Chunyang Ding

Mr. Kessler

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Investigation on Lacsap's Triangle

"Ahaha!" laughed the evil Lacsap as he brooded over his prisoner, the heroic mathematician. "This time, you are trapped in my domain, brother! You no longer have the advantage of using your triangle to get you everywhere! Even though your triangle is much more popular these days, one day, my triangle will gain just as much recognition! And what's more, if you can't figure out this triangle's maximum number, you won't ever be able to escape! Muahahaha!"

Our hero gulped as he looked around him at the pyramid of numbers that surrounded him. It was definitely labored over for many years by some twisted minds, but he was confident that he could see through this. After all, he was one of the most brilliant mathematical minds of his time, creating a masterful triangle of probabilities himself. Can you help him as he unravels this code?

A careful reproduction of the triangle our hero saw is below. What can we deduce about how it works?

$$1 \quad 1$$

$$1 \quad \frac{3}{2} \quad 1$$

$$1 \quad \frac{6}{4} \quad \frac{6}{4} \quad 1$$

$$1 \quad \frac{10}{7} \quad \frac{10}{6} \quad \frac{10}{7} \quad 1$$

$$1 \quad \frac{15}{11} \quad \frac{15}{9} \quad \frac{15}{9} \quad \frac{15}{11} \quad 1$$

One of the first patterns that we can discover about our series of fractions is the patterns within the numerator of Lacsap's triangle. If we were to isolate and only look at the numerator, we would see the following table:

Row	Numerator
1	1
2	3
3	6
4	10
5	15

As we look at this simple table, we would also search for a numerical algorithm to fit the row number to the numerator. One of the ways that we could do this is using our calculator to regress a function. This can be easily done with technology, with the simple creation of several lists. Before we do this, let's make a table to graph the difference in the numerator:

Row	Numerator
1	1
2	3
3	6
4	10
5	15

Difference in Numerator:
2
3
4
5

We can see a clear pattern in the first differences, which is also equivalent to the derivative of the actual function to be a function of one degree. If we were to take the anti-derivative, we would find the function for the numerator, and therefore, we would know that the actual function is a polynomial of the second degree. Let us now use our calculator to regress this function:

The equation given by the TI-89 Titanium calculator when given the previous data was $.5n^2 + .5\,n,$ or

$$N(n) = \frac{1}{2}(n+1)(n)$$

Another way to derive the correlation between the numerator and the row number is just to look at general patterns that seem to emerge. One such pattern is that the numerator is increased by the value of the row number each time. Arithmetically, this can be represented as a summation of the row numbers. For example, the numerator of the 4th row would be $1 + 2 + 3 + 4$, terminating at the row number of the current row. We can represent this in a similar table as before:

Row	Numerator
1	1
2	1 + 2
3	1 + 2 + 3
4	1 + 2 + 3 + 4
5	1 + 2 + 3 + 4 + 5

With this new information, we could formulate a summation equation for the numerator as the following:

$$N(n) = \sum_{i=1}^{n} i$$

, which is quite a lovely formula. However, how is this equation related to the algebraic equation we discovered? Let us contemplate the meaning of the sigma.

The sigma notation is shorthand for a summation of a large range of numbers, and fortunately for mathematicians dealing with series, there is a clear formula for finding sigma, given r. This formula is, in its general statement.

$$\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$$

By simple substitution of n, we can find that

$$N(n) = \sum_{i=1}^{n} i = \frac{1}{2}(n)(n+1)$$

,meaning that the algebraic equation is essentially the same as our sequential equation.

However, this numerator equation proves to be a little trickier than we expected. For instance, when we look at the 0^{th} element and the n^{th} element within each row, we can easily see that each of them does not have a large numerator, as we would determine from our numerator equation, but instead, is the number 1. What could 1 be in this triangle?

Thinking back to number theory, we realize that the number 1 is quite the ambiguous number. It can take on any form that we wish it to be, whether it be $\frac{36}{36}$ or even $\frac{1532}{1532}$. With this kind of ambiguity, it is actually possible to rewrite Lacsap's triangle so that it fits with our current numerator equation thoroughly.

$$\frac{1}{1} \qquad \frac{1}{1}$$

$$\frac{3}{3} \qquad \frac{3}{2} \qquad \frac{3}{3}$$

$$\frac{6}{6} \qquad \frac{6}{4} \qquad \frac{6}{4} \qquad \frac{6}{6}$$

$$\frac{10}{10} \qquad \frac{10}{7} \qquad \frac{10}{6} \qquad \frac{10}{7} \qquad \frac{10}{10}$$

$$\frac{15}{15} \qquad \frac{15}{11} \qquad \frac{15}{9} \qquad \frac{15}{9} \qquad \frac{15}{11} \qquad \frac{15}{15}$$

Now that we have the successful numerator equation, let us use it to check the 3rd row of numerators. As you can tell from the triangle, the numerator should be 6. Does this correspond with what our equation gives us?

$$\frac{(3)(3+1)}{2} = \frac{12}{2} = 6$$

Congratulations, the numerator equation does work! Let us move on to figuring out the significantly trickier denominator pattern.

