Math Camp

Homework 8

(1) Let A, B and C be defined by

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}.$$

Compute each of the following matrix products if possible. (Some are not possible.)

$$AB, BA, AC, CA, BC, \text{ and } CB$$

$$AB = \begin{pmatrix} 2 & 4 & 6 \\ 7 & 11 & 12 \\ 14 & 18 & 21 \end{pmatrix}$$

$$BA = \begin{pmatrix} 5 & 14 \\ 17 & 31 \end{pmatrix}$$

$$AC = \begin{pmatrix} 0 & 4 \\ 1 & 10 \\ 4 & 14 \end{pmatrix}$$

$$CB = \begin{pmatrix} 8 & 10 & 12 \\ 17 & 22 & 27 \end{pmatrix}$$

Neither CA nor BC is possible to compute.

(2) Find all solutions to each system of equations by Gaussian elimination.

(a)

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ 2x_1 - x_2 - x_3 &= -4 \\ 2x_1 - 2x_3 &= -2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 2 & -1 & -1 & -4 \\ 2 & 0 & -2 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -3 & -3 & -18 \\ 0 & -2 & -4 & -16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 6 \\ 0 & -2 & -4 & -16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

So $x_1 = 1$, $x_2 = 4$, and $x_3 = 2$.

(b)

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 3 \\ 1 & 0 & 2 & 1 & 8 \\ 1 & 1 & 1 & 0 & 5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 8 \\ 0 & 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 & 5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 8 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 & -3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 8 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & -2 & -2 & -6 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 8 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 8 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right)$$

So $a - 2d = 2$, $b = 0$, and $c + d = 3$. This means d is a free variable; we can choose any $d \in \mathbb{R}$ and then compute $a = 2 + d$, $b = 0$, $c = 3 - d$.

(c)

$$\begin{pmatrix} 5 & 1 \\ 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$$

This system has no solutions. The first two equations are uniquely solved by $x = 1$ and $y = 1$, but then this is inconsistent with the third equation.

(3) Compute the determinant of the matrix $\begin{pmatrix} 1 & -2 & 2 & 4 \\ 1 & 1 & 3 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$.

We expand down the first column of the matrix, then down the second row of each 3×3 matrix that appears.

$$1 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 2 & 4 \\ 2 & 0 & -1 \\ 1 & 0 & 2 \end{vmatrix} = (1)(-3) \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} - (1)(-2) \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -3(5) + 2(5) = -5$$

(4) Find the inverse of the matrix $A = \begin{pmatrix} 5 & 3 & 4 \\ 3 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix}$. Then use A^{-1} to solve the systems

$$Ax = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \text{ and } Ax = \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 5 & 3 & 4 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 5 & 3 & 4 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 & 1 & -3 \\ 0 & 3 & 4 & 1 & 0 & -5 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 & -3/2 \\ 0 & 3 & 4 & 1 & 0 & -5 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 & -3/2 \\ 0 & 0 & -1/2 & 1 & -3/2 & -1/2 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 & -3/2 \\ 0 & 0 & 1 & -2 & 3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 & -4 & -3 \\ 0 & 0 & 1 & -2 & 3 & 1 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 3 & -4 & -3 \\ -2 & 3 & 1 \end{pmatrix}$. We can solve the first system to get

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 3 & -4 & -3 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

and the second one to get

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 3 & -4 & -3 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \\ 2 \end{pmatrix}$$