Math Internal Assessment: Project

The Brachistochrone Puzzle

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1.0 Introduction

One of the most interesting solved problems of mathematics is the brachistochrone problem, first hypothesized by Galileo and rediscovered by Johann Bernoulli in 1697. The word brachistochrone, coming from the root words *brachistos* $(\beta\rho\alpha\chi\iota\sigma\tau o\varsigma)$, meaning shortest, and *chrone*$(\chi\rho o\nu o\varsigma)$, meaning time[1], is the curve of least time. This problem is not only beautiful in the simplicity of the question, but also elegant in the many solutions it invites. Through this puzzle, we can watch some of the greatest minds of mathematics wrestle and struggle to create more knowledge for all.

Simply stated, the brachistochrone problem asks the reader to find a line between two points. Euclid's first postulate states that a straight line segment can always be drawn joining any two points[2]. This line segment is naturally the shortest path, or distance, between two points on a Euclidian surface. What if we did not want to find the shortest path, but rather, the shortest *time* between these two points?

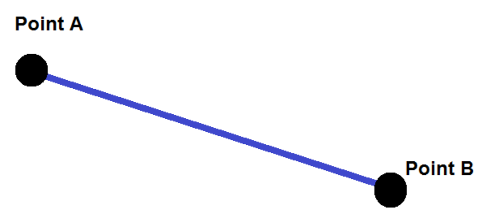

Fig. 1: Shortest Distance between Two Points

Suppose that there was a string with a bead threaded on it, such that the bead can freely move from point A to point B by negating friction and drag forces. In such a situation, with a constant acceleration downwards with a force g, what curve should the string be in order to minimize the travel time of the bead?

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[1] Weisstein, "Brachistochrone Problem"
[2] Weisstein, "Euclid's Postulates"

Fig. 2: Least Time Path?

This question may initially strike the reader as a simple minimization problem. All students of calculus understand the power that calculus has in this regard. If a function, $f(x)$, needs to be minimized, the derivative of $f(x)$, or $f'(x)$, indicates the minimum and maximum points on $f(x)$ when $f'(x) = 0$.

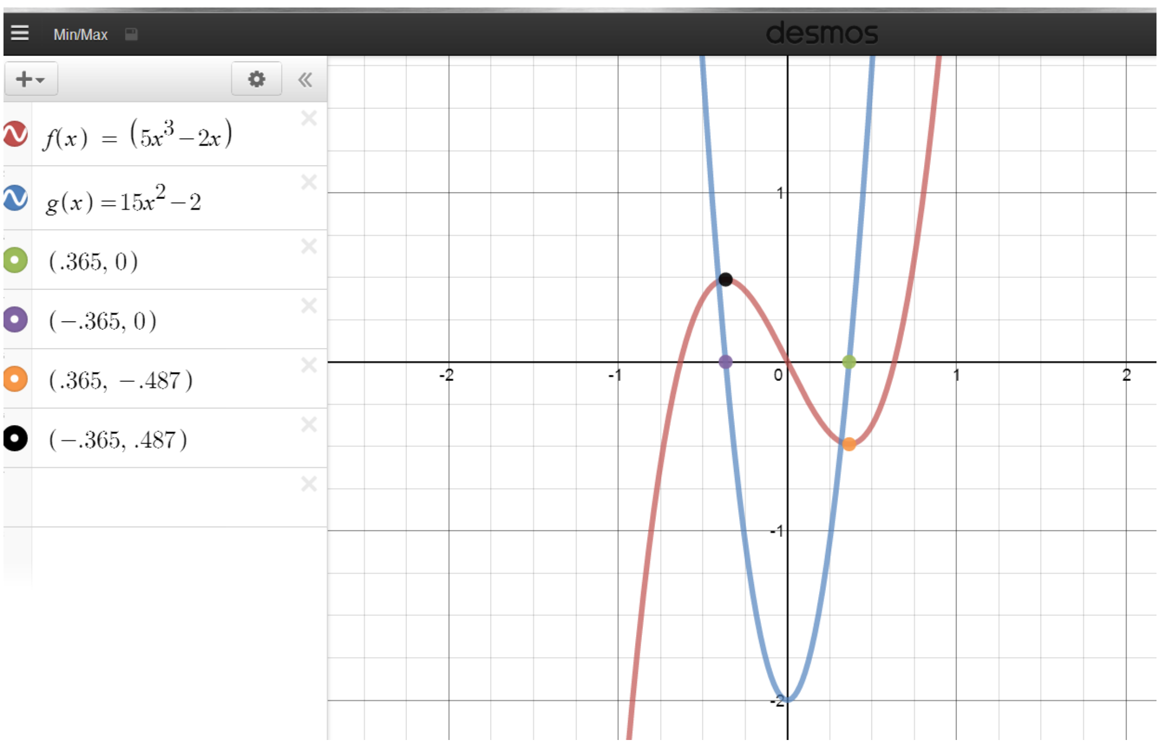

Fig 3: A Standard Use of the First Derivative to Identify Min/Max Points

Using this logic, we will first devise a formula for the time a bead takes to travel from point A to point B.

