these arguments can be expressed in certain higher-order intensional logics, such higher-order logics are essentially incomplete, to mention just one of their shortcomings. But things are better than they might seem. When an intensional abstraction operation is adjoined to first-order logic, the result is an intensional logic that is equipped to represent the above arguments—and indeed, nearly all problematic intensional arguments. At the same time, unlike higher-order intensional logics, this first-order intensional logic is, surprising as it might seem, provably complete.

In what follows I will show how to construct such a logic. The construction requires the development of both a new formal language and a new semantic method. The new semantic method does not appeal to possible worlds, even as a heuristic. The heuristic used is simply that of properties, relations, and propositions, taken at face value. And unlike the various possible-worlds approaches to intensional logic, the approach developed here is adequate for treating both modal and intentional matters. Initially, the intensional logic will have two parts, one for each of the traditional conceptions of PRPs identified in §2. At the end of this chapter the two parts will be integrated.

12. A Formal Intensional Language

I begin by specifying the syntax for a first-order language with intensional abstraction. This language will be called L_ω. Primitive symbols:

Logical operators:	&, $\neg$, $\exists$
Predicate letters:	$F_1^1, F_2^1, \ldots, F_p^q$
Variables:	$x, y, z, \ldots$
Punctuation:	(,), [,].

Simultaneous inductive definition of *term* and *formula* of L_ω:

(1) All variables are terms.
(2) If $t_1, \ldots, t_j$ are terms, then $F_i^j(t_1, \ldots, t_j)$ is a formula.
(3) If A and B are formulas and v_k a variable, then $(A \,\&\, B)$, $\neg A$, and $(\exists v_k)A$ are formulas.
(4) If A is a formula and $v_1, \ldots, v_m$, $0 \leq m$, distinct variables, then $[A]_{v_1 \ldots v_m}$ is a term.

In the limiting case where $m = 0$, $[A]$ is a term. All and only formulas and terms are *well-formed* expressions. An occurrence of a variable v_i in a well-formed expression is *bound* (*free*) if and only if it lies (does not lie) within a formula of the form $(\exists v_i)A$ or a term of the form $[A]_{v_1 \ldots v_i \ldots v_m}$. A variable is *free* (*bound*) in a well-formed expression if and only if it has (does not have) a free occurrence in that well-formed expression. A *sentence* is a formula having no free variables. The predicate letter F_1^2 is singled out as a distinguished logical predicate, and formulas of the form $F_1^2(t_1, t_2)$ are to be rewritten in the form $t_1 = t_2$. $\forall$, $\supset$, $\supset_{v_i \ldots v_j}$, $\vee$, $\equiv$, $\equiv_{v_i \ldots v_j}$ are to be defined in terms of $\exists$, &, and $\neg$ in the usual way. If v_i occurs free in A and is not one of the variables in the sequence of variables α, then v_i is an externally quantifiable variable in the term $[A]_\alpha$. Let the sequence δ be, in order, the externally quantifiable variables in $[A]_\alpha$; then $[A]_\alpha$ will sometimes be rewritten as $[A]_\alpha^\delta$ so that these variables can be identified at a glance.

Some observations are in order. First, on the intended informal interpretation of L_ω, a singular term $[A]_{v_1 \ldots v_m}$ denotes a proposition if $m = 0$, a property if $m = 1$, and an m-ary relation-in-intension if $m \geq 2$. Secondly, L_ω differs from a standard first-order language only in having these singular terms $[A]_{v_1 \ldots v_m}$. Thirdly, L_ω has a finite number of primitive constants, and hence, it satisfies desideratum 13, Davidson's learnability requirement. Of course, for purely mathematical purposes, one is free to adjoin an infinite number of additional primitive constants to L_ω. Yet if Davidson is right, such infinitistic extensions of L_ω will not qualify as idealized representations of natural language. Fourthly, L_ω contains no primitive names. My strategy with regard to names will be to proceed in two stages. First, I will study the logic of intensional language without names; that is, I will study the logic of L_ω as its stands. Once this task is completed, I will take up the question of how to treat names. There are two main competing theories of names—Frege's theory and Mill's theory. According to Frege's theory, names have descriptive content; according to Mill's theory, they do not. In §§38–9 it is shown that, given either theory, names can be successfully treated in the setting of L_ω. And finally, L_ω contains no functional constants: these are superfluous in L_ω since they can be contextually defined in terms of = and appropriate auxiliary predicates.[1]

Now let us reconsider the intuitively valid arguments mentioned at the outset of the chapter. In L_ω they can be represented as follows:

$$\begin{array}{l} (\forall z)(B(x, z) \supset B(y, z)) \\ \underline{B(x, [A])} \\ \therefore\ B(y, [A]) \end{array}$$

$$\begin{array}{l} [B(x)]_x = [U(x) \,\&\, M(x)]_x \\ \underline{N([(\forall x)(B(x) \equiv B(x))])} \\ \therefore\ N([(\forall x)(B(x) \equiv (U(x) \,\&\, M(x)))]).* \end{array}$$

Of course, to guarantee that these and other intuitively valid arguments come out valid in L_ω, I must first specify the semantics for L_ω.

* In order to enhance readability, I take the liberty here and elsewhere to use predicate letters (with or without indices) that do not strictly speaking belong to L_ω, and I occasionally delete some parentheses and commas.

13. A New Semantic Method

By what means should one characterize the semantics for L_ω? Since the aim is simply to characterize the logically valid formulas of L_ω, it will suffice to construct a Tarski-style definition of logical validity for L_ω. Such a definition will be built on Tarski-style definitions of truth for L_ω. The latter definitions will in turn depend in part on specifications of the denotations of the singular terms in L_ω. As already indicated, every formula of L_ω is just like a formula in a standard first-order extensional language except perhaps for the singular terms occurring in it. Therefore, once one has found a method for specifying the denotations of the singular terms of L_ω, the Tarski-style definitions of truth and validity for L_ω may be given in the customary way. What is being sought specifically is a method for characterizing the denotations of the singular terms of L_ω in such a way that a given singular term $[A]_{v_1 \ldots v_m}$ will denote an appropriate property, relation, or proposition, depending on the value of m.

Since L_ω has infinitely many complex singular terms $[A]_\alpha$, what is called for is a recursive specification of the denotation relation for L_ω. To do this I will arrange these singular terms into an order according to their syntactic kind and complexity. So, for example, just as the complex formula $((\exists x)Fx \;\&\; (\exists y)Gy)$ is the conjunction of the simpler formulas $(\exists x)Fx$ and $(\exists y)Gy$, I will say that the complex term $[(\exists x)Fx \;\&\; (\exists y)Gy]$ is the conjunction of the simpler terms $[(\exists y)Fx]$ and $[(\exists y)Gy]$. Similarly, just as the complex formula $\neg(\exists x)Fx$ is the negation of the simpler formula $(\exists x)Fx$, I will say that the complex term $[\neg(\exists x)Fx]$ is the negation of the simpler term $[(\exists x)Fx]$. The following are other examples: $[Rxy]_{yx}$ is the conversion of $[Rxy]_{xy}$; $[Sxyz]_{xzy}$ is the inversion of $[Sxyz]_{xyz}$; $[Rxx]_x$ is the reflexivization of $[Rxy]_{xy}$; $[Fx]_{xy}$ is the expansion of $[Fx]_x$; $[(\exists x)Fx]$ is the existential generalization of $[Fx]_x$; $[Fy]^y$ is the absolute predication of $[Fx]_x$ of y; $[F[Guvw]_{uvw}]$ is the absolute predication of $[Fx]_x$ of $[Guvw]_{uvw}$; $[F[Guvw]^{w}_{uv}]_w$ is the unary relativized predication of $[Fx]_x$ of $[Guvw]_{uvw}$; $[F[Guvw]^{vw}_{u}]_{vw}$ is the binary relativized predication of $[Fx]_x$ of $[Guvw]_{uvw}$; $[F[Guvw]^{uvw}]_{uvw}$ is the ternary relativized predication of $[Fx]_x$ of $[Guvw]_{uvw}$, and so on. In this way I isolate the following syntactic operations on intensional abstracts: conjunction, negation, conversion, inversion, reflexivization, expansion, existential generalization, absolute predication, unary relativized predication, binary relativized predication, ..., n-ary relativized predication, ...[2]

Those intensional abstracts whose form is $[F_h^m(v_1, \ldots, v_m)]_{v_1 \ldots v_m}$ are syntactically simpler than all others. I will call them *elementary*. And the denotation of an elementary intensional abstract $[F_h^m(v_1, \ldots, v_m)]_{v_1 \ldots v_m}$ is just the property or relation expressed by the primitive predicate F_h^m. The denotation of a more complex abstract $[A]_\alpha$ is defined in terms of the denotation(s) of the relevant syntactically simpler abstract(s). However, to state this definition, one must have a general technique for modeling PRPs.

