RUDIN'S DOWKER SPACE AND THE LINEARLY LINDELÖF PROBLEM

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1. Introduction

In this report, we present Mary Ellen Rudin's Dowker space and discuss the closely related linearly Lindelöf problem.

Definition 1.1. A *Dowker space* is a normal topological space X, whose product with the unit interval, $X \times [0,1]$, is not normal.

Dowker spaces are named after C.H. Dowker, whose name is attributed to the following theorem. Before stating the theorem, we introduce the relevant notion of countable paracompactness.

Definition 1.2. A space X is said to be *countably paracompact (metacompact)* iff every countable open cover of X has a locally-finite (point-finite) refinement.

Theorem 1.3 (Dowker's Theorem)**.** *Let X be a normal space. The following are equivalent:*
(i) If $D_0 \supseteq D_1 \supseteq \cdots$ is a nested sequence of closed sets in X with $\bigcap\{D_n | n < \omega\} = \emptyset$, then, for each $n < \omega$, there is an open $U_n \supseteq D_n$ such that $\bigcap\{U_n | n < \omega\} = \emptyset$;
(ii) X is countably paracompact;
(iii) $X \times [0,1]$ is normal.

Proof. See [DO]. □

The existence of Dowker spaces was first brought into question by Dowker in [DO]. The above theorem completely classifies Dowker spaces as spaces which are both normal and not countably paracompact.

Dowker spaces have proved to be exceedingly difficult to come by. This sheds some light on the subtlety between normality and countable paracompactness: it is quite hard to come up with a space that is both normal and not countably paracompact. In the next section, we exhibit Rudin's example of a Dowker space (see [RU]), which is a subspace of $\square_{i=1}^{\infty}[0,\omega_i]$. To prove that the space is not countably paracompact, we prove that condition (i) of the above theorem does not hold. This is often the condition that is worked with in practice when dealing with countable paracompactness in normal spaces.

Rudin's space is special because it was the first example of a Dowker space in ZFC. It was discovered some twenty years after Dowker published his paper.

A useful characterization of countable paracompactness (metacompactness) that is independent of normality is due to Ishikawa [IS].

Proposition 1.4. *Let X be a topological space.*
(i) X is countably paracompact iff whenever $D_0 \supseteq D_1 \supseteq \cdots$ is a nested sequence of closed sets in X with $\bigcap\{D_n|n < \omega\} = \emptyset$, there exists, for each $n < \omega$, an open $U_n \supseteq D_n$ such that $\bigcap\{\overline{U_n}|n < \omega\} = \emptyset$;
(ii) X is countably metacompact iff whenever $D_0 \supseteq D_1 \supseteq \cdots$ is a nested sequence of closed sets in X with $\bigcap\{D_n|n < \omega\} = \emptyset$, there exists, for each $n < \omega$, an open $U_n \supseteq D_n$ such that $\bigcap\{U_n|n < \omega\} = \emptyset$;

Thus, in a normal space, countable paracompactness and countable metacompactness are equivalent.

A related problem is the so called linearly Lindelöf problem.

Definition 1.5. Let X be a topological space. We say X is *Lindelöf* provided every open cover of X has a countable subcover. We say X is *linearly Lindelöf* iff every increasing open cover $\{U_\alpha|\alpha < \kappa\}$ has a countable subcover. Here, increasing means $U_\alpha \subseteq U_\beta$, whenever $\alpha < \beta$.

Problem 1.6. Is every normal, linearly Lindelöf space Lindelöf?

In section 3, we present Miščenko's example (see [MI]) of a linearly Lindelöf space that is not Lindelöf. This example, which is a subspace of $\prod_{i=1}^{\infty}[0, \omega_i]$, is similar to and helped inspire Rudin's Dowker space. In section 4, we discuss Buzyakova and Gruenhage's example (see [AR]) of a linearly Lindelöf space that is not Lindelöf. It is a subspace of $\{0, 1\}^{\aleph_\omega}$ and is slightly easier to describe than Miščenko's space.

The following proposition relates the linearly Lindelöf problem and Dowker spaces.

Proposition 1.7. *Every linearly Lindelöf, countably metacompact space is Lindelöf.*

Proof. Suppose X is linearly Lindelöf and countably metacompact. We will inductively show that every open cover of size $\kappa > \omega$ has a smaller subcover. Fix $\kappa > \omega$ and suppose $\{O_\alpha|\alpha < \kappa\}$ is an open cover of X. Assume that for all $\beta < \kappa$, any open cover of X having size β has a smaller subcover.
Case one: $cf(\kappa) > \aleph_0$. In this case, for $\beta < \kappa$, let $V_\beta = \bigcup\{O_\alpha|\alpha \leq \beta\}$. Then, $\{V_\beta|\beta < \kappa\}$ is an increasing open cover of X, which, by linear Lindelöfness, has a countable subcover $\{V_{\beta_n}|n < \omega\}$. Since $cf(\kappa) > \aleph_0$, $\gamma = \sup\{\beta_n|n < \omega\} < \kappa$, so that $\{V_\gamma\}$, and thus, $\{O_\alpha|\alpha \leq \gamma\}$ covers X.
Case two: $cf(\kappa) = \aleph_0$. In this case, let $\{\kappa_n|n < \omega\}$ be an increasing cofinal subset of κ. For each $n < \omega$, put $D_n = X \setminus \bigcup\{O_\alpha|\alpha < \kappa_n\}$. Then, $D_0 \supseteq D_1 \supseteq \cdots$ is a nested sequence of closed sets with $\bigcap\{D_n|n < \omega\} = \emptyset$. By 1.4, for each $n < \omega$, there is an open $U_n \supseteq D_n$ such that $\bigcap\{U_n|n < \omega\} = \emptyset$. Then, for each $n < \omega$, $\{O_\alpha|\alpha < \kappa_n\} \cup \{U_n\}$ is an open cover of X of size κ_n. By the inductive hypothesis, the is an $A_n \in [\kappa_n]^{\leq\aleph_0}$ such that $X \setminus U_n \subseteq \bigcup\{O_\alpha|\alpha \in A_n\}$. Let $A = \bigcup\{A_n|n < \omega\}$. Then, A is countable, and $X \subseteq \bigcup\{O_\alpha|\alpha \in A\}$. □

The significance of the above proposition is the following. Suppose there is a normal, linearly Lindelöf space X that is not Lindelöf. By the previous proposition, X would not be countably metacompact, hence, not countably paracompact. But then, since X is normal, Dowker's theorem gives us that $X \times [0, 1]$ is not normal. Hence, X would have to be a Dowker space.

2. Rudin's Dowker Space

In this section, we describe Rudin's famous ZFC Dowker space. We will follow her construction in [RU] very closely. For this proof, $\mathbb{N} = \{1, 2, \ldots\}$. Here's the setup.