As we begin, let us again remake Lacsap's triangle, but this time, only have the numerators as elements. This will help us clear up any misconceptions we may possible have prior to starting the work.

Row Number						
1			1	1		
2		3		2	3	
3		6	4	4	6	
4	10	7		6	7	10
5	15	11	9	9	11	15

As we observe this table, it is possible to see a pattern within each row. As we did with the correlation of the numerator with the row number, could we correlate the denominator with the row number? An immediate problem that we may see here is that the equation for each row would have to be different. However, let us put that to the side and use our calculators to regress each row's function, with our variable r being the element number.

Row Number													Equation
1						1		1					$r^2 - r + 1$
2					3		2		3				$r^2 - 2r + 3$
3				6		4		4		6			$r^2 - 3r + 6$
4			10		7		6		7		10		$r^2 - 4r + 10$
5		15		11		9		9		11		15	$r^2 - 5r + 15$

Now, there seems to be something interesting happening with our equation and our row number. Observe the coefficient for the x^1term. It seems to always be equal to the negative of the row number. Also, observe the constant term within each equation. If we could recall the numerator pattern, it would seem that the constant within the equation is the same as the numerator for that row. By simple replacement with n as the row number, we can get the following:

Row Number													Equation
1						1		1					$r^2 - (n)r + N(n)$
2					3		2		3				$r^2 - (n)r + N(n)$
3				6		4		4		6			$r^2 - (n)r + N(n)$
4			10		7		6		7		10		$r^2 - (n)r + N(n)$
5		15		11		9		9		11		15	$r^2 - (n)r + N(n)$

In fact, with the current equation, it would seem that we don't have any differences between the equation for one row and the equation for the next. Therefore, we can consolidate the equations and develop a general equation for the denominator given a row number n and an element r, so that $D_n(r)$ would represent the element at the nth row and the rth column. This equation is:

$$D_n(r) = r^2 - (n)r + N(n)$$

$$D_n(r) = r^2 - (n)r + \frac{1}{2}(n)(n+1)$$

$$D_n(r) = r^2 - (n)r + \frac{1}{2}(n^2 + n)$$

It seems that we are done with the basics of the denominator, but there is so much more that we could explore! For instance, let us begin by trying to find an analytical sequential general formula for the denominator, as we did for the numerator.

First of all, it seems that there would be more patterns per column of the Lacsap's triangle for the denominator, so let's begin by "slanting" the triangle.

	Column Number					
Row Number	**0**	**1**	**2**	**3**	**4**	**5**
1	1	1				
2	3	2	3			
3	6	4	4	6		
4	10	7	6	7	10	
5	15	11	9	9	11	15

From looking at this chart, two things are immediately clear. For one, each element in the 0^{th} column is the same as the numerator of that row. Applying symmetry, we can tell that each element in the n^th^ column of the n^th^ row is also the same as the numerator of that row. Therefore, we can formulate two statements from this:

$$D_n(0) = N(n)$$

$$D_n(n) = N(n)$$

Now, let's take a look at another diagonal that seems interesting within this triangle. What seems to be the correlation for the $D_n(n-1)$ term? Is there anything else within the triangle that we can relate it with?

	Column Number					
Row Number	**0**	**1**	**2**	**3**	**4**	**5**
1	1	1				
		+1				
2	3	2	3			
			+1			
3	6	4	4	6		
				+1		
4	10	7	6	7	10	
					+1	
5	15	11	9	9	11	15

One pattern that jumps out is how the $D_n(n-1)$ is actually just the $D_{n-1}(n-1)+1$. How interesting! Could we make a similar postulate for the $D_n(n-2)$ term?

	Column Number					
Row Number	**0**	**1**	**2**	**3**	**4**	**5**
1	1	1				
		+1				
2	3	2	3			
		+2	+1			
3	6	4	4	6		
			+2	+1		
4	10	7	6	7	10	
				+2	+1	
5	15	11	9	9	11	15

It would appear that the $D_n(n-2)$ term is just the $D_{n-1}(n-2)+2$. It seems that a very important factor in this equation is the number that is added to the previous term. Could we correlate this number with some other number in our equation?