Suppose that one were to use one of the previous approaches to this subject—namely, the approach of Russell, of Church, or of the possible-worlds theorists Montague, Kaplan, D. Lewis, *et al.* In that case one would be led to identify properties and relations with certain functions. I find such identification unintuitive. (The taste of pineapple, the missing shade of blue—are these functions?) Furthermore, the identification of properties and relations with functions leads naturally—and perhaps inevitably—to a hierarchy of artificially restricted logical types. (See desideratum 14, §4.) Since the thesis that properties and relations are functions is linked in this way to type theory, it proves to be more compatible with the higher-order approach to the logic of PRPs than it is with the first-order approach. In a first-order setting, such as that provided by L_ω, the identification of properties and relations with functions generates unwanted and unnecessary complications and restrictions. The alternative is to take properties and relations, as well as propositions, at face value, i.e., as real, irreducible entities. This is what I will do.

The identification of intensional entities with functions lies at the heart of the possible-worlds semantic method. If, as I have proposed, intensional entities are taken at face value and not as covert functions, then the possible-worlds semantic method will be of no use to us. But how, then, is the denotation of a given complex term $[A]_\alpha$ to be determined from the denotation(s) of the relevant syntactically simpler term(s)? My answer is that the new denotation is determined *algebraically*. That is, the new denotation is determined by the application of the relevant *fundamental logical operation* to the denotation(s) of the relevant syntactically simpler term(s). Let me explain.

Consider the following propositions, for example: $[(\exists x)Fx]$, $[(\exists y)Gy]$, $[(\exists x)Fx \,\&\, (\exists y)Gy]$. (Note: in this paragraph and the next I will be using—not mentioning—terms from L_ω.) What is the most obvious logical relation holding among these propositions? Answer: the third proposition is the conjunction of the first two. Similarly, what is the most obvious logical relation among the properties $[Fx]_x$, $[Gx]_x$, and $[Fx \,\&\, Gx]_x$? As before, the third is the conjunction of the first two. And what is the most obvious logical relation holding between the propositions $[(\exists x)Fx]$ and $[\neg(\exists x)Fx]$? Answer: the second is the negation of the first. Similarly, what is the most obvious logical relation holding between the properties $[Fx]_x$ and $[\neg Fx]_x$? As before, the second is the negation of the first. In a like manner I arrive at the following fundamental logical relationships: $[Rxy]_{yx}$ is the converse of $[Rxy]_{xy}$; $[Sxyz]_{xzy}$ is the inverse of $[Sxyz]_{xyz}$; $[Rxx]_x$ is the reflexivization of $[Rxy]_{xy}$; $[Fx]_{xy}$ is the expansion of $[Fx]_x$; $[(\exists x)Fx]$ is the existential generalization of $[Fx]_x$; $[Fy]^y$ is the absolute predication of $[Fx]_x$ of y; $[F[Guvw]_{uvw}]$ is the absolute predication of $[Fx]_x$ of $[Guvw]_{uvw}$; $[F[Guvw]^w_{uv}]_w$ is the unary relativized predication of $[Fx]_x$ of $[Guvw]_{uvw}$; $[F[Guvw]^{vw}_u]_{vw}$ is the binary relativized predication of $[Fx]_x$ of $[Guvw]_{uvw}$; $[F[Guvw]^{uvw}]_{uvw}$ is the ternary relativized predication of $[Fx]_x$ of $[Guvw]_{uvw}$, and so on. Thus, in one-to-one correspondence with the earlier syntactic operations on intensional abstracts there are fundamental logical operations on intensional entities: conjunction, negation, conversion, ...

The first two fundamental logical operations are intensional analogues of the two operations from Boolean algebra. A Boolean algebra having two elements (T and F) is an extensional model of first-order sentential logic. The next four operations are intensional analogues of operations from the algebra of relations, whose origins are found in the work of Peirce and Schröder. The algebra of relations, or transformation algebra as it is called, is the algebra for extensional relations. A transformation algebra is an extensional model of first-order predicate logic without quantifiers. The next operation, existential generalization, is an intensional analogue of the special new operation found in polyadic algebra. Polyadic algebra is just the algebra for extensional relations *with* quantification. A polyadic algebra is an extensional model of first-order predicate logic with quantifiers.[3] Finally, the predication operations, absolute predication and n-ary relativized predication, $n \geq 1$, are further operations that I have isolated for the purpose of modeling first-order quantifier logic with distinguished singular terms, including in particular intensional abstracts. Absolute predication is straightforward. As indicated above, the absolute predication of $[Fx]_x$ of y is $[Fy]^y$, i.e., the proposition that y is F. Similarly, the absolute predication of $[Fx]_x$ of $[Gy]_y$ is $[F[Gy]_y]$, i.e., the proposition that the property of being G is F. Relativized predication differs somewhat from absolute predication. It also predicates a property of an intension, but it involves in addition a simultaneous predication of which that intension is the result. So, for example, the unary relativized predication of $[Fx]_x$ of $[Gy]_y$ is $[F[Gy]^y]_y$, i.e., the property of being something y such that the proposition that y is G is F. To give a concrete example, the unary relativized predication of the property *being believed* of the property *being a spy* is the property *being believed to be a spy*. The other relativized predication operations behave analogously; of course, their second arguments must be intensions of appropriately higher degree.

Taken together, these fundamental logical operations have the following property. Choose any intensional abstract $[A]_\alpha$ in L_ω that is not elementary. If $[A]_\alpha$ is obtained from $[B]_\beta$ via the syntactic operation of negation (conversion, inversion, reflexivization, expansion, existential generalization), then the denotation of $[A]_\alpha$ is the result of applying the logical operation of negation (conversion, inversion, reflexivization, expansion, existential generalization) to the denotation of $[B]_\beta$. The same thing holds *mutatis mutandis* for abstracts that, syntactically, are conjunctions or predications (absolute or relativized). In this way, therefore, these fundamental logical operations make it possible to define recursively the denotation relation for all of the complex intensional abstracts $[A]_\alpha$ in L_ω.

The algebraic semantics for L_ω is thus to be specified in stages. First, an algebra of properties, relations, and propositions—or an algebraic model structure, as I will call it—is posited. Secondly, an intensional interpretation of the primitive predicates is given. Thirdly, the denotation relation for the terms of L_ω is recursively defined. Fourthly, the notion of truth for formulas is defined. Finally, in the customary Tarski fashion, the notion of logical validity for formulas of L_ω is defined.

Now a structure β is a Boolean algebra if and only if (i) β is an ordered set consisting of a universe or domain $\mathscr{D}$ and two operations on $\mathscr{D} \times \mathscr{D}$ and $\mathscr{D}$, respectively, and (ii) the elements of β satisfy certain specifiable conditions. By analogy, $\mathscr{M}$ is an algebraic model structure if and only if (i) $\mathscr{M}$ is an ordered set consisting of a universe or domain $\mathscr{D}$ and the fundamental logical operations on $\mathscr{D} \times \mathscr{D}, \mathscr{D}, \ldots$, respectively (plus certain supplementary elements), and (ii) the elements of $\mathscr{M}$ satisfy certain specifiable conditions. In §2 I mentioned that historically there have been two competing conceptions of intensional entities. According to conception 1, intensional entities are identical if and only if they are necessarily equivalent. According to conception 2, each definable intensional entity is such that, when it is defined completely, it has a unique, non-circular definition. By suitably adjusting the conditions imposed on the elements of a given algebraic model structure $\mathscr{M}$, one can fix the exact character of the intensional entities that $\mathscr{M}$ is designed to model. In particular, by suitably formulating the conditions imposed on the elements of $\mathscr{M}$, one can make precise what it takes for the intensional entities modeled by $\mathscr{M}$ to conform to conception 1 or conception 2.

In this way one actually arrives at two distinct types of algebraic model structures—type 1 and type 2. In turn, one arrives at two distinct notions of logical validity for L_ω—validity$_1$ and validity$_2$, i.e., truth-in-all-type-1-model-structures and truth-in-all-type-2-model-structures.

With these preliminary remarks in mind I will now use the new semantic method to lay out in detail the formal semantics for L_ω.