Let:

$$F = \{f : \mathbb{N} \to \omega_\omega |\text{ for all } n \in \mathbb{N}, f(n) \leq \omega_n\} = \prod_{n=1}^{\infty} [0, \omega_n];$$

$$X = \{f \in F |\text{ there exists } i \in \mathbb{N}, \text{ such that, for all } n \in \mathbb{N}, \omega < cf(f(n)) < \omega_i\}.$$

Suppose $f, g \in F$. Say:

$$f < g \text{ iff for all } n \in \mathbb{N}, f(n) < g(n);$$
$$f \leq g \text{ iff for all } n \in \mathbb{N}, f(n) \leq g(n);$$
$$f <_i g \text{ iff } i \in \mathbb{N} \text{ and, for all } n \geq i, f(n) < g(n);$$
$$U_{f,g} = \{h \in X | f < h \leq g\}.$$

Then, $\{U_{f,g} | f, g \in F\}$ is a basis for a topology on X. The aim of this section will be to prove that X, equipped with the topology generated by the aforementioned basis, is a Dowker space. We start by showing that X is not countably paracompact.

For $n \geq 1$, let:

$$D_n = \{f \in X |\text{ there exists an } i \geq n \text{ with } f(i) = \omega_i\}$$
$$C_n = \{f \in X |\text{ for all } i < n, f(i) = \omega_i \text{ and, for all } i \geq n, f(i) < \omega_i\}$$

Claim 2.1. *(i)* $D_1 \supseteq D_2 \supseteq \cdots$
(ii) $\bigcap\{D_n | n \in \mathbb{N}\} = \emptyset$
(iii) Each D_n *is closed.*

Proof. (i) is clear. To see (ii), note that if $f \in X$, there is an $n \in \mathbb{N}$ such that $cf(f(i)) < \omega_n$ for all $i \in \mathbb{N}$. In particular, $f(i) \neq \omega_i$ for all $i \geq n$, so that $f \notin D_n$. For (iii): if $f \in X \setminus D_n$, then, for all $i \geq n$, $f(i) < \omega_i$. Hence, $f \in U_{0,f} \subseteq X \setminus D_n$, so that D_n is closed. □

Now, for each $n \in \mathbb{N}$, let $U_n \supseteq D_n$ be open. To show that X is not countably paracompact, it suffices to show, by 1.4 and the above claim, that $\bigcap\{U_n | n \in \mathbb{N}\} = \emptyset$. We do this through the following three lemmas.

Lemma 2.2. *Suppose* $1 < n \in \mathbb{N}$*,* U *is an open set,* $f \in C_{n+1}$*, and* $U \supseteq \{h \in C_{n+1} | f <_{n+1} h\}$*. Then, there is a* $g \in C_n$ *such that* $U \supseteq \{h \in C_n | g <_n h\}$*.*

Proof. Define $k \in C_n$ by

$$k(i) = \begin{cases} f(i) & \text{if } i \neq n \\ 0 & \text{if } i = n \end{cases}.$$

Let $K = \{k_\alpha | \alpha < \lambda\}$ be a maximal well-ordered family of terms in $C_n \setminus U$ such that for all $\alpha < \beta < \lambda$, $k <_n k_\alpha <_n k_\beta$ - here we mean maximal in the sense that we cannot add a larger element to the collection. If $\lambda = 0$, then $g = k$ works. So, assume $\lambda > 0$. Since $k(n) < k_\alpha(n) < k_\beta(n)$ for all $\alpha < \beta < \lambda$, it must be that $\lambda \leq \omega_n$. We claim that $\lambda < \omega_n$. To see this, suppose on contrary that $\lambda = \omega_n$, and define $g' \in F$ by $g'(i) = \sup\{k_\alpha(i) | \alpha < \lambda\}$ for all $i \in \mathbb{N}$. Then, $g'(n) = \sup\{k_\alpha(n) | \alpha < \lambda = \omega_n\} = \omega_n$, and, for all $i > n$, $g'(i) < \omega_i$ since each $k_\alpha(i) < \omega_i$. Hence, $g' \in C_{n+1}$. But, we also have that $f <_{n+1} g'$, so, by assumption, $g' \in U$. Since U is open, there must be a $g^* < g'$ in F such that $U_{g^*,g'} \subseteq U$. By the definition of supremum, there is, for each $i \geq n$, an $\alpha_i < \lambda$ such that $k_{\alpha_i}(i) > g^*(i)$. Letting $\alpha = \sup\{\alpha_i | i \geq n\} < \omega_n = \lambda$, we get that $g^* < k_\alpha \leq g'$. Therefore, $k_\alpha \in U$, which is a contradiction. Hence, we have that $\lambda < \omega_n$. Now, let $g \in F$ be defined by

$$g(i) = \begin{cases} g'(i) + \omega_1 & \text{if } i \geq n \\ \omega_i & \text{if } i < n \end{cases}.$$

Then, $g \in X$ and, since $\lambda < \omega_n$, $g(i) < \omega_i$ for all $i \geq n$. Whence, $g \in C_n$. Furthermore, if $h \in C_n \setminus U$, satisfies $g <_n h$, then $h \in K$ by the maximality of K. But then, $h \leq g' < g$, which is impossible. Thus, $U \supseteq \{h \in C_n | g <_n h\}$, as desired. □

Lemma 2.3. *Suppose* $1 < n \in \mathbb{N}$. *There is an* $f' \in C_2$, *such that* $U_n \supseteq \{h \in C_2 | f' <_2 h\}$.

Proof. Note that the g in 2.2 depended only on the given function f and the given open set U. So, we denote a g that is given by 2.2 by $g_{f,U}$. We now define $k_i \in C_i$ for $2 \leq i \leq n+1$ by backwards induction. Let k_{n+1} be any function in C_{n+1}. Since $U_n \supseteq D_n \supseteq C_{n+1}$, it follows that $U_n \supseteq \{h \in C_{n+1} | k_{n+1} <_{n+1} h\}$. So, we can put $k_n = g_{k_{n+1},U_n}$, and, in general, we put $k_i = g_{k_{i+1},U_n}$ for $2 \leq i \leq n+1$. Then, $f' = k_2$ works. □

Lemma 2.4. $\bigcap\{U_n | n \in \mathbb{N}\} = \emptyset$.

Proof. For each $n > 1$, choose f_n as in 2.3; i.e., such that $U_n \supseteq \{h \in C_2 | f_n <_2 h\}$. For each $i > 1$, choose an ordinal α_i such that $cf(\alpha_i) = \omega_1$ and such that for all $n > 1$, $f_n(i) < \alpha_i < \omega_i$. Set $\alpha_1 = \omega_1$. Define $g \in X$ by $g(i) = \alpha_i$ for all $i \in \mathbb{N}$. Then, $g \in C_2$ and, for all $n > 1$, $f_n <_2 g$. By 2.3, it follows that $g \in U_n$ for all $n > 1$. Also, $g(1) = \omega_1$, so that $g \in D_1 \subseteq U_1$. Hence, $\bigcap\{U_n | n \in \mathbb{N}\} \ni g$. □

We have just shown that X is not countably paracompact. To prove that the X is normal requires a lot more work. Just as Rudin did in [RU], we will show that X is collectionwise normal; that is, for every discrete family of closed sets $\{F_i | i \in I\}$ there is a collection $\{U_i | i \in I\}$ of pairwise disjoint open sets, such that $U_i \supseteq F_i$. Let's get started!