2.0 Time of Travel between Two Points

Define T to be the total time the bead would take to travel, such that

$$T = \int dt \tag{2.1}$$

For some curve $y(x)$, the instantaneous speed of the ball at any time can be defined as

$$v = \frac{ds}{dt}$$

Where ds is the change in distance of travel and dt is the change in time. Rearranging terms leaves

$$dt = \frac{ds}{v}$$

$$T = \int \frac{ds}{v} \tag{2.2}$$

For any curve, $ds^2 = dx^2 + dy^2$, by the Pythagorean Theorem, so that

$$ds = \sqrt{dx^2 + dy^2}$$

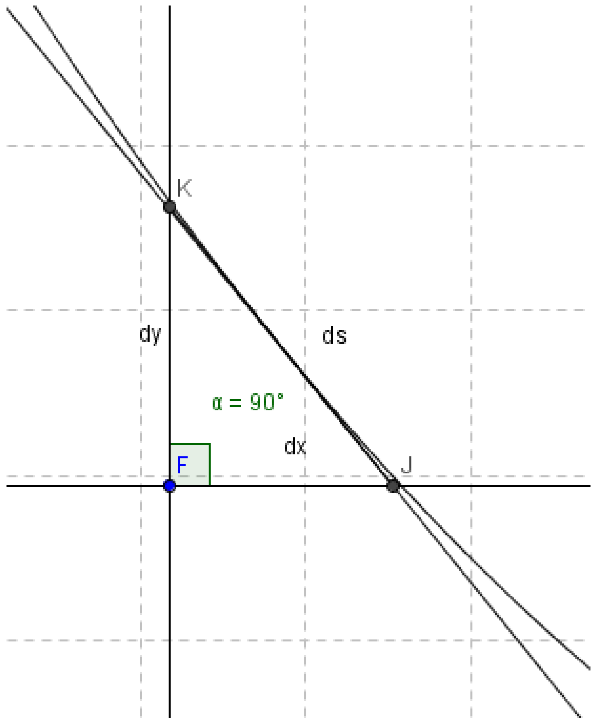

Fig. 4: A Sample Curve with Tangent

Rearranging these terms:

$$ds = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$ds = dx \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1 + y'^2} \cdot dx \qquad (2.3)$$

Next, this bead must obey the laws of energy. Therefore, comparing the kinetic energy of the bead with the gravitational potential energy, we realize that:

$$U = m \cdot gh$$

$$KE = \frac{1}{2}mv^2$$

$$U = KE \therefore\ m \cdot gh = \frac{1}{2} \cdot m \cdot v^2$$

$$2gh = v^2$$

$$v = \sqrt{2gh}$$

Defining h to be the distance y above the x axis,

$$v = \sqrt{2gy} \tag{2.4}$$

Therefore by substituting equations 2.4 and 2.3 in to equation 2.2, we have

$$T = \int \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}\, dx$$

$$T = \frac{1}{\sqrt{2g}} \cdot \int \frac{\sqrt{1 + (y')^2}}{\sqrt{y}}\, dx \tag{2.5}$$

Looking at this equation, we realize that conventional calculus methods do not apply here. Instead of minimizing a specific point, our task is to minimize *a family* of curves.

3.0 The Calculus of Variations – Euler's Method

Although Newton's answer to Bernoulli's challenge for this problem was stunning, writing a proof of construction for the problem in one single night, it was Euler who generalized the problem. This problem moved him into collaboration with J. L. Lagrange to investigate the *calculus of variations*, which is defined today as using "calculus to finding the maxima and minima of a function which depends for its values on another function or a curve[3]."In order to solve for the brachistochrone curve, we shall use their fundamental equation in this field, the *Euler-Lagrange Equation*[4]. It states that

[3]"Calculus of Variations", *Merriam-Webster.com*

[4] This equation is brilliantly deduced in Richard Feynman's lecture, "The Principle of Least Time" (*Feynman Lectures on Physics),* as well as in an excellent book by Paul J. Nahin, *When Least is Best*, "Beyond Calculus". In

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0}$$

Applying our equation for time, we substitute F to be

$$F = \frac{\sqrt{1 + (y')^2}}{\sqrt{y}}$$

So that we would evaluate

$$\frac{\partial F}{\partial y}_{|x,y(x),y'(x)} = \frac{d}{dx}\left[\frac{\partial F}{\partial p}_{|x,y(x),y'(x)}\right] \quad (3.0.1)$$

Where $p = y'$. In order to evaluate the partial derivatives, we will allow for the non-derived variable to be a constant, evaluating the derivative. In our work, allow for m to be a constant.

3.1 Evaluating the Left Hand Side of Euler-Lagrange Equation

Our first step will be to evaluate the Left Hand Side of the Euler-Lagrange Equation, as follows:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\sqrt{1 + p^2}}{\sqrt{y}}\right)$$

$$= \sqrt{1 + p^2} \cdot \frac{\partial}{\partial y} y^{-\frac{1}{2}}$$

$$= \sqrt{1 + p^2} \cdot \left(-\frac{1}{2} y^{-\frac{3}{2}}\right)$$

$$\therefore, \frac{\partial F}{\partial y} = -\frac{1}{2} \cdot \frac{\sqrt{1 + y'^2}}{y^{\frac{3}{2}}} \quad (3.1.1)$$

3.2 Evaluating the Right Hand Side of Euler-Lagrange Equation

In evaluating the Right Hand Side of this equation, we shall first find the partial derivative of F with respect to p, and then take the derivative of the result with respect to x.

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minimizing the work function between a "true" function and a variation of the function, this equation is clear to see. However, this paper will deal with the use of the equation and not necessarily the derivation of it.

Therefore:

$$\frac{\partial F}{\partial p} = \frac{\partial}{\partial p}\left(\frac{\sqrt{1+p^2}}{\sqrt{m}}\right)$$

$$= \frac{1}{\sqrt{m}} \cdot \frac{\partial}{\partial p}(1+p^2)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{m}} \cdot \frac{1}{2}(1+p^2)^{-\frac{1}{2}} \cdot 2p$$

$$\therefore, \frac{\partial F}{\partial p} = \frac{p}{\sqrt{y \cdot (1+p^2)}} \tag{3.2.1}$$

Recalling that $p = y'(x)$, if we then take the derivative of this function with respect to x, we would get the following:

$$\frac{d}{dx}\left[\frac{\partial F}{\partial p}\right] = \frac{d}{dx}\left(\frac{y'}{\sqrt{y(1+(y')^2)}}\right)$$

To simplify notation, allow for $a = y, b = y', c = y''$. Therefore,

$$\frac{d}{dx}\left[\frac{\partial F}{\partial p}\right] = \frac{d}{dx}\left(\frac{b}{\sqrt{a(1+b^2)}}\right)$$