14. The Formal Semantics*

Algebraic model structures. An algebraic model structure (or model structure, for short) is any structure

$$\langle \mathscr{D}, \mathscr{P}, \mathscr{K}, \mathscr{G}, \text{Id}, \text{Conj}, \text{Neg}, \text{Exist}, \text{Exp}, \text{Inv}, \text{Conv}, \text{Ref}, \text{Pred}_0, \text{Pred}_1, \ldots, \text{Pred}_k, \ldots \rangle$$

whose elements simultaneously satisfy the conditions set forth below. $\mathscr{D}$ is the domain of discourse and is non-empty. $\mathscr{P}$ is a relation on $\mathscr{D}$ that serves to partition $\mathscr{D}$ into a denumerable number of disjoint subdomains: $\mathscr{D}_{-1}, \mathscr{D}_0, \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3, \ldots$.[4] The elements

* Readers seeking a quick overview may skip this section.

of $\mathscr{D}_{-1}$ are to be thought of as particulars; the elements of $\mathscr{D}_0$, as propositions; the elements of $\mathscr{D}_1$, as properties, and the elements of $\mathscr{D}_i$, for $i \geq 2$, as i-ary relations. Although $\mathscr{D}_i$, $i \geq 0$, may not be empty, I do permit $\mathscr{D}_{-1}$ to be empty. $\mathscr{K}$ is a set of functions on $\mathscr{D}$. These functions are to be thought of as telling us the alternate or possible extensions of the elements of $\mathscr{D}$. Specifically, they tell us that the extension of a particular is itself, that the extension of a proposition is a truth value, that the extension of a property is a subset of $\mathscr{D}$, and that the extension of an i-ary relation is a set of ordered i-tuples of members of $\mathscr{D}$.[5] Thus, for $H \in \mathscr{K}$ and $x \in \mathscr{D}$, the following hold: if $x \in \mathscr{D}_{-1}$, then $H(x) = x$; if $x \in \mathscr{D}_0$, then $H(x) = \mathrm{T}$ or $H(x) = \mathrm{F}$; if $x \in \mathscr{D}_1$, then $H(x) \subseteq \mathscr{D}$; if, for $i > 1$, $x \in \mathscr{D}_i$, then $H(x) \subseteq {}^i\mathscr{D}$. The next element of a model structure is the function $\mathscr{G}$. $\mathscr{G}$ is a distinguished element of $\mathscr{K}$ and is to be thought of as the function that determines the *actual* extensions of the elements of $\mathscr{D}$. The element Id of a model structure is a distinguished element of $\mathscr{D}_2$ and is thought of as the fundamental logical relation-in-intension *identity*. Id must satisfy the following condition:

$$(\forall H \in \mathscr{K})(H(\mathrm{Id}) = \{xy \in \mathscr{D} : x = y\}).$$

That is, for every $H \in \mathscr{K}$, H singles out the extensional identity relation on $\mathscr{D}$ to be the extension of the intensional identity relation Id. The remaining elements of a model structure are functions which are thought of as fundamental logical operations on intensional entities. The domains and ranges of these operations are as follows:[6]

1.	Conj:	$\mathscr{D}_i \times \mathscr{D}_i \underset{\text{into}}{\rightarrow} \mathscr{D}_i$	for each $i \geq 0$
2.	Neg:	$\mathscr{D}_i \underset{\text{into}}{\rightarrow} \mathscr{D}_i$	for each $i \geq 0$
3.	Exist:	$\mathscr{D}_i \underset{\text{into}}{\rightarrow} \mathscr{D}_{i-1}$	for $i \geq 1$
		$\mathscr{D}_0 \underset{\text{into}}{\rightarrow} \mathscr{D}_0$	
4.	Exp:	$\mathscr{D}_i \underset{\text{into}}{\rightarrow} \mathscr{D}_{i+1}$	for $i \geq 0$
5.	Inv:	$\mathscr{D}_i \underset{\text{into}}{\rightarrow} \mathscr{D}_i$	for $i \geq 3$

6. Conv: $\mathscr{D}_i \underset{\text{into}}{\rightarrow} \mathscr{D}_i$ for $i \geq 2$

7. Ref: $\mathscr{D}_i \underset{\text{into}}{\rightarrow} \mathscr{D}_{i-1}$ for $i \geq 2$

8.0 Pred_0: $\mathscr{D}_i \times \mathscr{D} \underset{\text{into}}{\rightarrow} \mathscr{D}_{i-1}$ for $i \geq 1$

8.1 Pred_1: $\mathscr{D}_i \times \mathscr{D}_j \underset{\text{into}}{\rightarrow} \mathscr{D}_i$ for $i, j \geq 1$

8.2 Pred_2: $\mathscr{D}_i \times \mathscr{D}_j \underset{\text{into}}{\rightarrow} \mathscr{D}_{i+1}$ for $i \geq 1$ and $j \geq 2$

8.3 Pred_3: $\mathscr{D}_i \times \mathscr{D}_j \underset{\text{into}}{\rightarrow} \mathscr{D}_{i+2}$ for $i \geq 1$ and $j \geq 3$

....[7]

The following conditions specify how the extensions of elements in $\mathscr{D}$ are affected by each of these operations. For all $H \in \mathscr{K}$ and all $u, v, x_1, \ldots, x_i, x_{i+1}, y_1, \ldots, y_k \in \mathscr{D}$:

1. $H(\text{Conj}(u, v)) = \text{T} \equiv$
   $(H(u) = \text{T} \;\&\; H(v) = \text{T})$ (for $u, v \in \mathscr{D}_0$)

   $\langle x_1, \ldots, x_i \rangle \in H(\text{Conj}(u, v)) \equiv$
   $(\langle x_1, \ldots, x_i \rangle \in H(u) \;\&\; \langle x_1, \ldots, x_i \rangle \in H(v))$
   (for $u, v \in \mathscr{D}_i$, $i \geq 1$)

2. $H(\text{Neg}(u)) = \text{T} \equiv H(u) = \text{F}$ (for $u \in \mathscr{D}_0$)

   $\langle x_1, \ldots, x_i \rangle \in H(\text{Neg}(u)) \equiv$
   $\langle x_1, \ldots, x_i \rangle \notin H(u)$ (for $u \in \mathscr{D}_i$, $i \geq 1$)

3. $H(\text{Exist}(u)) = \text{T} \equiv H(u) = \text{T}$ (for $u \in \mathscr{D}_0$)

   $H(\text{Exist}(u)) = \text{T} \equiv (\exists x_1)(x_1 \in H(u))$ (for $u \in \mathscr{D}_1$)

   $\langle x_1, \ldots, x_{i-1} \rangle \in H(\text{Exist}(u)) \equiv$
   $(\exists x_i)(\langle x_1, \ldots, x_{i-1}, x_i \rangle \in H(u))$ (for $u \in \mathscr{D}_i$, $i \geq 2$)

4. $x_1 \in H(\text{Exp}(u)) \equiv H(u) = \text{T}$ (for $u \in \mathscr{D}_0$)

$\langle x_1, \ldots, x_i, x_{i+1}\rangle \in H(\text{Exp}(u)) \equiv \langle x_1, \ldots, x_i\rangle \in H(u)$ (for $u \in \mathscr{D}_i, i \geq 1$)

5. $\langle x_1, \ldots, x_{i-2}, x_i, x_{i-1}\rangle \in H(\text{Inv}(u)) \equiv \langle x_1, \ldots, x_{i-2}, x_{i-1}, x_i\rangle \in H(u)$ (for $u \in \mathscr{D}_i, i \geq 3$)

6. $\langle x_i, x_1, \ldots, x_{i-1}\rangle \in H(\text{Conv}(u)) \equiv \langle x_1, \ldots, x_{i-1}, x_i\rangle \in H(u)$ (for $u \in \mathscr{D}_i, i \geq 2$)

7. $\langle x_1, \ldots, x_{i-1}\rangle \in H(\text{Ref}(u)) \equiv \langle x_1, \ldots, x_{i-1}, x_{i-1}\rangle \in H(u)$ (for $u \in \mathscr{D}_i, i \geq 2$)

8.0 $H(\text{Pred}_0(u, y_1)) = \text{T} \equiv y_1 \in H(u)$ (for $u \in \mathscr{D}_1$)

$\langle x_1, \ldots, x_{i-1}\rangle \in H(\text{Pred}_0(u, y_1)) \equiv \langle x_1, \ldots, x_{i-1}, y_1\rangle \in H(u)$ (for $u \in \mathscr{D}_i, i \geq 2$)

8.1 $\langle x_1, \ldots, x_{i-1}, y_1\rangle \in H(\text{Pred}_1(u, v)) \equiv \langle x_1, \ldots, x_{i-1}, \text{Pred}_0(v, y_1)\rangle \in H(u)$ (for $u \in \mathscr{D}_i, i \geq 1$, and $v \in \mathscr{D}_j, j \geq 1$)

8.2 $\langle x_1, \ldots, x_{i-1}, y_1, y_2\rangle \in H(\text{Pred}_2(u, v)) \equiv \langle x_1, \ldots, x_{i-1}, \text{Pred}_0(\text{Pred}_0(v, y_2), y_1)\rangle \in H(u)$ (for $u \in \mathscr{D}_i, i \geq 1$, and $v \in \mathscr{D}_j, j \geq 2$)

. . . .[8]

This completes the characterization of what a model structure is.

Type 1 Model Structures

A model structure is type 1 *iff*$_{\text{df}}$ it satisfies the following auxiliary condition:

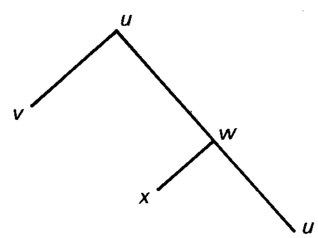
$$(\forall x, y \in \mathscr{D}_i)((\forall H \in \mathscr{K})(H(x) = H(y)) \supset x = y), \quad \text{for all } i \geq -1.$$

This condition provides us with a precise statement of conception 1.