Recall that the numbers $(n-1)$ and $(n-2)$ are really just shorthand for r. As we also see the correlation between the number subtracted from n, we can easily represent that as just $n-r$, as $n-(n-1)$ is the same as 1. Therefore, we can formulate a more general statement for the denominator as

$$D_n(r) = D_{n-1}(r) + (n-r)$$

However, this statement is still recursive, and quite difficult to immediately get an answer, especially for large values of n and r. What could we simplify this to in order to get a more friendly equation? One thing that we can notice is that not only is that $D_n(n)$ is not only equal to $D_{n-1}(r)+1$, but by substitution of $D_n(n)$, we know that $D_n(n-1) = N(n-1)+1$.

This also means that $D_n(n-2)$ is equal to $N(n-1)+1+2$, and so, we see a pattern emerge! Specifically, we can tell that

$$D_n(r) = N(r) + \sum_{i=1}^{(n-r)} i$$

Substituting for $N(n)$, we find that

$$D_n(r) = \sum_{i=1}^{r} i + \sum_{i=1}^{(n-r)} i$$

Applying a couple rules of sigma notation, we can find that

$$D_n(r) = \sum_{i=1}^{r} i + \sum_{i=1}^{n} i - \sum_{i=n-r+1}^{n} i$$

This is likely the simplest summation equation that we can get, so let's prove that this equation is actually the same as the algebraic equation that we have derived earlier. By applying rules of sigma notation, we can find that

$$D_n(r) = \frac{(r)(r+1)}{2} + \frac{(n)(n+1)}{2} - \frac{(n-(n-r+1)+1)(n-r+1+n)}{2}$$

$$D_n(r) = \frac{r^2+r+n^2+n}{2} - \frac{(r)(2n-r+1)}{2} - n$$

$$D_n(r) = \frac{r^2+r+n^2+n}{2} - \frac{2nr-r^2+r}{2}$$

$$D_n(r) = \frac{2r^2+n^2+n-2nr}{2}$$

$$D_n(r) = r^2 - nr + \frac{n^2 + n}{2}$$

$$D_n(r) = r^2 - nr + \frac{1}{2}(n^2 + n)$$

It looks like we can safely confirm that the sequential pattern that we have discovered works the same way as the algebraic definition that we have discovered. By combining both the numerator equation with the denominator equation, we would get the following:

$$E_n(r) = \frac{\frac{1}{2}(n^2 + n)}{r^2 - nr + \frac{1}{2}(n^2 + n)}$$

Or in summation notation,

$$E_n(r) = \frac{\sum_{i=1}^{n} i}{\sum_{i=1}^{r} i + \sum_{i=1}^{(n-r)} i}$$

At this point, an interesting pattern emerges that may remind readers of another famous mathematical triangle. Pascal's triangle, a seemingly simple triangle with many real world applications behind it, uses a very similar equation for its elements at the nth row and the rth column. The way to find an element in Pascal's triangle is to find the ${}_nC_r$ given n and r. This notation, which is a combinatory, actually means $\binom{n}{r}$, or $\frac{n!}{n!\cdot(n-r)!}$. As you may already know, $n!$ is also the same as $\prod_{i=1}^{n} i$, so the expanded equation would be

$$P_n(r) = \frac{\prod_{i=1}^{n} i}{\prod_{i=1}^{n} i \cdot \prod_{i=1}^{n-r} i}$$

Notice anything similar to the equation that we have just derived above?

It seems that the general equation for the denominator of Lacsap's triangle is almost identical to the general equation for Pascal's triangle, with the change that all Capital Pi's are changed to Capital Sigma's. This pattern may not be entirely relevant to the problems worked in the portfolio, but it is interesting to see how Lacsap apparently took quite a bit of inspiration in designing his triangle from Pascal.

Before we begin to test our general equation, we need to identify a couple of limitations that we may have. Thinking back to the original triangle, we notice that our row number must always be greater than 1, and that our column number must also be greater than 0. However, the column number is also limited by the row, in a way that the greatest the column number can be is that row's number. These limitations can be summarized by the following:

$$n \geq 1, \qquad r \geq 0, \qquad r \leq n \text{ for all n}$$

However, these are not the only limitations that we have. For instance, our row could not be a fractional number, or even an imaginary number. Instead, we are limited to the realm of integers, as if we used any other type of number, there wouldn't exist such an element. Therefore,

$$(r, n) \in \mathbb{Z}$$

However, it is interesting to note that we don't have an upper limit for our equation, as this triangle can, in theory, extend infinitely outwards.

