Math Camp

Homework 5

(1) Find the (global) maximum and minimum of each function on the given interval, if it exists:

(a) $f(x) = x^3 + x + 1$ on $[-1, 2]$

$f'(x) = 3x^2 + 1$, which is always defined and never zero. This means that the max and min can only occur at the endpoints. So the min is $f(-1) = -1$ and the max is $f(2) = 11$.

(b) $f(x) = x^2 - 8x + 15$ on $[1, 6]$

$f'(x) = 2x - 8$, so $x = 4$ is the only point where the first-order conditions apply. We check $f(1) = 1 - 8 + 15 = 8$, $f(4) = 16 - 32 + 15 = 1$, $f(6) = 36 - 48 + 15 = 3$. So the max is 8 and the min is 1.

(c) $f(x) = \dfrac{2x}{x^2 + 1}$ on $[0, 3]$

$f'(x) = \dfrac{2(x^2+1) - 2x(2x)}{x^2+1} = \dfrac{2 - 2x^2}{x^2+1} = \dfrac{2(1-x)(1+x)}{x^2+1}$. This is defined on all of $[0, 3]$, and zero at $x = 1$ (notice that we don't need to worry about the zero at -1 because it's outside our domain). We check: $f(0) = 0$, $f(1) = \frac{2}{2} = 1$, $f(3) = \frac{6}{10} = .6$. So the min is 0 and the max is 1.

(d) $f(x) = 3x^{2/3} + 2x$ on $[-1, 1]$ (Hint: be sure to check where $f'(x)$ is undefined)

$f'(x) = 2x^{-1/3} + 2$. At $x = 0$, $f'(x)$ is undefined. Setting $2x^{-1/3} + 2 = 0$, we write this as $\frac{2}{x^{1/3}} = -2$ and thus $1 = -x^{1/3}$. Cubing both sides, $-x = 1$ and so $x = -1$, which is an endpoint of our interval. So we check $f(-1) = 3 - 2 = 1$, $f(0) = 0$, and $f(1) = 3 + 2 = 5$. The min is 0 and the max is 5.

(e) $f(x) = x\ln(x)$ on $(0, \infty)$.

$f'(x) = \ln(x) + 1$, which is defined on the domain, and zero at $x = e^{-1}$. There is only one critical point, and using $f''(x) = 1/x$, we see $f''(e^{-1}) = e > 0$, so it is a local min. Thus the minimum is $f(e^{-1}) = e^{-1}\ln(e^{-1}) = -e^{-1}$. There is no maximum: as $x \to \infty$, both x and $\ln(x)$ increase without bound, so $f(x)$ does as well.

(f) $f(x) = e^x - e^{2x}$ on $(-\infty, \infty)$.

$f'(x) = e^x - 2e^{2x}$. This is defined on all of $\mathbb{R}$. Factoring out e^x, we get $e^x(1-2e^x) = 0$. This is zero when $1 = 2e^x$, or $x = \ln(1/2) = -\ln(2)$. Taking $f''(x) = e^x - 4e^{2x}$, we have $f''(\ln(1/2)) = 1/2 - 4e^{2\ln(1/2)} = 1/2 - 4(1/2)^2 = -1/2 < 0$. As in part (e), there is a maximum but no minimum, as $f(x) \to -\infty$ as $x \to \infty$. The max is $f(\ln(1/2)) = 1/2 - e^{2\ln(1/2)} = 1/2 - 1/4 = 1/4$.

(2) Find all local maximum and minimum values of $f(x) = x^4 - 2x^2$, and the x-coordinates where they occur. Does $f(x)$ have a global maximum or minimum on $\mathbb{R}$?

$f'(x) = 4x^3 - 4x = 0$ when $x = 0, 1, -1$. As $f''(x) = 12x^2 - 4$, we see that f has a local max at $x = 0$ and local mins at $x = \pm 1$. There is a global minimum, which we can compute by looking at end behavior. As $x \to \pm\infty$, $f(x)$ increases forever. The local minimum $f(1) = f(-1) = -2$ is attained twice, and is the global minimum.

(3) Of all the tangent lines to the graph of $y = \dfrac{1}{x^2+3}$, which are the lines with the largest and smallest slope?

The slope of the tangent line is, of course, $m(x) = f'(x) = \frac{-2x}{(x^2+3)^2}$. The problem is to minimize this quantity over $x \in \mathbb{R}$. So we take its derivative:

$$m'(x) = \frac{-2(x^2+3)^2 + 2x(2)(x^2+3)(2x)}{(x^2+3)^4} = \frac{(x^2+3)(-2(x^2+3)+8x^2)}{(x^2+3)^4} = \frac{6x^2-6}{(x^2+3)^3}$$

This is defined everywhere, and zero when $x = \pm 1$. So we compute $m(-1) = \frac{2}{16}$ and $m(1) = \frac{-2}{16}$. These are the min and max slope, respectively: looking at the sign of m', we see that m decreases on $(-\infty, -1)$ and $(1, \infty)$ and increases on $(-1, 1)$. However, the end behavior of m is that it decays to zero as $x \to \infty$ or $x \to -\infty$, so the local max and min are global max and min points. The actual lines are given by

$$y = \frac{1}{8}x + \frac{1}{8} \text{ and } y = -\frac{1}{8}x + \frac{3}{8}.$$

at $x = -1$ and $x = 1$, respectively.

(4) Find the point on the graph of $y = \sqrt{x}$ that is closest to the point $(4, 0)$.

We take $d(x) = \sqrt{(x-4)^2 + (\sqrt{x}-0)^2} = \sqrt{x^2 - 8x + 16 + x} = \sqrt{x^2 - 7x + 16}$. We want to maximum $d(x)$ on the interval $[0, \infty)$, as this is where $\sqrt{x}$ is defined. So

$$d'(x) = \frac{1}{2\sqrt{x^2 - 7x + 16}(2x-7)}$$

This is zero at $x = \frac{7}{2}$. It is defined everywhere, as is apparent from the fact that it's a distance if not from the formula. (The quantity under the square root is always positive.) From geometric considerations, the unique critical point corresponds to a minimum rather than a maximum. The point in question is $(7/2, \sqrt{7/2})$.

(5) What is the rectangle of largest area that can be formed with one edge aligned with the x axis and upper two corners lying on the graph of the circle $x^2 + y^2 = 4$?

If we label the upper right corner of the rectangle as (x, y), then the height is y, the width is $2x$, and we want to maximum $A = 2xy$ subject to the constraint that $y = \sqrt{4 - x^2}$. So of course we maximum $A(x) = 2x\sqrt{4 - x^2}$. We only need to worry about this when $x \in [0, 2]$, and in fact, the endpoints give degenerate area-zero rectangles where the max won't be attained. Take

$$A'(x) = 2\sqrt{4 - x^2} + 2x\frac{1}{2}(4 - x^2)^{-1/2}(-2x) = 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}}$$

This is defined on all of $(0, 2)$. Setting it equal to zero we have $2(4 - x^2) = 2x^2$, which solves to $x = \sqrt{2}$. This gives the maximum-area rectangle, which is when $(x, y) = (\sqrt{2}, \sqrt{2})$ and the area is $2(\sqrt{2})(\sqrt{2}) = 4$. This arises when the right half of the rectangle is a square of area 2.