Math Camp

Homework 7

(1) Suppose an experiment is run where four fair coins are flipped. Let

$$X = (\text{number of heads})^2 - (\text{number of tails}).$$

Compute the PMF of X and use it to compute $P(-3 \le X \le 3)$. Then compute the expectation and the variance of X.

There are 16 possible outcomes of the experiment:

1 outcome	with 0 heads and 4 tails
4 outcomes	with 1 heads and 3 tails
6 outcomes	with 2 heads and 2 tails
4 outcome	with 3 heads and 2 tails
1 outcome	with 4 heads and 1 tails

So $f(-4) = 1/16$, $f(-2) = 4/16$, $f(2) = 6/16$, $f(7) = 4/16$, and $f(16) = 1/16$. No other numbers are in the range of X, though of course we could declare that $f(x) = 0$ for all other real numbers x. Now we can carry out all the computations.

$$P(-3 \le X \le 3) = f(-2) + f(2) = \frac{10}{16}$$

because 2 and -2 are the only values of X in this range.

$$E[X] = (-4)\frac{1}{16} + (-1)\frac{4}{16} + (2)\frac{6}{16} + (7)\frac{4}{16} + (16)\frac{1}{16} = \frac{48}{16} = 3$$

$$Var(X) = E[X^2] - E[X]^2 = (16)\frac{1}{16} + (1)\frac{4}{16} + (4)\frac{6}{16} + (49)\frac{4}{16} + (256)\frac{1}{16} - 3^2 = \frac{496}{16} - 9 = 22$$

(2) Let the random variable X be given by the PDF

$$f(x) = \begin{cases} 1/2 & : -1 \le x \le 0 \\ 1/4 & : 0 \le x \le 2 \end{cases}$$

Compute $P(X \le 1)$, then compute the expectation and the variance of X.

The key to computing the integrals we need is to break them up into two intervals: -1 to 0 and 0 to 2.

$$P(X \le 1) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1/2)\,dx + \int_{0}^{1} (1/4)\,dx = (1/2)(1) + (1/4)(1) = 3/4$$

$$E[X] = \int_{-1}^{2} xf(x)\,dx = \int_{-1}^{0} x(1/2)\,dx + \int_{0}^{2} x(1/4)\,dx = \left.\frac{x^2}{4}\right|_{-1}^{0} + \left.\frac{x^2}{8}\right|_{0}^{2} = -\frac{1}{4} + \frac{4}{8} = \frac{1}{4}$$

$$E[X^2] = \int_{-1}^{2} x^2 f(x)\,dx = \int_{-1}^{0} x^2(1/2)\,dx + \int_{0}^{2} x^2(1/4)\,dx = \frac{x^3}{6}\Big|_{-1}^{0} + \frac{x^3}{12}\Big|_{0}^{2} = \frac{1}{6} + \frac{8}{12} = \frac{5}{6}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{5}{6} - \frac{1}{16} = \frac{15-3}{48} = \frac{12}{48} = \frac{1}{4}$$

(3) In lecture we discussed improper integrals where one or both bounds may be infinite. Another type of improper integral is when the bounds are real numbers, but the function blows up to infinity at one end. Compute

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

by first computing $\int_a^1 \frac{1}{\sqrt{x}}\,dx$, then taking a limit as $a \to 0^+$.

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \frac{x^{1/2}}{1/2}\Big|_a^1 = \lim_{a\to 0^+} (2\sqrt{1} - 2\sqrt{a}) = 2$$

Use the same idea to show

$$\int_0^2 \frac{1}{(x-2)^3}\,dx$$

is infinite. (For the second integral, you may need a u-substitution first.)

Taking $u = (x-2)$, we see that an antiderivative of $(x-2)^{-3}$ is $\frac{-1}{2(x-2)^2}$. So

$$\int_0^2 \frac{1}{(x-2)^3}\,dx = \lim_{b\to 2^-} \frac{-1}{2(b-2)^2} + \frac{1}{-8}.$$

As $b \to 2$ from the left, the quantity $(b-2)^2$ approaches zero, so the expression blows up to infinity.

(4) Let the random variable X be given by the PDF

$$f(x) = \begin{cases} \frac{2}{x^3} & : 1 \le x < \infty \\ 0 & : x < 1 \end{cases}$$

Compute $P(a \le X \le b)$ as a formula in terms of a and b. Then compute the expectation and variance of X (note that one or both of these might be infinite).

$$P(a \le X \le b) = \int_a^b 2x^{-3}\,dx = -\frac{1}{x^2}\Big|_a^b = \frac{1}{a^2} - \frac{1}{b^2}.$$

$$E[X] = \int_1^\infty x\frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x^2}\,dx = \lim_{b\to\infty} \frac{-2}{x}\Big|_1^b = \lim_{b\to\infty} \frac{-2}{b} + \frac{2}{1} = 2$$

$$E[X^2] = \int_1^\infty x^2\frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big|_1^b == \lim_{b\to\infty} \ln(b) - \ln(1) = \infty$$

As $E[X^2]$ is infinite, the variance will be infinite as well.