Specifically, this condition rules out the possibility of there being two (or more) elements of any given subdomain $\mathscr{D}_i$ that are necessarily equivalent.

Type 2 Model Structures

A model structure is type 2 *iff*$_{df}$ its operations Conj, Neg, Exist, Exp, Inv, Conv, Ref, Pred_0, Pred_1, $\text{Pred}_2, \ldots$ are (i) one-one, (ii) disjoint in their ranges, and (iii) non-cycling. Auxiliary conditions (i)–(iii) provide us with a precise formulation of conception 2. For, taken together, (i) and (ii) guarantee that the action of the inverses of the fundamental logical operations in a given type 2 model structure $\mathscr{M}$ is to decompose the elements of $\mathscr{D}$ into unique (possibly infinite) trees. And condition (iii) insures that, for each item u in such a decomposition tree, u cannot occur on any path descending from u. So the following is the sort of situation ruled out by condition (iii):



Hence, whereas conditions (i) and (ii) insure that the elements of $\mathscr{D}$ have at most one complete definition in terms of the elements of $\mathscr{D}$ plus the fundamental logical operations, condition (iii) insures that such definitions are never circular.

Notice by the way that in the formal characterizations of what it is to be a type 1 or type 2 algebraic model structure no use is made of any of the following intuitive notions: particular, property, relation, proposition, alternative or possible extension, actual extension, complete definition. For what it is worth, type 1 and type 2 model structures are characterized formally in exclusively set-theoretic terms.[9]

At the close of §2, I mentioned that there are various intermediate conceptions of PRPs between conceptions 1 and 2. To model such intermediate conceptions, one need only appropriately adjust the auxiliary conditions imposed on algebraic model structures.

Consider, for example, the conception that is like conception 2 except that it imposes less strenuous identity conditions on conjunctions so that $[A\alpha \& B\alpha]_\alpha = [B\alpha \& A\alpha]_\alpha$ and $[(A\alpha \& B\alpha) \& C\alpha]_\alpha = [A\alpha \& (B\alpha \& C\alpha)]_\alpha$. The model structures appropriate to this conception are just like type 2 model structures except for the auxiliary conditions imposed on the conjunction operation Conj. Specifically, we exempt Conj from condition (i) and instead require that items in its range (i.e., conjunctions) can be decomposed under its inverse into a unique set of items (i.e., conjuncts) but in no special order. Accordingly, the inverse of Conj behaves rather like the operation of prime factorization in number theory: every natural number is factorable into a unique set of primes, yet there is no special order in which these prime factors must be multiplied in order to obtain the original number.

The field here is very rich. But conceptions 1 and 2 are the motherlode, and we should be happy to explore there for quite a while.

Truth and Validity

An interpretation $\mathscr{I}$ for L_ω relative to model structure $\mathscr{M}$ is a function that assigns to the predicate letter F_1^2 (i.e., $=$) the element $\text{Id} \in \mathscr{M}$ and, for each predicate letter F_i^j in L_ω, assigns to F_i^j some element of the subdomain $\mathscr{D}_j \subset \mathscr{D} \in \mathscr{M}$. An assignment $\mathscr{A}$ for L_ω relative to model structure $\mathscr{M}$ is a function that maps the variables of L_ω into the domain $\mathscr{D} \in \mathscr{M}$. Truth $T_{\mathscr{I}\mathscr{A}\mathscr{M}}$ is defined in terms of denotation $D_{\mathscr{I}\mathscr{A}\mathscr{M}}$, which will be defined subsequently:

$$T_{\mathscr{I}\mathscr{A}\mathscr{M}}(A) \text{ } \mathit{iff}_{\text{df}} \text{ } \mathscr{G}(D_{\mathscr{I}\mathscr{A}\mathscr{M}}([A])) = T.$$

That is, formula A is true on interpretation $\mathscr{I}$ and assignment $\mathscr{A}$ relative to model structure $\mathscr{M}$ if and only if the actual extension of the proposition denoted by the term $[A]$ is the truth value T. (Of course, $T_{\mathscr{I}\mathscr{A}\mathscr{M}}$ could instead be given a standard Tarski-style recursive definition (see lemma 6 in §15), but in the algebraic setting a direct definition suffices. A recursive definition is needed only in the definition of $D_{\mathscr{I}\mathscr{A}\mathscr{M}}$.) Then I define the two notions of validity for L_ω:

> a formula A is *valid*$_1$ *iff*$_{\text{df}}$ for every type 1 model structure $\mathscr{M}$ and for every interpretation $\mathscr{I}$ and every assignment $\mathscr{A}$ relative to $\mathscr{M}$, $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$.

a formula A is *valid*$_2$ *iff*$_{df}$ for every type 2 model structure $\mathscr{M}$ and for every interpretation $\mathscr{I}$ and every assignment $\mathscr{A}$ relative to $\mathscr{M}$, $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$.

Denotation

It remains to define the denotation function $D_{\mathscr{I}\mathscr{A}\mathscr{M}}$, which was referred to in the truth definition. To do this, I must first define the basic syntactic operations on intensional abstracts that were mentioned informally in §13.* I begin by introducing some preliminary syntactic notions.

I will say that a term $[A]_\alpha$ is *normalized* if and only if all the variables in the sequence of variables α occur free in A and α displays the order in which these variables first occur free in A. If a variable occurs free in more than one of the terms $t_1, \ldots, t_j$ in the atomic formula $F_i^j(t_1, \ldots, t_j)$, then this variable will be called a *reflected variable* in $F_i^j(t_1, \ldots, t_j)$. If the formula A is atomic and if the variables in the sequence of variables α are all free in A, then the term $[A]_\alpha$ will be called a *prime term*. If α contains a variable that is reflected in atomic formula A, then a prime term $[A]_\alpha$ will be called a *prime reflection term*. Let $[F_i^j(t_1, \ldots, [B]_\gamma^\delta, \ldots, t_j)]_\alpha$ be a prime term that is not a prime reflection term. Then, if some variable occurs in both α and δ, the prime term will be called a *prime relativized predication term*, and the variable will be called a *relativized variable*.

Every term $[A]_\alpha$ has associated with it a certain permutation of the variables in α that I will designate as *primary relative* to $[A]_\alpha$. (I admit the possibility that α itself can be primary relative to $[A]_\alpha$.) There are three cases.

Case (1): prime reflection terms $[A]_\alpha$. Suppose that α is some permutation of the sequence of variables $v_1, \ldots, v_p$ and that $[A]_{v_1 \ldots v_p}$ is normalized. Suppose further that, among the variables in α that are reflected in A, v_k is the one that has the right most free occurrence in A. In this case, the sequence $v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p, v_k$ is *primary relative* to $[A]_\alpha$.

* In doing this, I encounter certain intricacies, which arise because of the need to keep track of the various permutations of the subscripted variables α in the terms $[A]_\alpha$. Most of the intricacies could be avoided here by adopting the alternate technique developed in my 'Completeness in the Theory of Properties, Relations, and Propositions'. In any event my general algebraic approach is wedded to no particular treatment of this matter.

Case (2): $[A]_\alpha$ is a prime relativized predication term $[F_i^j(t_1, \ldots, [B]_\gamma^\delta, \ldots, t_j)]_\alpha$. Let α be a permutation of the sequence of variables $u_1, \ldots, u_p$, $v_1, \ldots, v_q$, $w_1, \ldots, w_r$ such that the latter sequence displays the order in which these variables first occur free in A. Let $[B]_\gamma^\delta$ be the left most argument of F_i^j containing relativized variables. Finally, let $v_1, \ldots, v_q$ be all such relativized variables in $[B]_\gamma^\delta$. Then the sequence $u_1, \ldots, u_p, w_1, \ldots, w_r, v_1, \ldots, v_q$ is *primary relative* to $[A]_\alpha$.

Case (3): $[A]_\alpha$ is neither a prime reflection term nor a prime relativized predication term. Let α be a permutation of the sequence of variables $v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+k}$ where $[A]_{v_1 \ldots v_p}$ is normalized and $v_{p+1}, \ldots, v_{p+k}$ are in order of their occurrence in α the variables not occurring free in A. Then the sequence $v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+k}$ is *primary relative* to $[A]_\alpha$. (I allow that $v_1, \ldots, v_p$ or $v_{p+1}, \ldots, v_{p+k}$ is an empty sequence.)

I am now prepared to define the basic syntactic operations on intensional abstracts of L_ω.

(1) If $[(A \,\&\, B)]_\alpha$ is normalized, it is the *conjunction* of $[A]_\alpha$ and $[B]_\alpha$.

(2) If $[\neg A]_\alpha$ is normalized, it is the *negation* of $[A]_\alpha$.

(3) Let $[(\exists v_k)A]_\alpha$ be normalized. Then, if v_k is free in A, $[(\exists v_k)A]_\alpha$ is the *existential generalization* of $[A]_{\alpha v_k}$; otherwise, $[(\exists v_k)A]_\alpha$ is the *existential generalization* of $[A]_\alpha$.