In order to differentiate, we will use the quotient rule. Therefore, allow for

$$u(x) = b \qquad u'(x) = c$$

$$v(x) = \sqrt{a(1+b^2)}$$

$$v'(x) = \frac{d}{dx}\left(\sqrt{a} \cdot \sqrt{1+b^2}\right)$$

In order to differentiate $v(x)$, we must apply the product rule. Therefore:

$$s(x) = \sqrt{a} \qquad s'(x) = \frac{1}{2\sqrt{a}} \cdot b$$

$$t(x) = \sqrt{1+b^2} \qquad t'(x) = \frac{1}{2\cdot\sqrt{1+b^2}}\cdot 2b\cdot c$$

Such that $v'(x) = s(x)\cdot t'(x) + t(x)\cdot s'(x)$

$$v'(x) = \frac{\sqrt{a}\cdot bc}{\sqrt{1+b^2}} + \frac{b\cdot\sqrt{1+b^2}}{2\sqrt{a}}$$

$$= \frac{bc\cdot\sqrt{a}\cdot 2\sqrt{a} + \left(b\cdot\sqrt{1+b^2}\cdot\sqrt{1+b^2}\right)}{2\sqrt{a\cdot(1+b^2)}}$$

$$v'(x) = \frac{2abc + b\cdot(1+b^2)}{2\sqrt{a(1+b^2)}}$$

Applying the quotient rule, we have:

$$\frac{d}{dx}\left[\frac{\partial F}{\partial p}\right] = \frac{u'(x)\cdot v(x) - v'(x)\cdot u(x)}{v^2}$$

$$= \frac{c\cdot\sqrt{a(1+b^2)} - \left(b\cdot\frac{2abc + b\cdot(1+b^2)}{2\sqrt{a(1+b^2)}}\right)}{a\cdot(1+b^2)}$$

$$= \frac{\left(2\cdot c\cdot\sqrt{a(1+b^2)}\sqrt{a(1+b^2)} - \left(b\cdot\left(2abc + b(1+b^2)\right)\right)\right)}{2\sqrt{a(1+b^2)}}\cdot\frac{1}{a(1+b^2)}$$

$$= \frac{1}{\sqrt{a(1+b^2)}}\cdot\frac{2a(1+b^2)c - 2ab^2c - b^2(1+b^2)}{2a(1+b^2)}$$

$$= \frac{1}{\sqrt{a(1+b^2)}}\cdot\frac{2ac + 2ab^2c - 2ab^2c - b^2(1+b^2)}{2a(1+b^2)}$$

$$= \frac{1}{\sqrt{a(1+b^2)}}\cdot\frac{2ac - b^2(1+b^2)}{2\cdot a(1+b^2)}$$

$$= \frac{1}{\sqrt{a(1+b^2)}}\cdot(\frac{2ac}{2a(1+b^2)} - \frac{\left(b^2(1+b^2)\right)}{2\cdot a(1+b^2)}$$

$$= \frac{1}{\sqrt{a(1+b^2)}} \cdot \left(\frac{c}{1+b^2} - \frac{b^2}{2a}\right)$$

$$\therefore, \frac{d}{dx}\left[\frac{\partial F}{\partial p}\right] = \frac{1}{\sqrt{y(1+(y')^2)}} \cdot \left(\frac{y''}{1+y'^2} - \frac{y'^2}{2y}\right) \tag{3.2.2}$$

3.3 Solving the Euler-Lagrange Formula for the Brachistochrone Curve

Combining the Right Hand Side (3.2.2) and the Left Hand Side (3.1.1):

$$-\frac{1}{2} \cdot \frac{\sqrt{1+y'^2}}{y^{\frac{3}{2}}} = \frac{1}{\sqrt{y(1+(y')^2)}} \cdot \left(\frac{y''}{1+y'^2} - \frac{y'^2}{2y}\right) \tag{3.3.1}$$

Using the same substitution of $a = y, b = y', c = y''$,

$$\frac{1}{\sqrt{a(1+b^2)}} \cdot \left(\frac{c}{1+b^2} - \frac{b^2}{2a}\right) = -\frac{1}{2} \cdot \frac{\sqrt{1+b^2}}{a^{\frac{3}{2}}}$$

$$\left(\frac{c}{1+b^2} - \frac{b^2}{2a}\right) = -1 \cdot \frac{\left(\sqrt{a} \cdot \sqrt{a(1+b^2)} \cdot \sqrt{a(1+b^2)}\right)}{2a^{\frac{3}{2}}}$$

$$\frac{c}{1+b^2} - \frac{b^2}{2a} = -1 \cdot \frac{a^{\frac{1}{2}}}{2a^{\frac{3}{2}}} \cdot (1+b^2)$$

$$\frac{c}{1+b^2} = \frac{-(1+b^2)}{2a} + \frac{b^2}{2a}$$

$$\frac{c}{1+b^2} = -\frac{1}{2a}$$

$$2ac = -1 - b^2$$

$$\therefore, \qquad 2 \cdot y \cdot y'' + 1 + y'^2 = 0 \tag{3.3.2}$$

If we multiply the equation by y', we get

$$2 \cdot y \cdot y' \cdot y'' + {y'}^3 + y' = 0 \tag{3.3.3}$$

For the next step, we will work backwards a little. If we evaluate

$$\frac{d}{dx}\left[y + y \cdot {y'}^2\right]$$

It is clear that the result would be

$$2 \cdot y \cdot y' \cdot y'' + {y'}^3 + y'$$

This is the same as equation (3.3.3). Therefore, if we integrate both sides of (3.3.3) with respect to x, we have:

$$\int (2 \cdot y \cdot y' \cdot y'' + {y'}^3 + y')dx = \int 0\, dx$$

$$\int \frac{d}{dx}\left[y + y \cdot {y'}^2\right] dx \ = C$$

$$y + y \cdot {y'}^2 = C$$

This equation is a second order differential equation. Although second order differential equations can be difficult to solve, this equation is a special case, as:

$$y' = \sqrt{\frac{C - y}{y}}$$

$$\frac{dy}{dx} = \sqrt{\frac{C - y}{y}}, \qquad \frac{dx}{dy} = \sqrt{\frac{y}{C - y}} \tag{3.3.4}$$

