Math Camp

Homework 8

(1) Let X be a random variable with PDF given by

$$f(x) = \begin{cases} 0 & : x < 1 \\ ax & : 1 \leq x \leq 3 \\ 0 & : x > 3 \end{cases}$$

Compute the expectation, the variance, and the CDF of X. (You'll need to find a, which can be done by using the fact that the total integral of the PDF is 1.)

First, we must have $\int_{-\infty}^{\infty} f(x)\,dx = \int_1^3 ax\,dx = 1$. So

$$a\frac{x^2}{2}\Big|_1^3 - a\left(\frac{9}{2} - \frac{1}{2}\right) = 4a = a,$$

and we have $a = 1/4$. Now we can compute

$$E[X] = \int_1^3 \frac{1}{4}x^2\,dx = \frac{1}{4}\left(\frac{3^3}{3} - \frac{1^3}{3}\right) = \frac{1}{4}\frac{26}{3} = \frac{26}{12}$$

$$E[X^2] = \int_1^3 \frac{1}{4}x^3\,dx = \frac{1}{4}\left(\frac{3^4}{4} - \frac{1^4}{4}\right) = \frac{80}{16} = 5$$

$$Var(X) = E[X^2] - E[X]^2 = 5 - \left(\frac{26}{12}\right)^2 = 5 - \frac{169}{36} = \frac{11}{36}$$

$$CDF(x) = F(x) = \int_1^x \frac{1}{4}t\,dt = \frac{1}{4}\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

To be completely precise, the above F is the CDF in the interval $[1, 3]$; on the entire real number line the CDF is

$$F(x) = \begin{cases} 0 & : x < 1 \\ \frac{1}{4}\left(\frac{x^2}{2} - \frac{1}{2}\right) & : 1 \leq x \leq 3 \\ 1 & : x > 1 \end{cases}$$

(2) Let X be a uniform random variable on the interval $[2, 4]$. That is, the PDF of X is given by

$$f_X(x) = \begin{cases} 0 & : x < 2 \\ 1/2 & : 2 \leq x \leq 4 \\ 0 & : x > 4 \end{cases}$$

(a) Compute the CDF F_X of X.

On the interval $[2, 4]$, $F_X(x) = \int_2^x (1/2)\,dt = (1/2)t\Big|_2^x = \frac{x-2}{2}$. To the left of 2, $F(x) = 0$, and to the right of 4, $F(x) = 1$.

(b) Let Y be a random variable defined by $Y = X^2$. Compute the CDF F_Y of Y. (Hint: $F_Y(x) = P(Y \leq x) = P(X^2 \leq x) = P(X \leq \sqrt{x})$. The range of Y should be $[4, 16]$.)

$F_Y(x) = P(X \leq \sqrt{x}) = F_X(\sqrt{x}) = \frac{\sqrt{x}-2}{2}$. This formula is correct for $4 \leq x \leq 16$. To the left of $x = 4$, $F_Y(x) = 0$, and to the right of $x = 16$, $F_Y(x) = 1$.

(c) Compute the PDF f_Y of Y by taking the derivative of what you found in (b). Check your answer by showing that $\int_4^{16} f_Y(x)\, dx = 1$.

$f_y(x) = \frac{1}{4\sqrt{x}}$. The check:

$$\int_4^{16} \frac{1}{4\sqrt{x}}\, dx = \frac{1}{4}\frac{x^{1/2}}{1/2}\Big|_4^{16} = \sqrt{16}/2 - \sqrt{4}/2 = 2 - 1 = 1$$

(3) In a certain city, 90% of taxis are yellow, and the remaining 10% are blue. Suppose that all taxis are equally likely to be involved in an accident. For any accident involving a taxi of either color, an eyewitness correctly reports the color of the taxi 80% of the time, and reports the other possible color 20% of the time.

If an accident involving a taxi occurs and an eyewitness reports that the taxi is blue, what is the probability that the taxi is blue?

This is an application of conditioning and Bayes' rule. Let A be the event "taxi is blue" and let B be the event "eyewitness reports taxi is blue". We know

$$P(B \mid A) = \frac{80}{100} = .8.$$

We can compute

$$P(B) = P(B \mid A)P(A) + P(B \mid \overline{A})P(\overline{A}) = \frac{80}{100} \cdot \frac{10}{100} + \frac{20}{100} \cdot \frac{90}{100} = .08 + .18 = .26$$

By Bayes' rule,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} = \frac{(.8)(.1)}{.26} = \frac{8}{26} \approx 31\%$$

(4) Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$. Recall that a function $f : A \to B$ is a way of assigning an element of B to every element of A (an output to each input).

(a) How many different functions $f : A \to B$ are there?

For each element of A, we make a choice among the 4 elements of B. Thus there are $4^3 = 64$ such functions.

(b) How many different functions $g : B \to A$ are there?

By the same reasoning as in (a), there are $3^4 = 81$ such functions.

(c) How many injective (one-to-one) functions $f : A \to B$ are there?

There are 4 choices for $f(1)$. Having chosen $f(1)$, there are only 3 choices for $f(2)$, then only 2 for $f(3)$. So there are $(4)(3)(2) = 24$ such functions.

(d) How many surjective (onto) functions $g : B \to A$ are there?

This is harder to compute than (c). Perhaps the easiest approach is to consider the functions that are *not* surjective, then subtract this from the total 81.
If $g : B \to A$ is not surjective, then it misses something in its range, either $1, 2$, or 3. Let A_1 be the set of functions that do not have 1 in the range, and likewise for A_2 and A_3. Then $|A_1| = |A_2| = |A_3| = 4^2 = 16$, again by the reasoning as in (a) and (b). The set of non-surjective functions is the union $A_1 \cup A_2 \cup A_3$. By inclusion-exclusion,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

For each pair (i, j), $|A_i \cap A_j| = 1$. For example, the only function that omits 1 from the range and also omits 2 is the function that sends every element of B to 3. The triple intersection is the empty set, as a function must have something in its range. So the count of non-surjective functions is $16(3) - 1(3) = 45$. The count we actually want is $81 - 45 = 36$.

(e) Why are there no surjective functions $f : A \to B$ and no injective functions $g : B \to A$?

For any $f : A \to B$, there are not 4 elements of A that f can map to the 4 elements of B. For any function $g : B \to A$, there are more elements of the domain than the codomain, so there must be two elements of B that map to the same element of A.