(4) If $[A]_\alpha$ is normalized and if v_{s+1} is the alphabetically earliest variable not occurring in $[A]_{\alpha v_1 \ldots v_s}$, then $[A]_{\alpha v_1 \ldots v_s v_{s+1}}$ is the *expansion* of $[A]_{\alpha v_1 \ldots v_s}$.

(5)–(6) Suppose that the sequence $v_1, v_2, \ldots, v_{s-1}, v_s$ is not primary relative to $[A]_{v_1 v_2 \ldots v_{s-1} v_s}$. Suppose instead that the sequence $u_1, \ldots, u_{s-1}, u_s$ is primary relative to $[A]_{v_1 v_2 \ldots v_{s-1} v_s}$. In this case if, for some $h, k \geq 1$, $u_1, \ldots, u_k = v_h, \ldots, v_{h+k-1}$ and $u_{k+1} = v_s \neq v_{h+k}$, then $[A]_{v_1 v_2 \ldots v_{s-1} v_s}$ is the *inversion* of $[A]_{v_1 v_2 \ldots v_s v_{s-1}}$; otherwise, $[A]_{v_1 v_2 \ldots v_{s-1} v_s}$ is the *conversion* of $[A]_{v_2 \ldots v_{s-1} v_s v_1}$.

(7) Let $[F_i^j(t_1, \ldots, t_k(v_r), \ldots, t_j)]_{\alpha v_r}$ be a prime reflection term relative to which the sequence αv_r is primary. Suppose that $t_k(v_r)$ is the right most argument of F_i^j in which the reflected variable v_r has a free occurrence. And suppose, finally, that v_s is the alphabetically earliest variable not occurring in

$t_1, \ldots, t_j$. Then $[F_i^j(t_1, \ldots, t_k(v_r), \ldots, t_j)]_{\alpha v_r}$ is the *reflexivization* of $[F_i^j(t_1, \ldots, t_k(v_s), \ldots, t_j)]_{\alpha v_r v_s}$.

(8) Suppose that $[F_i^j(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_j)]_\alpha$ is a normalized non-prime-reflection term. Let the terms $t_1, \ldots, t_{k-1}$ be variables all of which occur in the sequence α. Suppose that no variable that occurs free in t_k also occurs in the sequence α, and let v_r be the alphabetically earliest variable not occurring in $t_1, \ldots, t_j$. Then, $[F_i^j(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_j)]_\alpha$ is the *predication*$_0$ of $[F_i^j(t_1, \ldots, t_{k-1}, v_r, t_{k+1}, \ldots, t_j)]_{\alpha v_r}$ of t_k. Alternatively, let $[F_i^j(t_1, \ldots, t_{k-1}, [B]_\gamma^\delta, t_{k+1}, \ldots, t_j)]_{\alpha v_1 \ldots v_m}$ be a prime relativized predication term, where $m \geq 1$. Suppose that the terms $t_1, \ldots, t_{k-1}$ are variables that occur in the sequence α. And suppose that the sequence $\alpha, v_1, \ldots, v_m$ is primary relative to this prime relativized predication term and that $v_1, \ldots, v_m$ are the relativized variables occurring in $[B]_\gamma^\delta$. Then, this term is the *predication*$_m$ of $[F_i^j(t_1, \ldots, t_{k-1}, v_1, t_{k+1}, \ldots, t_j)]_{\alpha v_1}$ of $[B]_{\gamma v_1 \ldots v_m}^{\delta'}$, where δ' is the result of deleting the relativized variables $v_1, \ldots, v_m$ from δ.

For each non-elementary intensional abstract in L_ω, either it or one of its alphabetic variants[10] falls into the range of one of these syntactic operations, and no two non-elementary abstracts that are alphabetic variants fall into the range of more than one of them. In this sense, these operations serve to partition the class of non-elementary abstracts into denumerably many disjoint *syntactic kinds*: conjunctions, negations, existential generalizations, expansions, inversions, conversions, reflexivizations, predications$_0$, predications$_1$, predications$_2$, ... Using these notions, I inductively define the *denotation* function $D_{\mathscr{I}\mathscr{A}\mathscr{M}}$:

Variables: $D_{\mathscr{I}\mathscr{A}\mathscr{M}}(v_i) = \mathscr{A}(v_i)$

Elementary complex terms: $D_{\mathscr{I}\mathscr{A}\mathscr{M}}([F_i^j(v_1, \ldots, v_j)]_{v_1 \ldots v_j}) = \mathscr{I}(F_i^j)$

Non-elementary complex terms:

1. If t is the conjunction of r and s, then $D_{\mathscr{I}\mathscr{A}\mathscr{M}}(t) = \text{Conj}(D_{\mathscr{I}\mathscr{A}\mathscr{M}}(r), D_{\mathscr{I}\mathscr{A}\mathscr{M}}(s))$.
2. If t is the negation of r, then $D_{\mathscr{I}\mathscr{A}\mathscr{M}}(t) = \text{Neg}(D_{\mathscr{I}\mathscr{A}\mathscr{M}}(r))$.
3. If t is the existential generalization of r, then $D_{\mathscr{I}\mathscr{A}\mathscr{M}}(t) = \text{Exist}(D_{\mathscr{I}\mathscr{A}\mathscr{M}}(r))$.

4. If t is an alphabetic variant of the expansion of r, then $D_{\mathcal{I}\mathcal{A}\mathcal{M}}(t) = \text{Exp}(D_{\mathcal{I}\mathcal{A}\mathcal{M}}(r))$.
5. If t is the inversion of r, then $D_{\mathcal{I}\mathcal{A}\mathcal{M}}(t) = \text{Inv}(D_{\mathcal{I}\mathcal{A}\mathcal{M}}(r))$.
6. If t is the conversion of r, then $D_{\mathcal{I}\mathcal{A}\mathcal{M}}(t) = \text{Conv}(D_{\mathcal{I}\mathcal{A}\mathcal{M}}(r))$.
7. If t is the reflexivization of r, then $D_{\mathcal{I}\mathcal{A}\mathcal{M}}(t) = \text{Ref}(D_{\mathcal{I}\mathcal{A}\mathcal{M}}(r))$.
8. If t is the predication$_k$ of r of s, then $D_{\mathcal{I}\mathcal{A}\mathcal{M}}(t) = \text{Pred}_k(D_{\mathcal{I}\mathcal{A}\mathcal{M}}(r), D_{\mathcal{I}\mathcal{A}\mathcal{M}}(s))$.

This completes the semantics for L_ω.

15. A Complete Logic for the First Conception

On conception 1 intensional entities are identical if and only if necessarily equivalent. Thus, on conception 1 the following abbreviation captures the properties usually attributed to the modal operator $\Box$:

$$\Box A \ \textit{iff}_{\text{df}} \ [A] = [[A] = [A]].$$

That is, necessarily A *iff* the proposition that A is identical to a trivial necessary truth. Since on conception 1 there is only one necessary truth, this definition is adequate. For the purpose of formulating the logic of L_ω on conception 1, this abbreviation will be adopted as a notational convenience. The modal operator $\Diamond$ is then defined in terms of $\Box$ in the usual way: $\Diamond A \ \textit{iff}_{\text{df}} \neg\Box\neg A$. By adopting these notational conventions, I am not reversing my earlier position on the parsing of natural language sentences such as 'it is necessary that A'. I would represent this sentence as $N([A])$. The 1-place predicate N may on conception 1 be defined as follows: $N(x) \ \textit{iff}_{\text{df}}\ x = [x = x]^x$.

The logic T1 for L_ω on conception 1 consists of the axiom schemas and rules for the modal logic S5 with quantifiers and identity and three additional axiom schemas for intensional abstracts.

Axiom Schemas and Rules of T1

A1: Truth-functional tautologies
A2: $(\forall v_i)A(v_i) \supset A(t)$ (where t is free for v_i in A)[11]
A3: $(\forall v_i)(A \supset B) \supset (A \supset (\forall v_i)B)$ (where v_i is not free in A)
A4: $v_i = v_i$

A5: $v_i = v_j \supset (A(v_i, v_i) \equiv A(v_i, v_j))$ (where $A(v_i, v_j)$ is a formula that arises from $A(v_i, v_i)$ by replacing some (but not necessarily all) free occurrences of v_i by v_j, and v_j is free for the occurrences of v_i that it replaces)

A6: $[A]_{u_1 \ldots u_p} \neq [B]_{v_1 \ldots v_q}$ (where $p \neq q$)

A7: $[A(u_1, \ldots, u_p)]_{u_1 \ldots u_p} = [A(v_1, \ldots, v_p)]_{v_1 \ldots v_p}$ (where these two terms are alphabetic variants)

A8: $[A]_\alpha = [B]_\alpha \equiv \Box(A \equiv_\alpha B)$

A9: $\Box A \supset A$

A10: $\Box(A \supset B) \supset (\Box A \supset \Box B)$

A11: $\Box A \supset \Box\Diamond A$

R1: if $\vdash A$ and $\vdash (A \supset B)$, then $\vdash B$.