Now that we have all of this information, it is possible to begin to test our equations. Let us first check on a few cases that were given, specifically the (4, 0) case and the (3, 2) cases.

For the (4, 0) case, we already know that the correct answer is $10/10$, or 1. What would our general equation give us?

$$E_4(4) = \frac{\frac{1}{2}(4^2 + 4)}{0^2 - 4 \cdot 0 + \frac{1}{2}(4^2 + 4)}$$

$$E_4(4) = \frac{10}{10}$$

Congratulations, this case works! Although the triangle provides us with the answer of 1, we know that $\frac{10}{10}$is equal to 1, so it checks out. Let us look at another case in the middle of the triangle, at (3, 2)

$$E_3(2) = \frac{\frac{1}{2}(3^2 + 3)}{2^2 - 3 \cdot 2 + \frac{1}{2}(3^2 + 3)}$$

$$E_3(2) = \frac{6}{-2 + 6}$$

$$E_3(2) = \frac{6}{4}$$

Yay, it worked again! These trials have proven the validity of our general statement.

Now, let’s try to forecast some more rows. The pattern for the numerator seems to be quite trivial, but using the summation pattern for the denominator, it is possible to find the following:

	Column Number							
Row Number	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1						
		+1						
2	3	2	3					
		+2	+1					
3	6	4	4	6				
		+3	+2	+1				
4	10	7	6	7	10			
		+4	+3	+2	+1			
5	15	11	9	9	11	15		
		+5	+4	+3	+2	+1		
6	21	16	13	12	13	16	21	
		+6	+5	+4	+3	+2	+1	
7	28	22	18	16	16	18	22	28

Now, let us use our denominator equation to test the validity of these new rows.

$$D_7(3) = 3^2 - 7 \cdot 3 + \frac{1}{2}(7^2 + 7)$$

$$D_7(3) = 9 - 21 + 28$$

$$D_7(3) = 16$$

This matches with the previous prediction, and demonstrates the prediction powers of our equations.

It is relatively safe to say that this triangle has been thoroughly explored, but there is more that we can do in order to find interesting facts about this pyramid. For example, what would be the maximum number produced by this triangle?

By deductive reasoning, and observing previous patterns, it is possible to notice that the largest number in each row would be in the middle, where $r = \frac{n}{2}$. Applying this relationship, we can find that the largest number per row would occur at

$$E_n\left(\frac{n}{2}\right) = \frac{\frac{1}{2}(n^2+n)}{\left(\frac{n}{2}\right)^2 - n\cdot\frac{n}{2} + \frac{1}{2}(n^2+n)}$$

$$E_n\left(\frac{n}{2}\right) = \frac{\frac{n^2+n}{2}}{\frac{n^2}{4} - \frac{2n^2}{4} + \frac{2(n^2+n)}{4}}$$

$$E_n\left(\frac{n}{2}\right) = \frac{\frac{n^2+n}{2}}{\frac{n^2+2n}{4}}$$

$$E_n\left(\frac{n}{2}\right) = \frac{2(n^2+n)}{(n^2+n)}$$

If we were to find the maximum this could reach, we would apply the limit properties of

$$\lim_{n\to\infty} E_n\left(\frac{n}{2}\right)$$

$$\lim_{n\to\infty} \frac{2(n^2+n)}{(n^2+n)}$$

$$\lim_{n\to\infty} 2 \cdot \lim_{n\to\infty} \frac{(n^2+n)}{(n^2+n)}$$

, which is then reduced to indeterminate form. However, with the use of our graphing calculator, it is possible to determine that $\lim_{n \to \infty} \frac{(n^2+n)}{(n^2+n)}$ is just equal to 1. This would mean that our maximum number reached in the graph would be 2. Quite an interesting relationship that emerges!

With that, our hero bravely carved the maximum number into the pyramid, using his mighty mathematical staff. Instantly, the pyramid broke open, revealing his sobbing brother. "Why, Pascal, why must you always see through my plans?! All I wanted was world domination!" Pascal grinned as he extended a hand. "Mathematics should always be used for good, whether it is to simply explore nature or to help with other branches of science. I think that this triangle you devised sure is interesting though…perhaps it does have some real world uses?" Lacsap looked up, tears still in his eyes. "You really think so? Would you help me work this out?" Pascal smiled and said, "Of course, that's what brothers are for!"