Math Camp

Homework 4

(1) Compute the derivative of each function.

(a) $f(x) = 3x^5 + 4x^3 - 7x^2 - 3$

$f'(x) = 15x^4 + 12x^2 - 14x$

(b) $f(x) = x^5 + 5^x + 5x + 5$

$f'(x) = 5x^3 + \ln(5)5^x + x$

(c) $f(x) = \sqrt[3]{e^x + x}$

$f'(x) = \frac{1}{3}(e^x + x)^{-2/3}(e^x + 1)$

(d) $f(x) = \frac{x^3+1}{x^3-1}$

$f'(x) = \frac{3x^2(x^3-1)-(x^3+1)3x^2}{(x^3-1)^2}$

(e) $f(x) = \ln(x^4 + 1)\sqrt{x^4 - 1}$

$f'(x) = \frac{4x^3}{x^4+1}\sqrt{x^4 - 1} + \ln(x^4 + 1)\frac{1}{2}(x^4 - 1)^{-1/2}(4x^3)$

(f) $f(x) = x^3 e^{4x^2}$

$f'(x) = 3x^2 e^{4x^2} + x^3 e^{4x^2}(8x)$

(g) $f(x) = \sin(\sqrt{x^2 + 4})$

$f'(x) = \cos(\sqrt{x^2 + 4})\frac{1}{2}(x^2 + 4)^{-1/2}(2x)$

(h) $f(x) = x\cos(x^5 e^x)$

$f'(x) = \cos(x^5 e^x) - x\sin(x^5 e^x)(5x^4 e^x + x^5 e^x)$

(2) Find the tangent line to the graph of $y = \dfrac{1}{x^2 + 1}$ at the point where $x = -1$.

The derivative is $y' = \frac{-2x}{(x^2+1)^2}$, which equals $-1/2$ when $x = 1$. The line through $(-1, 1/2)$ with slope $-1/2$ is $y = (-1/2)x - 1/2$.

(3) Find the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. It may be helpful to use the fact that the upper half of the circle is given by the equation $y = \sqrt{25 - x^2}$.

The derivative is $y' = \frac{-x}{\sqrt{25-x^2}}$, so $y'(3) = \frac{-3}{\sqrt{16}} = \frac{-3}{4}$. The line is $y = \frac{-3}{4}x + \frac{25}{4}$.

(4) For each of the following functions, find all intervals where f is increasing and decreasing, all intervals where f is convex (concave upward) and concave (concave downward), and all values of x where the f attains a local maximum or local minimum.

(a) $f(x) = x^3 - 12x$

The derivatives are $f'(x) = 3x^2 - 12$ and $f''(x) = 6x$. The function is increasing on $(-\infty, -2) \cup (2, \infty)$, decreasing on $(-2, 2)$, convex on $(0, \infty)$, and concave on $(-\infty, 0)$.

(b) $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$. Notice that $f'(x) = 0$ when $x = 1$ or $x = -1$, but f' can also change sign by passing through $x = 0$, where the function is not defined. f is increasing on $(-\infty, -1)$ and $(1, \infty$, and decreasing on $(-1, 0)$ and $(0, 1)$. It is convex on $(0, \infty)$ and concave on $(-\infty, 0)$.

(c) $f(x) = xe^{2x}$

$f'(x) = e^{2x} + 2xe^{2x} = e^{2x}(1 + 2x)$ and $f''(x) = 4e^{2x} + 4xe^{2x} = e^{2x}(4 + 4x)$. Noticing that e^{2x} is always positive, f' changes sign only at $x = -1/2$, and f'' changes sign only at $x = -1$. So f is increasing for $x > -1/2$ and decreasing for $x < -1/2$; f is convex for $x > -1$ and concave for $x < -1$.