Math 115 Calculus II

10.4 Polar Area

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OFFICE HOURS
UP TO THE EXAM,
WILL BE BY APPOINTMENT,
SEND ME EMAIL.
Table of Contents

Polar Curves

Polar Arc Length

Polar Area

Just for fun

MAYBE ON THE EXAM, MAYBE NOT

DEFINITELY ON THE EXAM
Matching.

I. \( r = 2 \sin(\theta) + 1 \)
II. \( r = 3 \sin(2\theta) \)
III. \( r = \sin(2\theta) + 2 \geq 1 \)
IV. \( r = 2 \sin(3\theta) \)

A. \( \text{I} \)
B. \( \text{II} \)
C. \( \text{II} \)
D. \( \text{III} \)

- I has 4 petals, general rule: if n even, 2n petals.
- II has 2 petals, general rule: if n is odd, n petals.
- III has 3 petals, general rule: if n is odd, n petals.

\( n \)
Table of Contents

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Polar Arc Length Formula

\[ x = r \cos \theta \]

\[ y = r \sin \theta \]

**Remember** \( r \) **is a function of** \( \theta \)

**But we are lazy and we write** \( r \)** instead of** \( r(\theta) \).

**Arc Length**

\[
\text{ARCLength} = \int_{a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta
\]

\[
= \int_{a}^{b} \sqrt{(r' \cos \theta - r \sin \theta)^2 + (r' \sin \theta + r \cos \theta)^2} \, d\theta
\]

\[
= \int_{a}^{b} \sqrt{(r')^2 + r^2} \, d\theta
\]

\[
= \int_{a}^{b} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta
\]

**Use product rule for derivatives:**

\[
\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta
\]

\[
\frac{dy}{d\theta} = r' \sin \theta + r \cos \theta
\]

... **(some basic algebra)**
Polar Arc Length

Example

Find the arc length of the curve \( r = 2 \sin(\theta) \) for \( 0 \leq \theta \leq 2\pi \).

\[
\text{Arc length} = \int_{0}^{2\pi} \sqrt{(\frac{dr}{d\theta})^2 + r^2} \, d\theta.
\]

\[
= \int_{0}^{2\pi} \sqrt{(2 \cos \theta)^2 + (2 \sin \theta)^2} \, d\theta.
\]

\[
= \int_{0}^{2\pi} 2 \, d\theta.
\]

\[
= 4\pi.
\]
Table of Contents

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Polar Area

Just for fun
As $\theta$ increases, the "radial arm" of a polar curve sweeps out some area.

**Area swept out by polar curve from $\theta$ to $\theta + \Delta \theta$**

Blue circular sector of radius $r$ and angle $\Delta \theta$

\[
eq \frac{1}{2} r^2 \theta.\]

Think of the total area swept out as the sum of

A bunch of these approximations in blue and let $\Delta \theta \to 0$...

Total area swept out by a polar curve

\[
= \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta.
\]
Example

Find the area of the 3 petals of the curve \( r = 2 \sin(3\theta) \).

\[
\text{AREA} = \int_0^\frac{\pi}{2} \frac{1}{2} (2\sin 3\theta)^2 \, d\theta = \ldots
\]

\( \text{(You do the computation.)} \)

Other ways that work:

1. All 3 petals are congruent. One petal is swept out by \( 0 \leq \theta \leq \frac{\pi}{3} \).

\[
\therefore \text{AREA} = 3 \int_0^{\frac{\pi}{3}} \frac{1}{2} (2\sin 3\theta)^2 \, d\theta
\]

2. As \( \theta \) changes from 0 to \( \frac{\pi}{2} \), 1.5 petals are swept out.

\[
\therefore \text{AREA} = 2 \int_0^{\frac{\pi}{2}} \frac{1}{2} (2\sin \theta)^2 \, d\theta
\]
Find an expression representing the area of the blue region.

\[ r = 2 \sin(\theta) + 1 \]

Again, imagine the radial arm moving and sweeping out an area.

Points of interest:

\[ r = 0 \Rightarrow 2 \sin \theta + 1 = 0 \]

\[ \Rightarrow \theta = \ldots, -\pi/6, 2\pi/6, 4\pi/6, \ldots \]

→ Don’t include the area inside the red “loop”.

\[ \int_{-\pi/6}^{7\pi/6} \frac{1}{2} r^2 \, d\theta \]

\( \text{BLUE + RED AREA} \)

\[ - \int_{7\pi/6}^{11\pi/6} \frac{1}{2} r^2 \, d\theta \]

\( \text{RED AREA} \)

\[ = \text{BLUE AREA} \]
Table of Contents

Polar Curves

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Just for fun
Complex numbers and the complex plane

\[ x^2 + 1 = 0 \]

\[ i = \sqrt{-1} \]

\[ z = a + bi \]

\[ (2 - 3i)i = 2i - 3i^2 = -3 + 2i \]
The complex exponential

In this course, we talked about power series of real numbers and viewing them as functions. For example, we have

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \]

If one wishes, we could in fact just take the power series as the definition of \( e^x \) as this power series (instead of defining \( \ln x \) as the integral of \( \frac{1}{x} \) and then defining \( e^x \) as the inverse function of \( \ln x \)).

Let's take this one step further and define a complex exponential function:

\[ e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \cdots \]

It turns out this converges for all complex numbers \( z \) (which is not a surprise given that \( e^x \) has radius of convergence \( \infty \)).
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The complex exponential and trigonometric functions

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Let's substitute \( z = i \theta \) in the above, where \( \theta \) is a real number.

\[ e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \]

\[ = 1 + i\theta - \theta^2 + \frac{i\theta^3}{3!} - \theta^4 + \frac{i\theta^5}{5!} - \theta^6 + \frac{i\theta^7}{7!} + \cdots \]

\[ = (1 - \theta^2 + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots) \]

\[ = \cos \theta + i \sin \theta \]

In particular, if we plug in \( \theta = \pi \) and rearrange, we get

\[ e^{i\pi} + 1 = 0 \]

This is the key takeaway from this course.
The complex exponential and trigonometric functions

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