R2: if $\vdash A$, then $\vdash (\forall v_i)A$.

R3: if $\vdash A$, then $\vdash \Box A$.

A1 is, of course, concerned with the truth-functional sentential connectives & and $\neg$. A2 and A3 are familiar axioms for first-order quantifiers. A4 asserts the reflexivity of identity. A5 is Leibniz's law. A6 asserts the distinctness of intensional entities having different degrees. A7 asserts the validity$_1$ of a change of bound variables within intensional abstracts. A8 asserts the necessary equivalence of identicals and the identity of necessary equivalents. This principle is, of course, the hallmark of conception 1. A9–A11 are the standard S5 axioms for $\Box$ and $\Diamond$. R1 is modus ponens. R2 is universal generalization. R3 is the necessitation rule from S5.[12]

Given the definition $\Box$ and $\Diamond$ in terms of identity and intensional abstraction, modal logic may be viewed as the identity theory for intensional abstracts. In this connection, notice that, whereas the principle of necessary identity

$$x = y \supset \Box x = y$$

is an immediate consequence of Leibniz's law (A5) (given the reflexivity of identity (A4)), the S5 axiom (A11) is just an instance of the principle of necessary distinctness

$$x \neq y \supset \Box x \neq y.$$

In fact, the S5 axiom and the principle of necessary distinctness are actually equivalent. For, given A1–A10 and R1–R3, not only is A11 derivable from the principle of necessary distinctness, but also the principle of necessary distinctness is derivable from A11.

Now I will state the primary result for T1:

Theorem (Soundness and Completeness)
For all formulas A in L_ω, A is valid$_1$ if and only if A is a theorem of T1 (i.e., $\vDash_1 A$ iff $\vdash_{T1} A$).

Proof (Soundness). First, the following lemmas are proved.
Lemma 1: T1 is equivalent to the theory that results when A5, A8, and A11 are replaced with the following simpler versions:

A5* $v_i = v_j \supset (A(v_i, v_i) \supset A(v_i, v_j))$ (where $A(v_i, v_i)$ and $A(v_i, v_j)$ are as in A5 except that A is atomic)
A8*(a) $\Box(A \equiv B) \equiv [A] = [B]$
A8*(b) $(\forall v_i)([A(v_i)]_\alpha = [B(v_i)]_\alpha) \equiv [A(v_i)]_{\alpha v_i} = [B(v_i)]_{\alpha v_i}$
A11* $v_i \neq v_j \supset \Box v_i \neq v_j$.

Lemma 2: Let v_h be an externally quantifiable variable in $[B(v_h)]_\alpha$, and let t_k be free for v_h in $[B(v_h)]_\alpha$. Consider any model structure $\mathscr{M}$ and any interpretation $\mathscr{I}$ and assignment $\mathscr{A}$ relative to $\mathscr{M}$. Let $\mathscr{A}'$ be an assignment that is just like $\mathscr{A}$ except that $\mathscr{A}'(v_h) = D_{\mathscr{I}\mathscr{A}\mathscr{M}}(t_k)$. Then, $D_{\mathscr{I}\mathscr{A}'\mathscr{M}}([B(v_h)]_\alpha) = D_{\mathscr{I}\mathscr{A}\mathscr{M}}([B(t_k)]_\alpha)$.

Lemma 3: For all $\mathscr{I}$, $\mathscr{A}$, $\mathscr{M}$ and for all $\mathscr{D}_k \subset \mathscr{D} \in \mathscr{M}$, $k \geq 0$, $D_{\mathscr{I}\mathscr{A}\mathscr{M}}([A]_{v_1 \ldots v_k}) \in \mathscr{D}_k$.

Lemma 4: For all $\mathscr{I}$, $\mathscr{A}$, $\mathscr{M}$ and for all terms t and t', if $\mathscr{M}$ is type 1, then $D_{\mathscr{I}\mathscr{A}\mathscr{M}}([t = t]) = D_{\mathscr{I}\mathscr{A}\mathscr{M}}([t' = t'])$.

Lemma 5: Let v_r be free in $[A(v_r)]_\alpha$. Then, for all $\mathscr{I}$, $\mathscr{A}$, $\mathscr{M}$, if $\mathscr{M}$ is type 1, $D_{\mathscr{I}\mathscr{A}\mathscr{M}}([A(v_r)]_\alpha) = \text{Pred}_0(D_{\mathscr{I}\mathscr{A}\mathscr{M}}([A(v_r)]_{\alpha v_r}), \mathscr{A}(v_r))$.

Lemma 6: For all $\mathscr{I}$, $\mathscr{A}$, $\mathscr{M}$,
(a) $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(F_i^j(t_1, \ldots, t_j))$ *iff* $\langle D_{\mathscr{I}\mathscr{A}\mathscr{M}}(t_1), \ldots, D_{\mathscr{I}\mathscr{A}\mathscr{M}}(t_j)\rangle \in \mathscr{G}(\mathscr{I}(F_i^j))$.
(b) $T_{\mathscr{I}\mathscr{A}\mathscr{M}}((A \mathbin{\&} B))$ *iff* $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$ and $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(B)$.
(c) $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(\neg A)$ *iff* it is not the case that $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$.
(d) $T_{\mathscr{I}\mathscr{A}\mathscr{M}}((\exists v_k)A)$ *iff* there is an assignment $\mathscr{A}'$ relative to $\mathscr{M}$ such that $\mathscr{A}'$ is just like $\mathscr{A}$ except perhaps in what it assigns to v_k and $T_{\mathscr{I}\mathscr{A}'\mathscr{M}}(A)$.

Then, given these lemmas, which are in most cases proofs by induction on the complexity of terms or formulas, the verification of the soundness of T1 is straightforward. (For example, the soundness of A6 follows directly from Lemma 3; the soundness of A8*(b), from Lemma 5, etc.)

Proof (Completeness). The proof is Henkin style. Let L_ω^* be any extension of L_ω. A sentence A is said to be derivable in T1 from set Γ of L_ω^*-sentences if, for some finite subset $\{B_1, \ldots, B_n\}$ of Γ, $\vdash_{T1} ((B_1 \& \ldots \& B_n) \supset A)$. A set $\mathscr{A}$ of sets of L_ω^*-sentences is said to be *perfect*$_1$ if (1) every set in $\mathscr{A}$ is maximal, consistent, and ω-complete; (2) for every identity sentence $t = t'$, if this sentence is in any set in $\mathscr{A}$, it is in all sets in $\mathscr{A}$; (3) for every sentence $[A]_{v_1 \ldots v_p} \neq [B]_{v_1 \ldots v_p}$ $(p \geq 0)$, if this sentence belongs to some $\Delta \in \mathscr{A}$, then there is some set $\Delta' \in \mathscr{A}$ (where possibly $\Delta = \Delta'$) such that the sentence $(\exists v_1) \ldots (\exists v_p) \neg (A \equiv B)$ belongs to Δ'; (4) for every closed term $[B]_{v_1 \ldots v_p}$, there is a primitive predicate letter F_q^p such that the sentence $[B]_{v_1 \ldots v_p} = [F_q^p(v_1, \ldots, v_p)]_{v_1 \ldots v_p} \in \Delta$, for some $\Delta \in \mathscr{A}$. The completeness of T1 follows from two lemmas.

Lemma 1: For every consistent set Γ of sentences in L_ω, there is a (denumerable) extension of L_ω relative to which there is a perfect$_1$ set $\mathscr{A}$ one of whose members Δ includes Γ.

Lemma 2: For every extension of L_ω relative to which $\mathscr{A}$ is a perfect$_1$ set, every set Δ in $\mathscr{A}$ has a type 1 model (whose cardinality is that of Δ).

To prove Lemma 1, I first form an extension L_ω^* of L_ω that has denumerably many primitive names and denumerably many i-ary primitive predicates for each $i \geq 0$. The sentences of L_ω^* are then arranged into a sequence of consecutive sentences $A_1, A_2, A_3, \ldots$, having the following property: $A_1 = A_2$ and for every closed term $[B]_{v_1 \ldots v_p}$ in L_ω^*, there is at least one j such that A_j is the sentence $[B]_{v_1 \ldots v_p} = [F_q^p(v_1, \ldots, v_p)]_{v_1 \ldots v_p}$ where F_q^p is a primitive predicate letter that does not occur in B, Γ, or any A_h, $h < j$. Relative to this sequence, I use certain rules to construct an array of sets of L_ω^*-sentences:

$$
\begin{array}{llllll}
\Delta_1 & \Delta_3 & \Delta_7 & \cdots & \Delta_{n^2+n+1} & \cdots \\
\Delta_2 & \Delta_4 & \Delta_8 & & \Delta_{n^2+n+2} & \cdots \\
\Delta_5 & \Delta_6 & \Delta_9 & & \Delta_{n^2+n+3} & \cdots \\
\vdots & & \ddots & & \vdots & \\
 & & & \Delta_{n^2} & \Delta_{n^2+2n} & \cdots \\
\Delta_{n^2+1} & \Delta_{n^2+2} & \Delta_{n^2+3} \cdots & \Delta_{n^2+n} & \Delta_{(n+1)^2} & \cdots \\
\cdots & & & & &
\end{array}
$$