Suppose $\mathcal{H} = \{H_j | j \in J\}$ is a discrete collection of closed sets in X. Let $H = \bigcup\{H_j | j \in J\}$. If $U \subseteq F$, define $t_U \in F$ by $t_U(n) = \sup\{f(n) | f \in U\}$ for all $n \in \mathbb{N}$, and note that $U \supseteq V$ implies $t_U \geq t_V$.

Goal.

For each $\alpha < \omega_1$, we will define, by induction, a cover $\mathcal{J}_\alpha$ of H consisting of *disjoint open sets* having the following property:
If $\beta < \alpha < \omega_1$ and $V \in \mathcal{J}_\alpha$, then there exists a $U \in \mathcal{J}_\beta$ such that:
(1) $V \subseteq U$; i.e., $\mathcal{J}_\alpha$ refines $\mathcal{J}_\beta$;
(2) if V intersects at least two members of $\mathcal{H}$, then $t_V \neq t_U$;
(3) if U intersects at most one member of $\mathcal{H}$, then $U = V$.

We will first show that the existence of the $\mathcal{J}_\alpha$, as described in Goal, is enough to find a set of pairwise disjoint open sets $\{U_j | j \in J\}$ such that $U_j \supseteq H_j$, which is what we need in order to show X is collectionwise normal.

Suppose $f \in H$. If $\alpha < \omega_1$, then, since $\mathcal{J}_\alpha$ covers H with disjoint open sets, there is a unique $U_\alpha \in \mathcal{J}_\alpha$ such that $f \in U_\alpha$. If $\beta < \alpha < \omega_1$, then, by (1), $U_\alpha \subseteq U_\beta$; hence, $t_{U_\alpha} \leq t_{U_\beta}$. By (2), if U_α intersects more than one term of $\mathcal{H}$, there is an n such that $t_{U_\alpha}(n) < t_{U_\beta}(n)$. Since there is no infinite descending sequence of ordinals, for each $n \in \mathbb{N}$, an inequality of the above form can only hold for finitely many $\beta < \alpha < \omega_1$. So, considering all $n \in \mathbb{N}$ and all $\beta < \alpha < \omega_1$, an inequality of the form $t_{U_\alpha}(n) < t_{U_\beta}(n)$ can hold at most countably many times. So, since ω_1 is uncountable, there must be an $\alpha_f < \omega_1$ such that U_{α_f} intersects at most one term of $\mathcal{H}$. Therefore, by (3), if $\alpha_f < \beta < \omega_1$, then $U_\beta = U_{\alpha_f}$. Define $U_j = \bigcup\{U_{\alpha_f} | f \in H_j\}$ for all $j \in J$. Note that for each $j \in J$, $H_j \subseteq U_j$. We will now show that the U_j's are pairwise disjoint. It suffices to show that $U_{\alpha_f} \cap U_{\alpha_g} = \emptyset$, whenever f and g belong to different terms of $\mathcal{H}$. To this end, choose $\gamma < \omega_1$ greater than α_f and α_g. Then, by the above, the term of $\mathcal{J}_\gamma$ which contains f is U_{α_f}, and the term of $\mathcal{J}_\gamma$ which contains g is U_{α_g}. If f and g do not belong to the same term of $\mathcal{H}$, then $U_{\alpha_f} \neq U_{\alpha_g}$. Thus, since $\mathcal{J}_\gamma$ is a cover of *disjoint* open sets, U_{α_f} and U_{α_g} must, in fact, be disjoint.

So, we need to prove the existence of the $\mathcal{J}_\alpha$ as described in Goal. We proceed by induction.

Define $\mathcal{J}_0 = \{X\}$. Fix $\alpha < \omega_1$ and suppose that $\mathcal{J}_\beta$ has been defined and satisfies the properties in Goal for all $0 \leq \beta < \alpha$. We first deal with the case when α is a limit ordinal. For $f \in H$ and $\beta < \alpha$, let $U_f(\beta)$ denote the term of $\mathcal{J}_\beta$ to which f belongs, and set $U_f = \bigcap\{U_f(\beta) | \beta < \alpha\}$. Put $\mathcal{J}_\alpha = \{U_f | f \in H\}$. We need a lemma to ensure that each U_f is open.

Lemma 2.5. *X is a P-space; that is, the intersection of a countable collection of open sets is open.*

Proof. Suppose $\{O_n | n \in \mathbb{N}\}$ is a collection of open sets in X. Suppose $h \in \bigcap\{O_n | n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, choose a function $g_n \in F$ such that $U_{g_n,h} \subseteq O_n$. Define $g \in F$ by $g(i) = \sup\{g_n(i) | n \in \mathbb{N}\}$ for all $i \in \mathbb{N}$. Then, $g \leq h$. If $g(i)$ is a maximum instead of a true supremum, then $g(i) = g_n(i)$ for some n, and hence, $g(i) < h(i)$. Otherwise, $cf(g(i)) \leq \omega < \omega_1 \leq cf(h(i))$, and thus, $g(i) < h(i)$ in this case, too. Therefore, $g < h$ and $h \in U_{g,h} \subseteq \bigcap\{O_n | n \in \mathbb{N}\}$, so that $\bigcap\{O_n | n \in \mathbb{N}\}$ is open. □

We now know that $\mathcal{J}_\alpha = \{U_f | f \in H\}$ is a collection of open sets; they clearly cover H. To see that the sets in $\mathcal{J}_\alpha$ are pairwise disjoint, suppose $U_f \neq U_g$ and $h \in U_f \cap U_g$. Then, $h \in U_f(\beta) \cap U_g(\beta)$ for all $\beta < \alpha$. So, by the inductive hypothesis, $U_f(\beta) = U_g(\beta)$ for all $\beta < \alpha$, and hence, $U_f = U_g$, which is a contradiction. Now, fix $\beta < \alpha$. It suffices to verify the other three properties listed in Goal with $V = U_f$ and $U = U_f(\beta)$. Clearly $U_f(\beta) \supseteq U_f$, so that (1) is satisfied. If U_f intersects two terms in $\mathcal{H}$, then, since α is a limit, so do $U_f(\beta)$ and $U_f(\beta+1)$. Since $\mathcal{J}_\beta$ and $\mathcal{J}_{\beta+1}$ consist of disjoint sets, the inductive hypothesis gives that $U_f(\beta+1) \subsetneq U_f(\beta)$, and hence, $t_{U_f(\beta+1)} \neq t_{U_f(\beta)}$. Therefore, there is an $n \in \mathbb{N}$ such that $t_{U_f}(n) \leq t_{U_f(\beta+1)}(n) < t_{U_f(\beta)}(n)$, so that (2) holds. To see (3), suppose $U_f(\beta)$ intersects at most one member of $\mathcal{H}$. Then, for all $\beta < \gamma < \alpha$, $U_f(\beta) = U_f(\gamma)$ by condition (3) for $\mathcal{J}_\gamma$. So, since $\delta_1 < \delta_2$ implies that $U_f(\delta_1) \supseteq U_f(\delta_2)$, it follows that $U_f = \bigcap\{U_f(\delta) | \delta < \alpha\} = \bigcap\{U_f(\delta) | \delta \leq \beta\} = U_f(\beta)$, which is (3). This concludes the case where α is a limit.

