Math 115 Calculus II
10.2 Arc Length with Parametric Functions

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NEXT WEEK
OFFICE HOUR
MTW@6

Midterm Discussion

NEXT WEEK
Tu: 10.3, 10.4 Polar
Th: ???

Coordination
Formula for Arc Length of a Parametric Curve

In 1-dimensional setting: Position of a particle is $x(t)$. And the velocity is $v(t) = x'(t)$.

Total distance travelled from $a \leq t \leq b$:
$$\int_a^b |v(t)| \, dt = \int_a^b |x'(t)| \, dt.$$

In 2-dimensional setting: Position of a particle is $(x(t), y(t))$

What is the length of a "small" piece of the curve?

Straight line approximation:

$$(x(t+\Delta t), y(t+\Delta t))$$

$$L \approx \sqrt{(x(t+\Delta t) - x(t))^2 + (y(t+\Delta t) - y(t))^2}$$

$$= \sqrt{(\frac{x(t+\Delta t) - x(t)}{\Delta t})^2 + (\frac{y(t+\Delta t) - y(t)}{\Delta t})^2} \Delta t.$$

Length of curve $a \leq t \leq b$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$
Example

Sketch the parametric curve described by $x = 1 + 4t^2$, $y = 4 + 2t^3$. Find the length of the curve if $0 \leq t \leq 1$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x(t)$</th>
<th>$y(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>37</td>
<td>-50</td>
</tr>
<tr>
<td>-2</td>
<td>17</td>
<td>-12</td>
</tr>
<tr>
<td>-1</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>37</td>
<td>58</td>
</tr>
</tbody>
</table>

Length from $t = 0$ to $t = 1$

\[
\int_{0}^{1} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt
\]

\[
= \int_{0}^{1} \sqrt{8t} + 6t^2 \, dt
\]

\[
= \int_{0}^{1} \sqrt{64t^2 + 36t^4} \, dt
\]

\[
= \frac{1}{54} (64 + 36t^2)^{3/2} \bigg|_{0}^{1}
\]

\[
= \frac{1}{54} (1000 - 512)
\]

\[
= \frac{488}{54}
\]
Choose a parametrization of the unit circle and find its circumference using the parametrization.

\[(x, y) = (\cos \theta, \sin \theta) \quad \text{or} \quad (\sin \theta, \cos \theta) \quad 0 \leq \theta \leq 2\pi \]

\[0 \leq \theta \leq 2\pi \]

\[0 \leq \pi \leq \theta \leq 3\pi \]

\[\text{or Etc.} \]

CIRCUMFERENCE = \[
\int_{0}^{2\pi} \sqrt{(x'(\theta))^2 + (y'(\theta))^2} \, d\theta
\]

\[\int_{0}^{2\pi} \sqrt{\sin^2 \theta + \cos^2 \theta} \, d\theta
\]

\[= \int_{0}^{2\pi} 1 \, d\theta = 2\pi.\]
Example

Sketch the parametric curve described by $x = t \cos t$, $y = t \sin t$. Find the length of the curve if $0 \leq t \leq \pi$. 

\[ x^2 + y^2 = t^2 \]

**EXERCISE:** Graph the curve for $-4\pi \leq t \leq 0$.

**LENGTH FROM $0 \leq t \leq 4\pi$**

\[ L = \int_{0}^{4\pi} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \]

\[ = \int_{0}^{4\pi} \sqrt{(t \sin t - t \cos t)^2 + (t \cos t + t \sin t)^2} \, dt \]

\[ = \int_{0}^{4\pi} \sqrt{1 + t^2} \, dt \]

$t = \tan \theta$
Arc Length of the Graph of a Function

Graphs of functions \( y = f(x) \) are all parametric curves. For example, we can view the graph as the parametric equations:

\[
x = t \quad y = f(t)
\]

\((x \text{ is the parameter}, \ x = x \quad y = f(x))\).

Length of graph of \( f(x) \) for \( a \leq x \leq b \):

\[
= \int_a^b \sqrt{\left( x'(t) \right)^2 + \left( y'(t) \right)^2} \, dt
= \int_0^b \sqrt{1 + (f'(x))^2} \, dx
= \int_a^b \sqrt{1 + (f'(x))^2} \, dx
\]
Example

Find the length of the curve for the function \( f(x) = 1 + 6x^{3/2} \) with \( 0 \leq x \leq 2 \).

\[
\text{LENGTH OF GRAPH OF } y = f(x) \text{ FOR } 0 \leq x \leq 2
\]

\[
= \int_0^2 \sqrt{1 + (f'(x))^2} \, dx
\]

\[
= \int_0^2 \sqrt{1 + (9x^{1/2})^2} \, dx
\]

\[
= \int_0^2 \sqrt{1 + 81x} \, dx
\]

\[
= \frac{2}{243} (1 + 81x)^{3/2} \bigg|_0^2
\]

\[
= \ldots
\]
Example

Find the length of the curve for the curve \( x = \frac{1}{3} \sqrt{y(y - 3)} \) with \( 1 \leq y \leq 9 \).