The rules are these. (1) $\Delta_1 = \Gamma$. (2) If A_n, $n \geq 1$, is $[A]_\alpha \neq [B]_\alpha$ and $A_n \in \Delta_{n^2}$, then $\Delta_{n^2+1} = \{(\exists\alpha)\neg(A \equiv B)\}$; otherwise, $\Delta_{n^2+1} = \Delta_{n^2}$. (3) Let Δ_m, $m > 1$, be in column $i > 1$ and row $k \geq 1$. Then if $m^+ \cup m^* \cup \{A_i\}$ is consistent, $\Delta_m = m^+ \cup m' \cup \{A_i\}$; otherwise, $\Delta_m = m^+ \cup m'$. The sets m^+, m^*, and m' are:

$m^+ =_{\text{df}}$ the set in row k and column i–1
$m^* =_{\text{df}} \{[B]_\alpha = [C]_\beta : (\exists n < m)(\Delta_n \vdash_{\text{T1}} [B]_\alpha = [C]_\beta)\}$
$m' =_{\text{df}} \{C_1(a_1), \ldots, C_s(a_s)\}$

where the sentences $C_1(a_1), \ldots, C_s(a_s)$ are determined as follows: in the order in which they first occur in the sequence $A_1, A_2, \ldots, A_i, \ldots$, the sentences $(\exists v_1)C_1(v_1), \ldots, (\exists v_s)C_s(v_s)$ exhaust the existential sentences in m^+ that occur before A_i, and $C_1(a_1), \ldots, C_s(a_s)$ are the earliest substitution instances of $(\exists v_1)C_1(v_1), \ldots, (\exists v_s)C_s(v_s)$ occurring after A_i such that in order each $C_r(a_r)$, $1 \leq r \leq s$, contains the first occurence of the primitive name a_r anywhere in the sequence $A_1, A_2, \ldots, A_i, \ldots$. Now the set Δ^j is defined to be the union of all sets in row j, $j \geq 1$. And the set $\mathscr{A}$ is defined to be the set of all sets Δ^j, $j \geq 1$. Claim: $\mathscr{A}$ is perfect$_1$. This claim, which entails Lemma 1, can be proved once we have the following sublemma: for all $m \geq 1$, $\Delta_m \cup m^*$ is consistent. This sublemma is proved by induction on m.

Lemma 2 is proved as follows. Let L_ω^* be any extension of L_ω relative to which $\mathscr{A}$ is a perfect$_1$ set. For each $\Delta \in \mathscr{A}$, I construct a separate type 1 model $\langle \mathscr{M}_\Delta, \mathscr{I}_\Delta \rangle$. Choose some well-ordering $<$ of the union of the class of individual constants and the class of primitive predicate letters in L_ω^*, where $=$ is the least primitive predicate letter in this well-ordering. The domain $\mathscr{D}_\Delta$ is then identified with the following union:

$\{F_i^j \in L_\omega^*$: there is no $F_h^k \in L_\omega^*$ such that $F_h^k < F_i^j$ and the sentence $[F_h^k(v_1, \ldots, v_k)]_{v_1 \ldots v_k} = [F_i^j(u_1, \ldots, u_j)]_{u_1 \ldots u_j} \in \Delta\} \cup$
$\{a_j \in L_\omega^*$: there is no $F_h^k \in L_\omega^*$ such that the sentence $[F_h^k(v_1, \ldots, v_k)]_{v_1 \ldots v_k} = a_j \in \Delta$, and there is no $a_i \in L_\omega^*$ such that $a_i < a_j$ such that the sentence $a_i = a_j \in \Delta\}$.

The subdomain $\mathscr{D}_{-1}$ is the set of primitive names in $\mathscr{D}_\Delta$, and the subdomain $\mathscr{D}_i$, $i \geq 0$, is the set of primitive i-ary predicates in $\mathscr{D}_\Delta$. The prelinear ordering $\mathscr{P}$ is defined as follows: $\mathscr{P}(x, y)$ *iff*$_{\text{df}}$ for some i and j, $i < j$, $x \in \mathscr{D}_i$ and $y \in \mathscr{D}_j$. The set $\mathscr{K}$ of alternate extension functions $H_{\Delta'}$, is determined by the atomic sentences belonging to the various sets Δ' belonging to $\mathscr{A}$. The actual extension function $\mathscr{G} =_{\text{df}} H_\Delta$. The identity element Id $\in \mathscr{M}_\Delta$ is just the identity predicate $=$. And the fundamental logical operations Conj$_\Delta$, Neg$_\Delta$, Exist$_\Delta$, ... are determined by the identity sentences in Δ. Finally, the interpretation $\mathscr{I}_\Delta$ may be defined as follows:

$\mathscr{I}_\Delta(`a_i') =_{\text{df}}$ the individual constant 'a_j' $\in \mathscr{D}$ such that '$a_i = a_j$' $\in \Delta$

$\mathscr{I}_\Delta(`F_i^j') =_{\text{df}}$ the primitive predicate 'F_k^j' $\in \mathscr{D}$ such that '$[F_i^j(v_1, \ldots, v_j)]_{v_1 \ldots v_j} = [F_k^j(v_1, \ldots, v_j)]_{v_1 \ldots v_j}$' $\in \Delta$.

With $\mathscr{M}_\Delta$ and $\mathscr{I}_\Delta$ so specified, it is then shown by induction on the complexity of formulas that $\langle \mathscr{M}_\Delta, \mathscr{I}_\Delta \rangle$ is a model of Δ, for all $\Delta \in \mathscr{A}$.

By the way, the completeness theorem for T1 yields an interesting corollary. Notice that L_ω is a notational variant of a first-order extensional language that is fitted out with identity and extensional abstracts $\{v_1 \ldots v_j: A\}$, for $j \geq 0$. Let an extensional type 1 model structure be defined to be a type 1 model structure in which the class $\mathscr{K}$ of alternate extension functions is just $\{\mathscr{G}\}$ (i.e., the singleton of the actual extension function). Thus, in an extensional type 1 model structure the following holds for all $i \geq -1$:

$$(\forall x, y \in \mathscr{D}_i)(\mathscr{G}(x) = \mathscr{G}(y) \supset x = y).$$

And hence, the elements of $\mathscr{D}$ behave as extensional entities do. The semantics is done in precisely the same way in which the semantics for L_ω is done except that only extensional type 1 model structures are considered. This yields the notion of *extensional validity*. And the formal logic consists of the axiom schemas and rules for standard first-order quantifier logic with identity (i.e., A1–A5, R1–R2) plus three axioms schemas for extensional abstracts:

(i) $\{u_1 \dots u_p: A\} \neq \{v_1 \dots v_q: B\}$ (where $p \neq q$)
(ii) $\{u_1 \dots u_p: A(u_1, \dots, u_p)\} = \{v_1 \dots v_p: A(v_1, \dots, v_p)\}$ (where the externally quantifiable variables in these two complex terms are the same and, for each k, $1 \leq k \leq p$, u_k is free in A for v_k and conversely)
(iii) $(A \equiv_{v_1 \dots v_p} B) \equiv \{v_1 \dots v_p: A\} = \{v_1 \dots v_p: B\}$.

Schema (i) asserts the distinctness of truth values from sets and relations-in-extension, the distinctness of sets from relations-in-extension, and the distinctness of m-ary relations-in-extension from n-ary relations-in-extension ($m \neq n$). Schema (ii) asserts the validity of a change of bound variables within extensional abstracts. And schema (iii) asserts the equivalence of identicals and the identity of equivalents. This property is the hallmark of extensional entities. The primary result for this extensional logic is the following corollary of the completeness and soundness theory for T1:

Corollary (Soundness and Completeness)
For all formulas A in a first-order extensional language with identity and extensional abstraction, A is extensionally valid if and only if A is a theorem of the logic for the language.

Thus, when $\in$ is treated as an arbitrary first-order predicate, set theory with identity and extensional abstraction is sound and complete.

16. A Complete Logic for the Second Conception

On conception 2 each definable intensional entity is such that when it is defined completely, it has a unique, non-circular definition. The logic T2 for L_ω on conception 2 consists of axioms A1–A7 and rules R1–R2 from T1, five additional axiom schemas for intensional abstracts, and one additional rule. In stating the additional principles, I write $t(F^n_m)$ to indicate that t is a complex term of L_ω in which the primitive predicate F^n_m occurs.