We now move on to the case when α is a successor. Suppose $\alpha = \beta + 1$. We begin with a claim.

Claim 2.6. *Suppose that for all $U \in \mathcal{J}_\beta$, there is a collection $\mathcal{J}_U$ of disjoint open subsets of U covering $U \cap H$, such that $V \in \mathcal{J}_U$ implies (2) and (3) of Goal are satisfied with respect to U. Then, $\mathcal{J}_\alpha := \bigcup\{\mathcal{J}_U | U \in \mathcal{J}_\beta\}$ will have the desired properties of Goal.*

Proof. Let $\gamma < \alpha$ and let $V \in \mathcal{J}_\alpha$. Then, there is a $U \in \mathcal{J}_\beta$ such that $V \in \mathcal{J}_U$. Choose a set $U' \in \mathcal{J}_\gamma$ containing U. We'll show that V and U' satisfy the properties of Goal. We have $V \subseteq U \subseteq U'$, so that (1) is satisfied. If V intersects at least two terms of $\mathcal{H}$, then $t_V \neq t_U$ by the assumption of the claim. So, $t_V < t_U \leq t_{U'}$, which proves that (2) holds. Finally, if U' intersects at most one terms of $\mathcal{H}$, then, so does U. Thus, by the assumption of the claim, $V = U$. If $\gamma = \beta$, then $U = U'$. Otherwise, by the inductive hypothesis applies to $\mathcal{J}_\beta$, $U = U'$. In either case, $V = U'$, which is (3). □

For the rest of the proof, we hereby fix a $U \in \mathcal{J}_\beta$ and let $t := t_U$. We need to construct a collection $\mathcal{J}_U$ as specified by the above claim. We consider three cases.

Case One: *U intersects at most one term of $\mathcal{H}$.*
In this case, define $\mathcal{J}_U = \{U\}$. Then, if $V \in \mathcal{J}_U$, $V = U$, so (3) is satisfied. And, (2) holds trivially.

Case Two: *U intersects at least two members of $\mathcal{H}$ and there is an $i \in \mathbb{N}$ such that $cf(t(i)) \leq \omega$.*
In this case, (3) never holds, so we only need to ensure (2). Because U intersects two members of $\mathcal{H}$, $U \neq \emptyset$, and hence, $t(i) \neq 0$. We claim that $cf(t(i)) = \omega$. Otherwise, $0 < cf(t(i)) < \omega$, and so, $t(i) = \gamma$ for some ordinal γ. If $f \in U \subseteq X$, then $cf(f(i)) > \omega > cf(t(i))$ and $f(i) \leq t(i)$. Hence, $f(i) < t(i)$. But then, $f(i) \leq \gamma$, and, since this holds for all $f \in U$, it follows that $t(i) \leq \gamma < \gamma + 1 = t(i)$, which is impossible. Thus, $cf(t(i)) = \omega$. Now, let $\{\lambda_n | n \in \mathbb{N}\}$ be a cofinal increasing subset of $t(i)$. Define $V_1 = \{f \in U | f(i) \leq \lambda_1\}$, and, for $1 < n \in \mathbb{N}$, set $V_n = \{f \in U | \lambda_{n-1} < f(i) \leq \lambda_n\}$. Clearly each V_n is open, and the V_n's are disjoint. Thus, $\mathcal{J}_U = \{V_n | n \in \mathbb{N}\}$ is a set of disjoint open subsets of U, which cover $U \cap H$. Furthermore, to see that (2) holds, note that for each $n \in \mathbb{N}$, $t_{V_n}(i) \leq \lambda_n < t(i)$.

Case Three: *U intersects at least two members of $\mathcal{H}$ and $cf(t(n)) > \omega$ for all $n \in \mathbb{N}$.*
This case will require substantially more work than the previous two. Consider the following lemma.

Lemma 2.7. *There is an $f \in F$ such that $f < t$ and $\{h \in U | f < h\}$ intersects at most one term of $\mathcal{H}$.*

Claim 2.8. *If the above lemma holds, then we can define $\mathcal{J}_U$ with the desired properties of 2.6.*

Proof. Suppose that the above lemma holds, and let $f \in F$ be given as in the lemma. We define $\mathcal{J}_U$ as follows. For $M \subseteq \mathbb{N}$, set

$$V_M = \{h \in U | h(n) \leq f(n) \text{ for all } n \in M \text{ and } h(n) > f(n) \text{ for all } n \in \mathbb{N} \setminus M\}.$$

Define $\mathcal{J}_U = \{V_M | M \subseteq \mathbb{N}\}$. Since U intersects at least two members of $\mathcal{H}$, (3) is satisfied trivially, just as in Case Two. To see that (2) holds, we prove the contrapositive. Suppose $t_{V_m} = t$. Then,

$$\begin{aligned} & t_{V_m}(n) = t(n) && \forall n \in \mathbb{N} \\ \implies & t_{V_m}(n) = t(n) && \forall n \in M \\ \implies & t(n) \leq f(n) && \forall n \in M \\ \implies & M = \emptyset && \text{since } f < t. \end{aligned}$$

But, $V_\emptyset = \{h \in U | h > f\}$, which, by the above lemma, intersects at most one on term of $\mathcal{H}$. This proves (2) via the contrapositive. So, since $\mathcal{J}_U$ is a set of disjoint open sets, $\mathcal{J}_U$ satisfies the condition of 2.6. □

By the above claim, it suffices to prove 2.7 in order to conclude the proof. In order to do this, we will appeal to the following lemma.

For each $n \in \mathbb{N}$, we define $U_n = \{h \in U |$ for all $i \in \mathbb{N}, cf(h(i)) \leq \omega_n\}$.

Lemma 2.9. *Suppose $n \in \mathbb{N}$. There exists a $g \in F$ with $g < t$ such that $\{h \in U_n | g < h\}$ intersects at most one term of $\mathcal{H}$.*

Claim 2.10. *If the above lemma holds, then so does 2.7.*

Proof. Suppose the above lemma holds, and, for each $n \in \mathbb{N}$, let g_n be given as in the lemma. Define $f \in F$ by $f(i) = \sup\{g_n(i) | n \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Then, $f \leq t$. In fact, $f < t$. This follows by an argument similar to the one in the proof of 2.5 since $cf(t(n)) > \omega$ for all $n \in \mathbb{N}$ by the assumption of Case Three. We will now show that $\{h \in U | f < h\}$ intersects at most one term of $\mathcal{H}$. Suppose $h_0, h_1 \in \{h \in U | f < h\} \cap H$. We'll show that h_0 and h_1 belong to the same term of $\mathcal{H}$. There are $i, j \in \mathbb{N}$ such that $h_0 \in U_i$ and $h_1 \in U_j$. Let $n = i + j$. Then, $h_0, h_1 \in U_n$ and $g_n \leq f < h_0, h_1$. Hence, by 2.9, h_0 and h_1 belong to the same element of $\mathcal{H}$. □

Therefore, by the above claim, in order to conclude the proof that X is collectionwise normal, we must prove 2.9. The rest of this section is devoted to this proof.