From this point, we shall introduce a new variable, ϕ, such that

$$\tan(\phi) = \frac{dx}{dy}$$

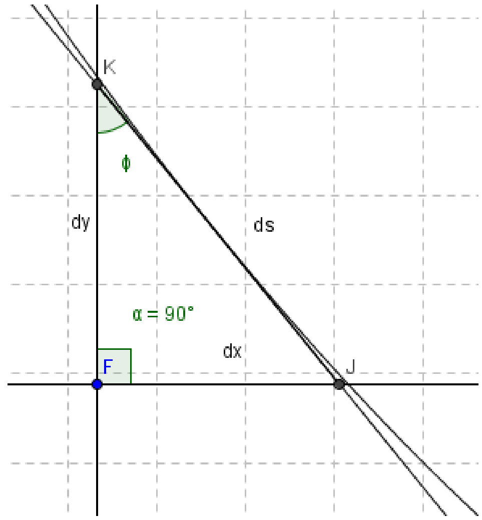

Fig. 5: Defining the angle ϕ

We shall now solve the parametric equation for this curve using the variable ϕ.

Therefore,

$$\tan\phi = \sqrt{\frac{y}{C-y}}$$

$$\frac{\sin^2\phi}{\cos^2\phi} = \frac{y}{C-y}$$

$$C\cdot\sin^2\phi - y\cdot\sin^2\phi = y\cdot\cos^2\phi$$

$$y = C\cdot\sin^2\phi$$

$$y = C\cdot(1-(1-\sin^2\phi))$$

$$\boxed{y = \frac{C}{2}(1-\cos 2\phi)} \qquad (3.3.5)$$

Solving for the parametric equation in the x direction,

$$\frac{dx}{d\phi} = \frac{dx}{dy}\cdot\frac{dy}{d\phi}$$

$$\frac{dy}{d\phi} = C \cdot 2\sin\phi\cos\phi\,, \qquad \frac{dx}{dy} = \sqrt{\frac{y}{C-y}}$$

$$\frac{dx}{d\phi} = \sqrt{\frac{y}{C-y}} \cdot 2C\sin\phi\cos\phi$$

$$\frac{dx}{d\phi} = \sqrt{\frac{C \cdot \sin^2\phi}{C - C \cdot \sin^2\phi}} \cdot 2C\sin\phi\cos\phi$$

$$= C \cdot \sqrt{\frac{\sin^2\phi}{1-\sin^2\phi}} \cdot 2C\sin\phi\cos\phi$$

$$= C \cdot \frac{\sin\phi}{\cos\phi} \cdot 2\sin\phi\cos\phi$$

$$= C\big(1 - (1 - 2\sin^2\phi)\big)$$

$$\frac{dx}{d\phi} = C(1-\cos 2\phi)$$

$$dx = C(1-\cos(2\phi))\, d\phi$$

$$\int dx = C\int (1-\cos 2\phi)\ d\phi$$

$$x = C \cdot \left(\int 1\, d\phi - \int \cos 2\phi\ d\phi\right)$$

$$x = C\left(\phi - \frac{1}{2}\sin 2\phi\right)$$

$$\boxed{x = \frac{C}{2}(2\phi - \sin 2\phi)} \qquad (3.3.6)$$

Therefore, our final two parametric formulas for this curve would be of the following:

$$x = \frac{C}{2}(2\phi - \sin 2\phi)$$

$$y = \frac{C}{2}(1 - \cos 2\phi)$$

4.0 The Cycloid

These equations must have surprised Bernoulli, Newton, Lagrange, and Euler when they discovered it, for these are the parametric equations of a cycloid. The cycloid is a curve that was so fiercely debated among 18th century mathematicians that it was frequently called the "Helen of Geometers[5]", and was even alluded to in Moby Dick[6].This curve is also simply constructed, but clearly has some fascinating properties.

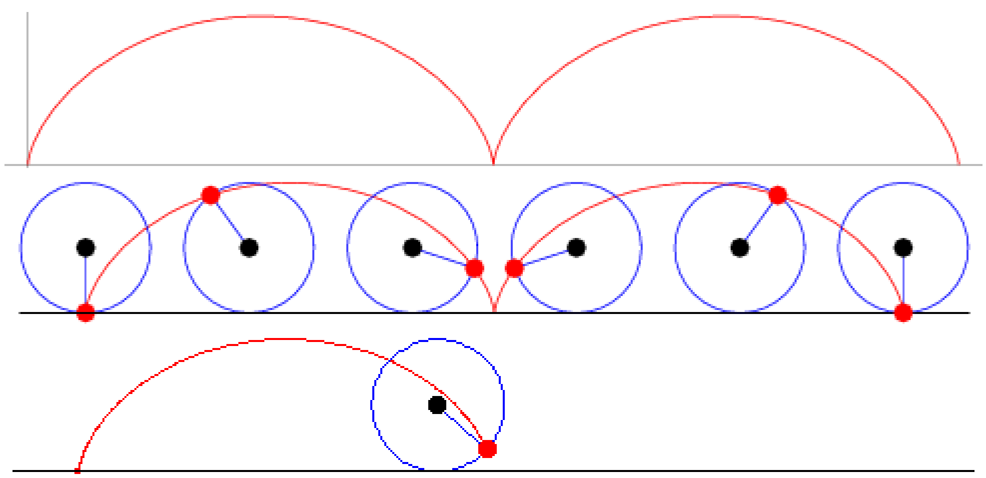

Fig. 6: The Ever Elusive Cycloid[7]