Additional Axiom Schemas and Rules for T2

$\mathscr{A}8$: $[A]_\alpha = [B]_\alpha \supset (A \equiv B)$

$\mathscr{A}9$: $t \neq r$ (where t and r are non-elementary complex terms of different syntactic kinds)

$\mathscr{A}10$: $t = r \equiv t' = r'$ (where t and r are the negations (existential generalizations, expansions, inversions, conversions, reflexivizations) of t' and r', respectively)

$\mathscr{A}11$: $t = r \equiv (t' = r' \;\&\; t'' = r'')$ (where t is the conjunction of t' and t'' and r is the conjunction of r' and r'', or t is the predication$_k$ of t' of t'' and r is the predication$_k$ of r' of r'', for $k \geq 0$)

$\mathscr{A}12$: $t(F_i^j) = r(F_h^k) \supset q(F_i^j) \neq s(F_h^k)$ (where t and s are elementary and r and q are not)

$\mathscr{R}3$: Let F_m^n be a non-logical predicate that does not occur in $A(v_i)$; let $t(F_m^n)$ be an elementary complex term, and let t' be any complex term of degree n that is free for v_i in $A(v_i)$. If $\vdash A(t)$, then $\vdash A(t')$.

$\mathscr{A}8$ affirms the equivalence of identical intensional entities. Schemas $\mathscr{A}9$–$\mathscr{A}11$ capture the principle that a complete definition of an intensional entity is unique. And schema $\mathscr{A}12$ captures the principle that a definition of an intensional entity must be non-circular. $\mathscr{R}3$ says roughly that if $A(t)$ is valid$_2$ for an arbitrary elementary n-ary term t, then $A(t')$ is valid$_2$ for any n-ary term t'.

Now recall the two intuitively valid arguments mentioned at the outset of this chapter. As we have seen, these arguments may be symbolized in L_ω as follows:

$$\begin{array}{l} (\forall z)(B(x,z) \supset B(y,z)) \\ \underline{B(x,[A])} \\ \therefore\; B(y,[A]) \end{array}$$

$$\begin{array}{l} [B(x)]_x = [U(x) \;\&\; M(x)]_x \\ \underline{N([(\forall x)(B(x) \equiv B(x))])} \\ \therefore\; N([(\forall x)(B(x) \equiv (U(x) \;\&\; M(x)))]). \end{array}$$

These arguments are both valid$_1$ and valid$_2$, and relatedly, in both T1 and T2 the conclusion of each argument is derivable from its premises.

To bring out the difference between T1 and T2 (and between validity$_1$ and validity$_2$), consider the following intuitively *invalid* argument involving the intentional predicate 'wonders':

x wonders whether there is a trilateral that is not a triangle.
Necessarily, all and only trilaterals are triangles.

∴ x wonders whether there is a triangle that is not a triangle.

Let this argument be symbolized as follows:

$$xW[(\exists y)(\text{Trilateral}\,(y)\ \&\ \neg\text{Triangle}\,(y))]$$
$$\Box(\forall y)(\text{Trilateral}\,(y) \equiv \text{Triangle}\,(y))$$
$$\therefore\ xW[(\exists y)(\text{Triangle}\,(y)\ \&\ \neg\text{Triangle}\,(y))].$$

In T1, but not T2, the conclusion of this argument is derivable from the two premises. And relatedly, the argument is valid$_1$, but not valid$_2$. So only the formal logic and semantics that are based on conception 2 could be appropriate for the treatment of intentional matters. The fact that Church's Alternative (2) and the various possible-worlds constructions of intensional logic (including Carnap's original construction in *Meaning and Necessity*) are all based on conception 1 is what lies at the root of their failure to provide adequate treatments of intentional matters. (See desideratum 2 on the chart in §4.)

The following is the primary result for T2:

Theorem (Soundness and Completeness)
For all formulas A in L_ω, A is valid$_2$ if and only if A is a theorem of T2 (i.e., $\models_2 A$ iff $\vdash_{T2} A$).

Proof. The proof of the soundness of T2 is quite straightforward. For example, the soundness of $\mathscr{A}8$ follows directly from Lemma 6 (stated earlier); $\mathscr{A}9$, from the fact that the fundamental logical operations Conj, Neg, Exist, ... in a type 2 model structure have disjoint ranges; $\mathscr{A}10$ and $\mathscr{A}11$, from the fact that these functions are one-one; $\mathscr{A}12$, from the fact that they are non-cycling. The soundness proofs for R1 and R2 are standard. For the soundness of $\mathscr{R}3$, the induction hypothesis yields $\models_2 A(t(F^n_m))$. Hence, by the soundness of R2, A2, and A5 (Leibniz's law), we have $\models_2 t(F^n_m) = t' \supset A(t')$. But since F^n_m is a non-logical predicate and does not occur in $A(t')$, $\models_2 A(t')$. The completeness proof is again

Henkin style. A set of L^*_ω-sentences is said to be *perfect*$_2$ if (1) it is maximal, consistent, ω-complete and (2) for every closed term $[B]_{v_1 \ldots v_p}$ in L^*_ω, there is a primitive predicate letter F^p_k such that the sentence $[B]_{v_1 \ldots v_p} = [F^p_k(v_1, \ldots, v_p)]_{v_1 \ldots v_p}$ belongs to the set. I show, first, that every consistent set of L_ω-sentences is included in some perfect$_2$ set of L^*_ω-sentences and, secondly, that every perfect$_2$ set has a type 2 model. The argument, while parallel to the argument used for T1, is routine.

The completeness problem for T2 is essentially simpler than the one for T1. This is no reflection on the relative importance of T2, though, for T2, not T1, provides a logic for intentional matters.

17. A Complete Logic for Modal and Intentional Matters

The conception 1 intensional logic T1 is ideally suited for treating modal matters. And the conception 2 intensional logic T2 is ideally suited for treating intentional matters. I will now formulate a richer conception 2 logic T2′ that is ideally suited for treating both modal and intentional matters. This simultaneous treatment is achieved by adjoining to L_ω a 2-place logical predicate $\approx_N$ which is intended to express the relation of necessary equivalence. T2′ succeeds in providing a single logic for both modal and intentional matters by having what are in effect two sorts of "identity"—one weak and one strong. The former is necessary equivalence; the latter, strict identity. In §46 I will show that, when conceptions 1 and 2 are synthesized, necessary equivalence (and also necessity) can be defined. So we should not feel hesitant to adjoin $\approx_N$ to L_ω here.

I begin by defining a new type of model structure. A *type 2′* model structure

$$\langle \mathscr{D}, \mathscr{P}, \mathscr{K}, \mathscr{G}, \text{Id}, \text{Eq}_N, \text{Conj}, \text{Neg}, \text{Exist}, \text{Exp}, \text{Inv}, \text{Conv}, \text{Ref}, \text{Pred}_0, \text{Pred}_1, \ldots \rangle$$

is any structure that satisfies all the conditions imposed on type 2 model structures plus one additional condition. The element Eq_N must be a distinguished element of $\mathscr{D}_2$ that satisfies the following principle:

$$(\forall H \in \mathscr{K})(H(\text{Eq}_N) = \{xy\colon (\exists i \geq -1)x, y \in \mathscr{D}_i \;\&\; (\forall H' \in \mathscr{K})H'(x) = H'(y)\}).$$

Thus $\mathrm{Eq_N}$ is to be thought of as the distinguished logical relation-in-intension *necessary equivalence*. Now an interpretation $\mathscr{I}$ relative to a type 2′ model structure is just like an interpretation relative to a type 1 or type 2 model structure except that $\mathscr{I}(\approx_N) = \mathrm{Eq_N}$. Then type 2′ denotation, truth, and validity are defined *mutatis mutandis* as in §14. The following abbreviations are introduced for notational convenience:

$\Box A$ *iff*$_{\text{df}}$ $[A] \approx_N [[A] \approx_N [A]]$
$\Diamond A$ *iff*$_{\text{df}}$ $\neg\Box\neg A$.

The intensional logic T2′ simply consists of the axioms and rules for T2 plus the following additional axioms and rules for $\approx_N$:

$\mathscr{A}13$: $x \approx_N x$
$\mathscr{A}14$: $x \approx_N y \supset y \approx_N x$
$\mathscr{A}15$: $x \approx_N y \supset (y \approx_N z \supset x \approx_N z)$
$\mathscr{A}16$: $x \approx_N y \supset \Box x \approx_N y$
$\mathscr{A}17$: $\Box(A \equiv_\alpha B) \equiv [A]_\alpha \approx_N [B]_\alpha$
$\mathscr{A}18$: $\Box A \supset A$
$\mathscr{A}19$: $\Box(A \supset B) \supset (\Box A \supset \Box B)$
$\mathscr{A}20$: $\Box A \supset \Box\Diamond A$
$\mathscr{R}4$: if $\vdash A$, then $\vdash \Box A$.

Notice that these axioms and rules for $\approx_N$ are just analogues of the special T1 axioms and rules for $=$. Finally, the soundness and completeness of T2′ can be shown by applying the methods of proof used for T1 and T2.

Intensional logic constitutes the first stage in the theory of PRPs. Why is it that complete intensional logics can be achieved in the setting of a first-order language such as L_ω but not in the setting of a higher-order language? The answer lies in the opposing treatments of *predication*. To this, the second stage in the theory of PRPs, I will soon turn. But first I must address a problem that is perhaps the major outstanding problem in intensional logic, namely, the paradox of analysis.