Proof of 2.9. Fix $n \in \mathbb{N}$. We shall proceed by contradiction, assuming that 2.9 is false. I.e., assume that for all $f \in F$ with $f < t$, there are terms $h, k \in U_n$ such that $f < h$ and $f < k$, but h and k belong to different terms of $\mathcal{H}$. Before proceeding with the proof, we need to introduce some necessary notation.

For $i \leq n$, define $M_i = \{j \in \mathbb{N} | cf(t(j)) = \omega_i\}$. Let $M = \{j \in \mathbb{N} | cf(t(j)) > \omega_n\}$. By the assumption of Case Three, $\mathbb{N} = \bigcup\{M_i | i \leq n\} \cup M$. Let $R = \{r : (1, 2, \ldots, n) \to \omega_n | r(i) < \omega_i$ for all $i \leq n\}$. Note that $|R| = |\omega_1 \times \cdots \times \omega_n| = \omega_n$. Since ω_n is regular, we can enumerate R by $\{r_\lambda | \lambda < \omega_n\}$, such that if $\lambda < \omega_n$ and $r \in R$, then there is a γ with $\lambda < \gamma < \omega_n$ and $r_\gamma = r$. Finally, for all $i \leq n$ and $j \in M_i$, choose an increasing cofinal subset $\{s_{j,\sigma} | \sigma < \omega_i\}$ of $t(j)$.

In order to help aid with the proof later on, we will define by induction, for each ordinal $\lambda < \omega_n$, an $f_\lambda \in F$ and $h_\lambda, k_\lambda \in U_n$ as follows. Let $f_0 \in F$ be given by

$$f_0(j) = \begin{cases} s_{j,r_0(i)} & \text{if } i \leq n \text{ and } j \in M_i \\ 0 & \text{if } j \in M \end{cases}.$$

Then, $f_0 < t$, so, since we are assuming the negation of 2.9, we can choose $h_0, k_0 \in U_n$ with $f_0 < h_0$ and $f_0 < k_0$ such that h_0 and k_0 belong to different terms of $\mathcal{H}$. Now, fix $0 < \lambda < \omega_n$ and suppose $h_\gamma, k_\gamma \in U_n$ have been chosen for all $\gamma < \lambda$. Define $f_\lambda \in F$ by

$$f_\lambda(j) = \begin{cases} s_{j,r_\lambda(i)} & \text{if } i \leq n \text{ and } j \in M_i \\ \sup\left\{h(j) | h \in \bigcup\{\{h_\gamma, k_\gamma\} | \gamma < \lambda\}\right\} & \text{if } j \in M \end{cases}.$$

Note that if $h = h_\gamma$ or $h = k_\gamma$ for some $\gamma < \lambda$, then $h \in U_n$ by our inductive hypothesis. Hence, for all $j \in M$, $cf(h(j)) \leq \omega_n < cf(t(j))$, so, $h(j) < t(j)$ because $h \in U$. Therefore, if $j \in M$, $f_\lambda(j) < t(j)$ since $cf(t(j)) > \omega_n$ and $\lambda < \omega_n$. Thus, $f_\lambda < t$. So, since we are assuming 2.9 is false, we can pick $h_\lambda, k_\lambda \in U_n$ with $f_\lambda < h_\lambda, k_\lambda$, such that h_λ and k_λ belong to different members of $\mathcal{H}$.

We need to remember three important facts about the functions just defined:
(a) If $\lambda < \omega_n$, $i \leq n$, and $j \in M_i$, then $f_\lambda(j) = s_{j,r_\lambda(i)} < h_\lambda(j) \leq t(j)$ and $f_\lambda(j) = s_{j,r_\lambda(i)} < k_\lambda(j) \leq t(j)$;
(b) If $\gamma < \lambda < \omega_n$ and $j \in M$, then $h_\gamma(j) < k_\lambda(j) < h_{\lambda+1}(j) < t(j)$ and $k_\gamma(j) < h_\lambda(j) < k_{\lambda+1}(j) < t(j)$;
(c) h_λ and k_λ belong to different terms of $\mathcal{H}$.

Now, let $g \in F$ be given by

$$g(j) = \begin{cases} t(j) & \text{if } j \in \mathbb{N} \setminus M \\ \sup\{h_\lambda(j) | \lambda < \omega_n\} & \text{if } j \in M \end{cases}.$$

If $j \in \mathbb{N} \setminus M$, then, by the assumption of Case Three and the definition of M, $\omega < cf(g(j)) = cf(t(j)) \leq \omega_n$. If $j \in M$, then, by (b), $cf(g(j)) = \omega_n$ since ω_n is regular. It follows that $g \in X$, and, by (a), $g \leq t$. Since $\mathcal{H}$ is a discrete family of closed sets, and since $g \in X$, there is an $f \in F$ with $f < g$, such that the basic open set $U_{f,g}$ intersects at most one term of $\mathcal{H}$. Note, too, that $f < t$, since $f < g$ and $g \leq t$.

Now, given an $i \leq n$, recall that for $j \in M_i$, $\{s_{j,\sigma} | \sigma < \omega_i\}$ is an increasing cofinal subset of $t(j)$. So, for each $j \in M_i$, given that $f(j) < t(j)$, there is a $\sigma_j < \omega_i$ such that $f(j) < s_{j,\sigma_j}$. Put $\mu_i = \sup\{\sigma_j | j \in M_i\} < \omega_i$. Define $r \in R$ by $r(i) = \mu_i$ for all $i \leq n$. Then, for all $j \in M_i$, we have $f(j) < s_{j,\sigma_j} \leq s_{j,\mu_i} = s_{j,r(i)}$.

Furthermore, since $f < g$, the definition of g gives, for each $j \in M$, a $\delta_j < \omega_n$ such that $f(j) < h_{\delta_j}(j)$. Putting $\delta = \sup\{\delta_j | j \in M\} < \omega_n$, (b) gives us that $k_\gamma(j), h_\gamma(j) > h_\delta(j) > f(j)$ for all $\gamma > \delta$ and $j \in M$.

Finally, by the way we enumerated R, there is a γ such that $\delta < \gamma < \omega_n$ and $r_\gamma = r$. We claim that $f < h_\gamma, k_\gamma \leq g$. If $j \in M$, then, by the above, $f(j) < h_\gamma(j) < g(j)$ since $\gamma > \delta$. If instead $j \in M_i$ for some $i \leq n$, then, by (a), $f(j) < s_{j,r(i)} = s_{j,r_\gamma(i)} = f_\gamma(j) < h_\gamma(j) \leq t(j) = g(j)$. Therefore, $f < h_\gamma \leq g$ and a symmetrical argument shows that $f < k_\gamma \leq g$. Thus, by (c), the basic open set $U_{f,g}$ intersects at least two terms of $\mathcal{H}$. This contradicts that assumption that $U_{f,g}$ is contained in at most one term of $\mathcal{H}$, which completes the proof. $\square$

3. Miščenko's Finally Compact Space

Before describing Miščenko's space, we introduce a few notions and preliminary results. Among these is the notion of complete accumulation points and their relation to the linear Lindelöf property.