The cycloid is created if we could imagine a pen stuck to the edge of a circle as the circle rotates forwards in the x-direction. Therefore, we are able to construct parametric equations for such a curve by merely studying a circle. While the standard equations for circles are

[5] Cajori, Florin
[6] Melville, Herman
[7] Weisstein, "Cycloid"

$x = r\cos\theta, y = r\sin\theta,$ if we allow for the circle to rotate clockwise with angle t from the bottom of the circle, the following equations must be used to correct for the change in position:

$$x = -r\sin t\,, y = r\cos t$$

Because the circle that constructs the cycloid moves in the positive x direction, we must add this motion into the parametric function of the cycloid. Therefore, allowing for t to represent the number of radians that the circle has moved,

$$\Delta x = 2\pi r \cdot \frac{t}{2\pi}$$

$$x = -r\sin t + rt$$

$$\boxed{x = r(t - \sin t)} \qquad (4.0.1)$$

In the y direction, the only correction that needs to be made is that for this circle, we shall assume that the center of the circle is not at (0,0), but rather at (r, r) so that the bottom of the cycloid rests at the x axis. Therefore,

$$y = r - r\cos t$$

$$\boxed{y = r(1 - \cos t)} \qquad (4.0.2)$$

Before we proceed, the similarity between this equation and the equation derived for the brachistochrone curve is jarring. There is no mistake; the two curves are the same!

4.1 The Differential Equation of the Cycloid

If we take the differential of these equations with respect to t, then we have

$$\frac{dx}{dt} = \frac{d}{dt}(r(t - \sin(t)) = r \cdot (1 - \cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt}(r(1 - \cos(t)) = r \cdot (\sin t)$$

Therefore, we can determine the derivative of the cycloid to be

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = r \sin t \cdot \frac{1}{r(1-\cos t)}$$

$$= \frac{\sin t}{1-\cos t} \tag{4.1.1}$$

If we the square both sides, then

$$\left(\frac{dy}{dx}\right)^2 = \frac{\sin^2 t}{(1-\cos t)^2}$$

$$= \frac{1-\cos^2 t}{(1-\cos t)^2}$$

$$= \frac{(1-\cos t)(1+\cos t)}{(1-\cos t)^2}$$

$$= \frac{(1+\cos t)}{(1-\cos t)}$$

Recalling equation (4.0.2), such that

$$\cos t = 1 - \frac{y}{r}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1+\left(1-\frac{y}{r}\right)}{1-\left(1-\frac{y}{r}\right)}$$

$$= \frac{2-\frac{y}{r}}{\frac{y}{r}}$$

$$= \frac{2r-y}{r} \cdot \frac{r}{y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{2r-y}{y} \tag{4.1.2}$$

$$\boxed{y(y'^2+1)=2r} \tag{4.1.3}$$

This equation was used when the mathematician Johann Bernoulli attempted to solve this problem. Rather than using the string and bead method, he imagined a beam of light traveling through a “variable density” medium. Because light will always “choose” the path of least time, he followed light using Snell’s law to find the general path of least time. The proof is simple and elegant, combining fields of geometry and physics. In doing so, he found light to obey the same differential equation as stated in (4.1.3), proving that light would travel in a cycloid path.

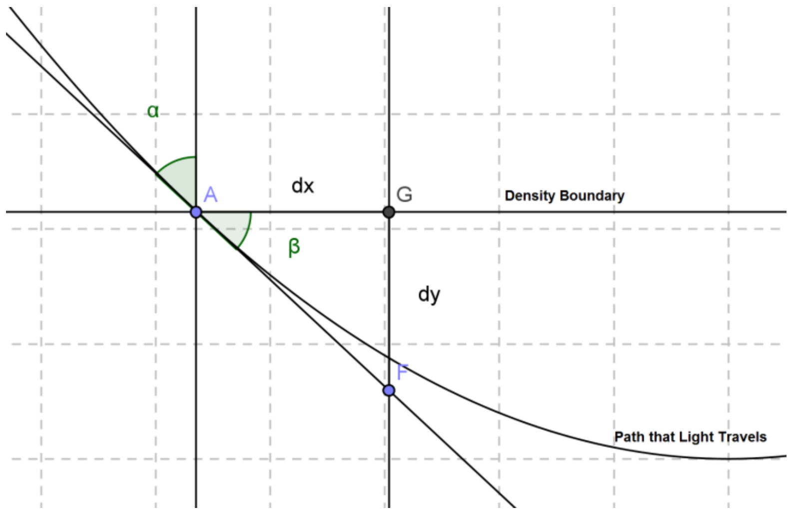

Fig. 8: Snell’s Law in a Variable Density Glass

5.0 Solving for the Time of Travel

At this point, we would like to solve for the time that travelling along a cycloid would take, as stated in equation (2.5). Restating it here shows that:

$$T=\frac{1}{\sqrt{2g}}\cdot\int_0^{x_1}\sqrt{\frac{1+y'^2}{y}}\,dx$$

And since a modification of equation (4.1.2) states:

$$\frac{dy}{dx}=\sqrt{\frac{2r-y}{y}}$$

Separating variables results in

$$dy \cdot \frac{1}{\sqrt{\frac{2r-y}{y}}} = dx$$

Substituting this to the equation for time of travel leads to

$$T = \frac{1}{\sqrt{2g}} \cdot \int_{0}^{x_1} \sqrt{\frac{1 + \left(\sqrt{\frac{2r-y}{y}}\right)^2}{y}} \cdot \frac{1}{\sqrt{\frac{2r-y}{y}}} \cdot dy$$

$$= \frac{1}{\sqrt{2g}} \cdot \int_{0}^{y(x_1)} \sqrt{\frac{\frac{y+2r-y}{y}}{y \cdot \frac{2r-y}{y}}} \cdot dy$$

$$= \frac{1}{\sqrt{2g}} \cdot \int_{0}^{y(x_1)} \sqrt{\frac{2r}{y} \cdot \frac{1}{2r-y}}\, dy$$

$$T = \frac{1}{\sqrt{2g}} \cdot \int_{0}^{y(x_1)} \sqrt{\frac{2r}{y(r-y)}}\, dy \qquad (5.1)$$