Definition 3.1. Let X be a topological space and let $M \subseteq X$. A point $x \in X$ is said to be a *complete accumulation point of* M iff for every neighbourhood U of x, $|U \cap M| = |M|$.

Definition 3.2. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are infinite cardinals with $\mathfrak{a} \leq \mathfrak{b}$.
(a) We say a space X is $[\mathfrak{a}, \mathfrak{b}]^{AC}$*-compact* iff for all regular cardinals $\mathfrak{c}$ with $\mathfrak{a} \leq \mathfrak{c} \leq \mathfrak{b}$, every $M \subseteq X$ of size $\mathfrak{c}$ has a complete accumulation point. We write $[\mathfrak{a}, \infty]^{AC}$*-compact* iff X is $[\mathfrak{a}, \mathfrak{b}]^{AC}$-compact for all $\mathfrak{b} \geq \mathfrak{a}$.
(b) We say a space X is $[\mathfrak{a}, \mathfrak{b}]$*-compact* iff for each cardinal $\mathfrak{c}$ with $\mathfrak{a} \leq \mathfrak{c} \leq \mathfrak{b}$, every open cover of X of size $\mathfrak{c}$ has a subcover of size $< \mathfrak{a}$. $[\mathfrak{a}, \infty]$-compact is defined similarly as in (a).

The following result, due to Alexandrov and Urysohn (see [AL]), relates the notions of complete accumulation points and open covers in the case of regular cardinals.

Theorem 3.3. *Let X be a space and suppose κ is a regular cardinal. Then, every set $A \subseteq X$ of cardinality κ has a complete accumulation point iff every open cover of size κ has a subcover of strictly smaller size.*

Proof. Fix a regular cardinal κ. Suppose first that every open cover of X of size κ has a subcover of strictly smaller size. Assume to the contrary that X contains a subset $A = \{x_\alpha | \alpha < \kappa\}$ of size κ without a complete accumulation point. For each $x \in X$, choose an open set $N(x) \ni x$ such that $|N(x) \cap A| < \kappa$. Let $\beta(x) = \min\{\xi < \kappa | N(x) \cap A \subseteq \{x_\alpha | \alpha < \xi\}\}$, and, for $\alpha < \kappa$, put $V_\alpha = \bigcup\{N(x) | \beta(x) \leq \alpha\}$. Then, $\{V_\alpha | \alpha < \kappa\}$ is an increasing open cover of X. By assumption, there is smaller subcover $\{V_{\alpha_\xi} | \xi < \lambda\}$. Let $\delta = \sup\{\alpha_\xi | \xi < \lambda\}$. Then, $\delta < \kappa$ since κ is regular, and $V_\delta = X$. Hence, $|A| = |A \cap V_\delta| \leq |\{x_\alpha | \alpha < \delta\}| \leq \delta < \kappa$, which is a contradiction.
Conversely, suppose that every subset of X of size κ has a complete accumulation point, and let $\{U_\alpha | \alpha < \kappa\}$ be an open cover of X. For $\beta < \kappa$, let $V_\beta = \bigcup\{U_\alpha | \alpha \leq \beta\}$. Since κ is regular, it suffices to show that $\{V_\beta | \beta < \kappa\}$ has a smaller subcover. So, suppose on contrary that $\{V_\beta | \beta < \kappa\}$ does not have a smaller subcover. We may assume, without loss of generality, that $V_\alpha \subsetneq V_\beta$ whenever $\alpha < \beta$. For each $\alpha < \kappa$, choose $x_\alpha \in V_{\alpha+1} \setminus V_\alpha$ and let $A = \{x_\alpha | \alpha < \kappa\}$. By assumption, A has a complete accumulation point x. There is $\beta < \kappa$ such that $x \in V_\beta$. But then, $|V_\beta \cap A| = |\{x_\alpha | \alpha < \beta\}| \leq \beta < \kappa$, contradicting that x is a complete accumulation point of A. □

Corollary 3.4. *A space is $[\mathfrak{a}, \mathfrak{b}]^{AC}$-compact whenever it is $[\mathfrak{a}, \mathfrak{b}]$-compact.*

This motivates the question of whether these two notions of compactness are in fact the same.

Definition 3.5. A topological space X is said to be *finally compact in the sense of accumulation points (FCAP)* iff every $M \subseteq X$ of uncountable regular cardinality has a complete accumulation point. In the previous terminology, this amounts to saying X is $[\aleph_1, \infty]^{AC}$-compact.

Miščenko's example is that of a space which is $[\aleph_1, \infty]^{AC}$-compact, but not $[\aleph_1, \infty]$-compact. His example, therefore, proves that the converse of 3.4 does not hold. Furthermore, as the next Corollary shows, Miščenko's example provides us with a space that is linearly Lindelöf but not Lindelöf.

Corollary 3.6. *X is linearly Lindelöf iff X is FCAP.*

Proof. Suppose first that X is linearly Lindelöf. If κ is an uncountable regular cardinal, $\{U_\alpha | \alpha < \kappa\}$ is an open cover of X, and $\{V_\beta | \beta < \kappa\}$ is the corresponding increasing open cover, then $\{V_\beta | \beta < \kappa\}$ has a countable subcover. Hence, since κ is regular, $\{U_\alpha | \alpha < \kappa\}$ must have a subcover of smaller size. Since κ was arbitrary, it follows, by 3.3, that X is FCAP.
Conversely, suppose that X is FCAP. Then, by 3.3, every increasing open cover of regular cardinality has a strictly smaller subcover. Also, every increasing open cover of singular cardinality has a strictly smaller subcover. Since there is no infinite decreasing sequence of ordinals, it follows that every increasing open cover has a coutnable subcover. □

We are almost ready to describe Miščenko's example, but we need to introduce one more notion first.

Definition 3.7. Let X be a space and let $\mathfrak{m}$ be an infinite cardinal. Then, X is said to be $\mathfrak{m}$-*bounded* iff for every $A \subseteq X$ with $|A| \leq \mathfrak{m}$, there is a compact $C \subseteq X$ with $A \subseteq C$.