Recall that, by equation (4.0.2), we know that

$$y = r(1 - \cos t)$$

Therefore, the equation for time of travel can be reorganized to be

$$T = \frac{1}{\sqrt{2g}} \cdot \int_{0}^{y(x_1)} \sqrt{\frac{2r}{-1 \cdot [(r-y)^2 - r^2]}}\, dy$$

If we focus our attention the bottom of the fraction, we realize that

$$(r-y)^2 - r^2 = (r - r + r \cdot \cos t)^2 - r^2$$

$$= \left(r \cdot (1 - 1 + \cos t)\right)^2 - r^2$$

$$= r^2 \cos^2 t - r^2$$

$$= r^2(\cos^2 t - 1)$$

$$= -r^2 \sin^2 t$$

Therefore, the integral is simplified to be

$$T = \frac{1}{\sqrt{2g}} \cdot \int_0^{y(x_1)} \sqrt{\frac{2r}{-1 \cdot -r^2 \sin^2 t}} dy$$

$$= \frac{1}{\sqrt{2g}} \cdot \int_0^{y(x_1)} \frac{\sqrt{2r}}{r \cdot \sin t}\, dy$$

Substituting this equation to be in terms of r, t results in

$$dy = r \sin t\, dt$$

Such that

$$T = \frac{1}{\sqrt{2g}} \cdot \int_0^{t(y(x_1))} \frac{\sqrt{2r}}{r \cdot \sin t} \cdot r \sin t\, dt$$

$$T = \frac{1}{\sqrt{2g}} \cdot \int_0^{t(y(x_1))} \sqrt{2r}\, dt$$

$$\boxed{T = \sqrt{\frac{r}{g}} \cdot t} \tag{5.2}$$

As t, r are variables for the parametric equation of the brachistochrone curve, given any equation and a gravitational acceleration, we can calculate the shortest time of travel.

6.0 Calculation of Sample Path

Up to this point, all of our work sought to understand the nature of the brachistochrone curve. We have explored differential equations as well as parametric forms of this curve. However, rather than leave the curve as a hypothetical cycloid, we shall define a real curve using points and investigate the time it takes for an object to follow this path.

. Let us investigate the brachistochrone curve between the points (x_1, y_1) and (x_2, y_2), where $x_1 = 1, y_1 = 1, x_2 = 5, y_2 = -5$. Before evaluating, we first make an adjustment in notation. It will always be most optimal for the beginning point to be placed at one of the peaks of the cycloid, as the tangent line to the peak would be vertical, providing for the largest initial velocity. However, this requires the parametric equation to be $t \in 2\pi n, where\ n \in \mathbb{Z}^+$.Therefore, using the value $t = 0$ in the equations

$$x_1 = r(t - \sin t), \qquad y_1 = r(1 - \cos t)$$

Results in the calculation of $r = \infty, \to x_1, y_1 \neq 0$. A simple solution is to adjust our frame of reference so that the starting point is always centered at $(0, 0)$.

Our new cycloid must therefore pass through the points $(0, 0), (4, -4)$. We shall note that a key limitation for the second point is that $y_2 \leq y_1$, which we will discuss in depth later. This allows for the brachistochrone curve to be the *inverted* cycloid. However, it is more difficult to calculate the parametric equations for such a curve. Therefore, we shall solve this problem by reflecting (x_2, y_2) across the x-axis, so that we look for the cycloid between the points $(0, 0), (4, 4)$.