Lemma 3.8. *Every $\mathfrak{m}$-bounded space is $[\aleph_0, \mathfrak{m}]$-compact.*

Proof. Suppose X is an $\mathfrak{m}$-bounded space. Assume to the contrary that X is not $[\aleph_0, \mathfrak{m}]$-compact. Let $\mathcal{O}$ be an open cover of X with $|\mathcal{O}| = \kappa \in [\aleph_0, \mathfrak{m}]$ having no finite subcover. For each $N \in [\mathcal{O}]^{<\omega}$, choose $x_N \in X \setminus \bigcup N$. Then, $\{x_N | N \in [\mathcal{O}]^{<\omega}\}$ is a set of size at most κ. Since X is $\mathfrak{m}$-bounded, there is a compact $C \subseteq X$ containing A. Since $\mathcal{O}$ covers C, there is a finite subcover $U \in [\mathcal{O}]^{<\omega}$ covering C, too. But then, $x_U \notin A$, since $x_U \notin \bigcup U$ and $A \subseteq C \subseteq \bigcup U$. This contradiction shows that X must be $[\aleph_0, \mathfrak{m}]$-compact. □

Lemma 3.9. *Let $\mathfrak{m}$ be an infinite cardinal. If $\{X_\alpha | \alpha < \lambda\}$ are a collection of $\mathfrak{m}$-bounded topological spaces, then $X = \prod\{X_\alpha | \alpha < \lambda\}$ is $\mathfrak{m}$-bounded.*

Proof. Let $\pi_\alpha : X \to X_\alpha$ denote the projection map onto the α-coordinate. Suppose $A \subseteq X$ is of size $|A| \leq \mathfrak{m}$. For each $\alpha < \lambda$, $|\pi_\alpha(A)| \leq \mathfrak{m}$, so, since each X_α is $\mathfrak{m}$-bounded, there is a compact $C_\alpha \subseteq X_\alpha$ containing $\pi_\alpha(A)$. By the Tychonoff theorem, $C = \prod_\alpha C_\alpha \subseteq X$ is compact and $A \subseteq C$. Thus, X is $\mathfrak{m}$-bounded. □

Example 3.10 (Miščenko's Space)**.** The space will be a subset of the product space

$$R = \prod_{k=1}^{\infty} [0, \omega_k].$$

For $1 \leq k < \omega$, let

$$R_k = \prod_{i=1}^{k} [0, \omega_i] \times \prod_{i=k+1}^{\infty} [0, \omega_i),$$

and set

$$R^* = \bigcup_{k=1}^{\infty} R_k.$$

We will show that R^* is $[\aleph_1, \infty]^{AC}$-compact. The topology on R^* will be the subspace topology of the ordinary product topology on R with the usual Tychonoff basis $\mathcal{B}$. Note that $|\mathcal{B}| = \aleph_\omega$. Hence, R^* is $[\aleph_{\omega+1}, \infty]^{AC}$-compact. To see this, take a subset $A \subseteq X$ with $|A|$ regular and $|A| \geq \aleph_{\omega+1}$. If A has no complete accumulation point in X, then, for each $x \in X$, we can find a basic open set $N(x) \in \mathcal{B}$ containing x, such that $|A \cap N(x)| < |A|$. But then, $|A| = |A \cap \bigcup\{N(x)|x \in X\}| = |\bigcup\{A \cap N(x)|x \in X\}| < |A|$, since $|\mathcal{B}| = \aleph_\omega$ and $|A| \geq \aleph_{\omega+1}$ is regular, which is a contradiction.

Now, fix $1 \leq n < \omega$ and suppose $k \geq n$. Each factor in R_k is $\aleph_n$-bounded, thus, by 3.9, R_k is $\aleph_n$-bounded, too. Applying 3.8 then gives that R_k is $[\aleph_0, \aleph_n]$-compact. Since R^* is an increasing union, $R^* = \bigcup_{k=n}^{\infty} R_k$. So, R^* is $[\aleph_1, \aleph_n]$-compact, and hence, $[\aleph_1, \aleph_\omega)^{AC}$-compact by 3.4. Thus, R^* is $[\aleph_1, \infty]^{AC}$-compact.

We now show that R^* is not $[\aleph_1, \infty]$-compact. To do this, we will exhibit an open cover of R^* of size $\aleph_\omega$ having no subcover of smaller size. Here's the setup. For $1 \leq k < \omega$ and for each $\alpha < \omega_k$, let $\Gamma_{\alpha,k} = \{x = \{x(i)|i < \omega\} \in R^*|x(k) < \alpha\}$. Let $\pi_k = \{\Gamma_{\alpha,k}|\alpha < \omega_k\}$, and set $\pi = \bigcup_{1 \leq k < \omega} \pi_k$. Then, π is an open cover of R^* of size $\aleph_\omega$. Suppose π had a subcover π^* of smaller size. Then, there must be an $n < \omega$ with the property that for all $k > n$, there is an $\alpha_k < \omega_k$, such that for all $\alpha > \alpha_k$, $\Gamma_{\alpha,k} \notin \pi^*$. Define $x = \{x(i)\}$ as follows:

$$x(i) = \begin{cases} \omega_i & \text{if } 1 \leq i \leq n \\ \alpha_i + 1 & \text{if } i > n \end{cases}.$$

Then, $x \in R_n \subseteq R^*$, but $x \notin \bigcup \pi^*$, which is a contradiction. Thus, R^* is not $[\aleph_1, \infty]$-compact.

Of course, at this point, one might very well ask whether Miščenko's example provides us with a counterexample to settle the famous linearly Lindelöf problem. Note that R^* is completely regular, being the subspace of a completely regular space, but is it normal? The answer is no.

Claim 3.11. *R^* is not normal.*

Proof. For $1 \leq k < \omega$, let $p_k = \prod_{i=1}^{k}\{\omega_i\} \times \prod_{i=k+1}^{\infty}\{0\}$. Let $A = \{p_k|1 \leq k < \omega\}$ and $B = \{x = \{x(i)\} \in R^*|$ for all $i, x(i) \neq 0\}$. It is easy to see that A and B are disjoint closed subsets of R^*. Suppose we are given open sets $U \supseteq A$ and $V \supseteq B$. We'll show that $U \cap V \neq \emptyset$. For each $1 \leq k < \omega$, there exist $\alpha_1^k, \ldots, \alpha_k^k$ such that

$$p_k \in \prod_{i=1}^{k}(\alpha_i^k, \omega_i] \times \prod_{i=k+1}^{\infty}\{0\} \subseteq U.$$

For $1 \leq k < \omega$, let $\theta_k = \sup\{\alpha_k^j|j \geq k\} < \omega_k$. Then, for each $1 \leq k < \omega$,

$$H_k := \prod_{i=1}^{k}(\theta_i, \omega_i] \times \prod_{i=k+1}^{\infty}\{0\} \subseteq U.$$

Now, consider $z = \prod_{i=1}^{\infty}\{\theta_i + 1\}$. Clearly, $z \in B \subseteq V$. But, we also have that $z \in \overline{\bigcup_{1\leq k<\omega} H_k} \subseteq \overline{U}$. Hence, $U \cap V \neq \emptyset$, which is what we set out to show. $\square$

4. Buzyakova and Gruenhage's Space

In this section, we present another interesting example of a linearly Lindelöf space that is not Lindelöf. This example, attributed independently to Buzyakova and Gruenhage, is somewhat simpler and more self-contained than Miščenko's. Let's get right to it.

Example 4.1 (Buzyakova and Gruenhage's Space)**.** Consider the two-point discrete space $D = \{0, 1\}$, and the corresponding product $D^{\aleph_\omega}$. The space in question will be

$$X = \{x = x(\alpha) \in D^{\aleph_\omega} || \operatorname{supp}(x)| < \aleph_\omega\},$$

where $\operatorname{supp}(x) = \{\alpha < \aleph_\omega | x(\alpha) = 1\}$. To help define the topology on X, we introduce the following definition.

Definition 4.2. Given a function f, we let

$$[f] = \{g \in X | \text{ for all } j \in dom(f), g(j) = f(j)\}.$$

We declare sets of the form $[f]$ with $dom(f) \subseteq \omega_\omega$ and $|dom(f)| < \omega$ to be elementary open sets. The topology on X will be generated by the collection $\mathcal{B}$ consisting of all such elementary open sets.

We first show that X is linearly Lindelöf. Note that $\mathcal{B}$ has cardinality $\aleph_\omega$. Thus, X is $[\aleph_{\omega+1}, \infty]^{AC}$-compact, just as in Miščenko's example. So, by 3.6, it suffices to show that X is $[\aleph_1, \aleph_\omega)^{AC}$-compact. To this end, let $A \subseteq X$ and suppose $|A| = \aleph_k$ for some $1 \leq k < \omega$. We want to show A has a complete accumulation point in X. For $1 \leq n < \aleph_\omega$, let $X_n = \{x \in X || \operatorname{supp}(x)| = \aleph_n\}$ and let $A_n = X_n \cap A$. Since A is uncountable, there is an $1 \leq n < \omega$ with $|A_n| = |A|$. Let $S = \bigcup_{x \in A_n} \operatorname{supp}(x)$, and set $H = \{x \in X | \operatorname{supp}(x) \subseteq S\}$. Note that H is a closed subset of the compact space $D^{\aleph_\omega}$, and hence, is compact. Thus, every infinite subset of H has a complete accumulation point in H. In particular, there is a complete accumulation point $a \in H$ of A_n. But, $H \subseteq X$. Indeed, $|S| \leq \max\{|A_n|, \sup\{|\operatorname{supp}(x)||x \in A_n\}\} = \max\{\aleph_k, \aleph_n\} < \aleph_\omega$. Hence, $a \in X$, so that A_n has a complete accumulation point in X. Since $|A_n| = |A|$, a is a complete accumulation point of A, too. Thus, X is linearly Lindelöf.

To see that X is not Lindelöf, let $\pi_\alpha : X \to D$ be the projection onto the α-coordinate, and consider the open cover $\{\pi_\alpha^{\leftarrow}\{0\} | \alpha < \aleph_\omega\}$ of X. If X were Linedlöf, there would be a countable subcover $\{\pi_{\alpha_n}^{\leftarrow}\{0\} | n < \omega\}$ that would cover X. But this cannot be, for the point $x = x(\beta)$ given by $x(\beta) = 1$ iff $\beta = \alpha_n$ for some $n < \omega$ is missed by the subcover, albeit still in X.

Note that just like Miščenko's space, X is completely regular. Unfortunately, though, just as with Miščenko's space, X fails to be normal, as we shall now see.

Claim 4.3. *X is not normal.*

Proof. Let $\pi_\alpha : X \to D$ denote the projection onto the α-coordinate. Let $H = \bigcap\{\pi_{\omega_n}^{\leftarrow}\{1\} | n \geq 0\}$, and let $K = \{p_n | n < \omega\}$, where $p_n(j) = 1$ iff $j < \omega_n$. It is easy to see that H and K are closed subsets of X, which are disjoint. Suppose $U \supseteq H$ and

$V \supseteq K$ are open. We'll show that $U \cap V \neq \emptyset$. For each $n < \omega$, let $[f_n]$ be an elementary open set such that $p_n \in [f_n] \subseteq V$, and define $y_n \in X$ by

$$y_n(j) = \begin{cases} 1 & \text{if } f_k(j) = 1 \text{ for some } 0 \leq k \leq n \\ 1 & \text{if } j = \omega_k \text{ for some } k < n \\ 0 & \text{otherwise} \end{cases}.$$

Then, each $y_n \in [f_n]$. Indeed, since $p_k \in [f_k]$ for all $k < \omega$, it follows that $f_k(j) \neq 0$ for all $j < \omega_k$. Note, too, that the y_n's build upon each other; i.e., if $k < l$ and $y_k(j) = 1$, then $y_l(j) = 1$. Now, let $y \in X$ be defined by $y(j) = \max\{y_n(j) | n < \omega\}$. By construction, $y \in H \subseteq U$. If $[f]$ is any elementary open set containing y, then, for each $\alpha \in dom(f)$ with $f(\alpha) = 1$, there is an $n_\alpha < \omega$ such that $y_{n_\alpha}(\alpha) = y(\alpha) = f(\alpha)$. Since $|dom(f)| < \omega$ and since the y_n's build upon each other, we can let $N = \max\{n_\alpha | \alpha \in dom(f) \text{ and } f(\alpha) = 1\}$, so that $y_N \in [f]$. Thus, $y \in \overline{\bigcup_{n=1}^{\infty}[f_n]} \subseteq \overline{V}$. Hence, $U \cap V \neq \emptyset$, as desired. □

5. Concluding Remarks

It is worth pointing out that in both Miščenko's and Buzyakova and Gruenhage's example, we proved something slightly stronger than non-normality. In both cases, one of the closed sets witnessing non-normality was countable. Thus, neither of the two spaces is even pseudonormal; a space is *pseudonormal* iff for every two disjoint closed sets, one of which is countable, there are disjoint open sets separating them. An example of a linearly Lindelöf space that is not Lindelöf, but that is pseudonormal would, therefore, be quite interesting. As far as I know, there is no such example of a space in the literature.

References

[AL] P.S. Alexandroff, P.S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Kon. Akad. Wetensch. Amsterdam **14** (1929)m no. 1, 96 pp

[AR] A.V. Arhangel'skii, R.Z. Buzyakova, *On Linearly Lindelöf and Strongly Discretely Lindelöf Spaces*, Proc. Amer. Math. Soc. **127** (1999), no. 8, 2449-2458.

[DO] C.H. Dowker, *On Countably Paracompact Spaces*, Canad. J. Math. **3** (1951), 219-224.

[GU] S.I. Gulden, W.F. Fleischman, J.H. Weston, *Linearly Ordered Topological Spaces*, Proc. Am. Math Soc. **24** (1970), 197-203.

[HOD] R.E. Hodel, J.E. Vaughan, *A note on* $[\mathfrak{a}, \mathfrak{b}]$*-Compactness* Gen. Top. and its App.**4** (1974), 179-189.

[HOW] N.R. Howes, *A Note on Transfinite Sequences*, Fund. Math. **106** (1980), 213-216.

[IS] F. Ishikawa, *On Countably Paracompact Spaces*, Proc. Japan Acad. **39:** 95-97.

[MI] A. Miščenko, *On Finally Compact Spaces*, Dokl. Akad. Nauk SSSR **145** (1962), 1224-1227, Translation: Soviet Math. Dokl. 3 (1962) 1199-1202.

[RU] M.E. Rudin, *A normal space X for which $X \times I$ is not normal*, Fund. Math. **73** (1971), 179-186.