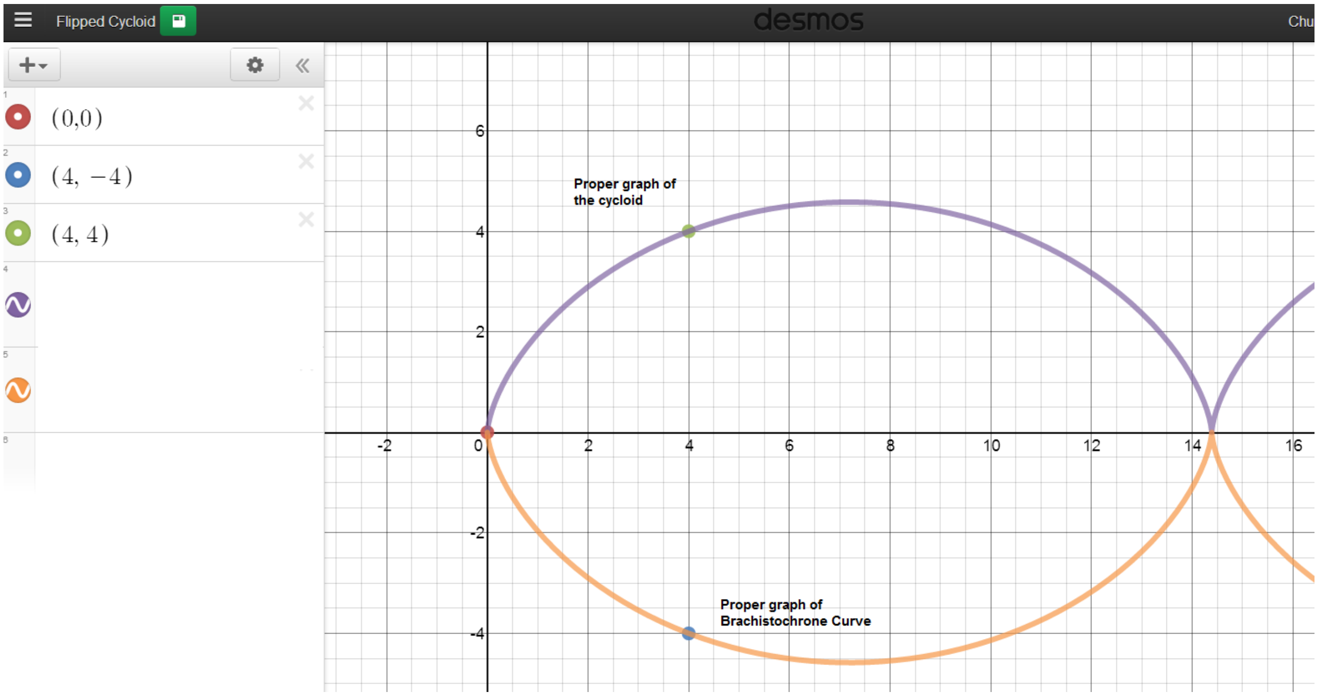

Fig. 9: The Inverted Cycloid is the Brachistochrone Curve

Solving for this equation is considerably simpler, as

$$4 = r(t - \sin t)$$

$$4 = r(1 - \cos t)$$

$$\frac{4}{1 - \cos t} = r$$

$$4 = \left(\frac{4}{1 - \cos t}\right) \cdot (t - \sin t)$$

$$1 - \cos t = t - \sin t$$

$$1 - t = \cos t - \sin t \qquad (6.1)$$

At this point, the simplest way to solve for t is to either use a calculator or graph the two equations. Using Mathematica for this function, we find that

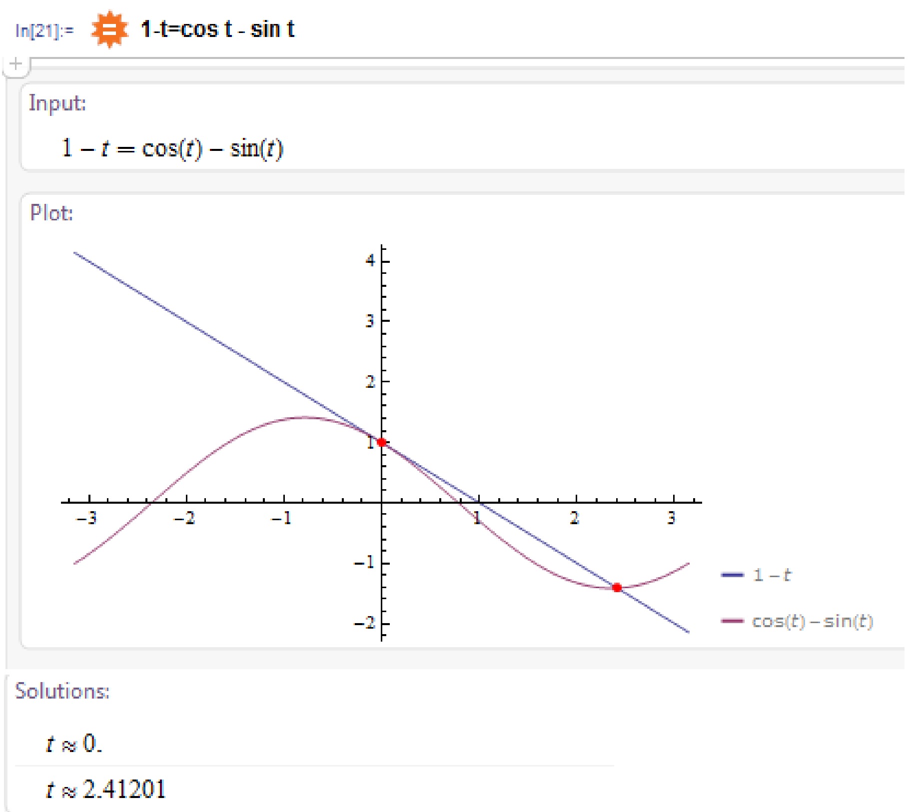

Fig. 10: Mathematica Solving for Solution of Two Lines

Such that $t \approx 2.41201$. From here, it is nearly trivial to solve for r, as

$$r = \frac{4}{1 - \cos 2.41201} \approx 2.29167$$

In order to find the equation for the brachistochrone curve, we reflect this curve across the x-axis again to get

$$x = 2.29(t - \sin t)$$

$$y = -2.29(1 - \cos t)$$

Plotting this on a graph yields:

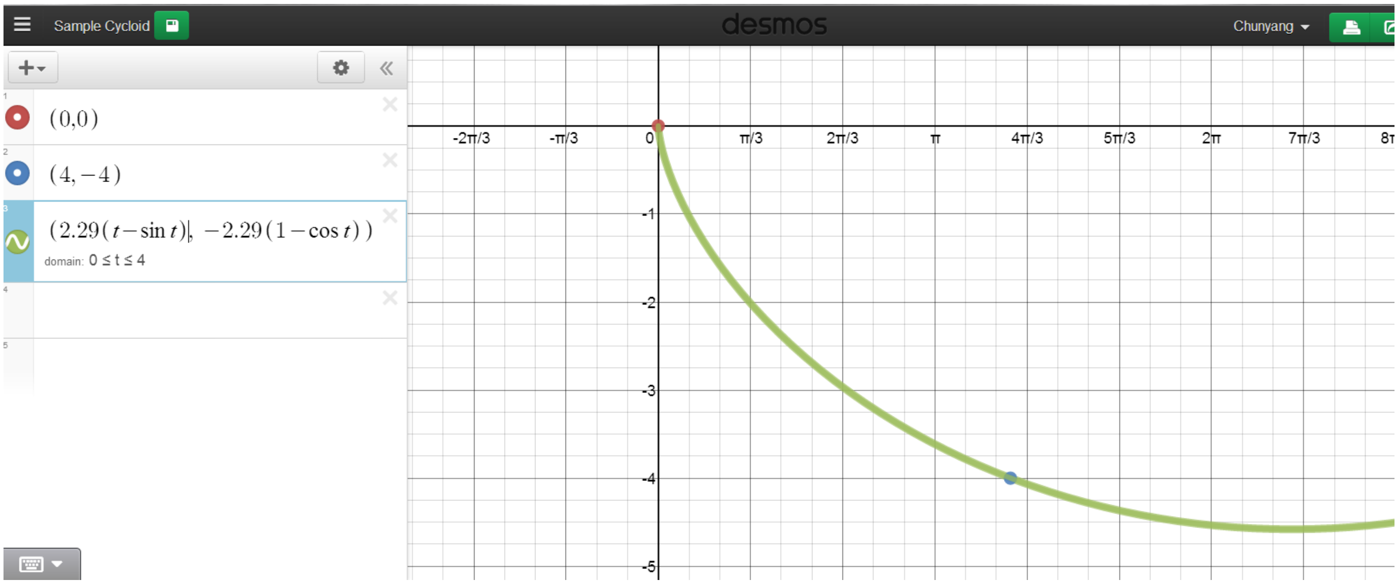

Fig. 11: Plot of Cycloid and Original Points

We conclude from Fig. 10 that our calculated parametric equation passes through the points of interest. However, is it the path of least time? We shall define this curve in Earth conditions, such that one unit is one meter in length, and $g = 9.81\frac{m}{s^2}$. We therefore apply equation (6.2) to calculate the time function for this curve to be:

$$T = \sqrt{\frac{2.29167}{9.81}} \cdot 2.412$$

$$T = 1.16579\ seconds$$

Although we will not calculate the time for all possible curves, it may be beneficial to see how much time is saved by the cycloid as compared to the straight line that contains these two points.

Solving the time for the straight line path is relatively trivial, as it only requires basic physics knowledge to understand that

$$distance = \sqrt{4^2 + (-4)^2} = \sqrt{32}$$

$$acceleration = g \sin\frac{\pi}{4} = \frac{9.81}{\sqrt{2}}$$

$$d = v_i t + \frac{1}{2}at^2$$

As $v_i = 0$,

$$\sqrt{32} = \frac{1}{2} \cdot \frac{9.81}{\sqrt{2}} \cdot t^2$$

$$t = \sqrt{\frac{2\sqrt{64}}{9.81}}$$

$$t = \sqrt{1.63099} = 1.2771\ seconds$$

$$\Delta T = t_{line} - t_{cycloid} = 1.2771 - 1.16579 = 0.111312\ seconds$$

At this point, it may seem somewhat pointless. We did all of these calculations just to save 0.11 seconds while traveling? However, it may be interesting to see what would happen if we scale this second point farther, such that the slope of the line is constant, but the distance between the two points is different. In doing so, we shall compare the percent of time gained, as determined by

$$\%\ Gain = \frac{\Delta T}{t_{linear}}$$

For our previous situation,

$$\%\ Gain = \frac{0.111312}{1.2771} \approx 8.72\%$$

We can quickly calculate the same time difference using the point (1000, 1000) for the second point, in order to evaluate how the ratio of time changes as the linear time increases. Processing, we find that

$$1000 = r(t - \sin t)$$

$$1000 = r(1 - \cos t)$$

$$1 - t = \cos t - \sin t$$

$$t \approx 2.41201$$

$$r = \frac{1000}{1 - \cos 2.41201} \approx 572.917$$

$$T = \sqrt{\frac{572.917}{9.81}} \cdot 2.41201 \approx 18.43\ seconds$$

For the linear path,

$$d = 1000\sqrt{2}\ meters$$

$$a = \frac{9.81}{\sqrt{2}} \cdot \frac{m}{s^2}, v_i = 0\frac{m}{s}$$

$$1000\sqrt{2} = \frac{1}{2} \cdot \frac{9.81}{\sqrt{2}} \cdot t^2$$

$$\frac{1000 \cdot 2 \cdot 2}{9.81} = t^2$$

$$t = \sqrt{\frac{1000}{9.81}} \approx \sqrt{407.7} \approx 20.19\ seconds$$

Calculating for the percent gain results in

$$\%\ Gain = \frac{20.19 - 18.43}{20.19} \approx 8.72\%$$

It seems as if regardless the amount of scaling that the percent gain in time is constant for a cycloid vs. a linear line! We can attempt to generalize this for all different slopes.

For some points (0, 0) and (x_1, y_1), we find that the equation of the parametric is $x_1 = r(t - \sin(t), y_1 = r(1 - \cos(t))$. Therefore, $r = \frac{y_1}{1-\cos(t)}$. We understand that for a given ratio of $\frac{y_1}{x_1}$, t will be constant. Therefore, if the (x_1, y_1) point is scaled by some factor s, such that the new points are $(s \cdot x_1, s \cdot y_1)$, the new r is

$$r = \frac{s \cdot y_1}{1 - \cos(t)}$$

Therefore,

$$T_{cycloid} = \sqrt{s} \cdot \sqrt{\frac{y}{g}} \cdot \frac{t}{1 - \cos(t)}$$

Calculating the time for a general linear form would simply be

$$d = \frac{1}{2} a t^2$$

$$T_{linear} = \sqrt{\frac{2\sqrt{s^2x^2 + s^2y^2}}{g \sin(\theta)}}$$

$$T_{linear} = \sqrt{\frac{2 \cdot s \cdot \sqrt{x^2 + y^2}}{g \frac{sy}{s \cdot \sqrt{x^2 + y^2}}}}$$

Therefore,

$$\% \, Gain = \frac{\left(\sqrt{s} \cdot \sqrt{\frac{2\sqrt{x^2 + y^2}}{g \frac{y}{\sqrt{x^2 + y^2}}}} - \sqrt{s} \cdot \sqrt{\frac{y}{g}} \cdot \frac{t}{1 - \cos(t)} \right)}{\sqrt{s} \sqrt{\frac{2\sqrt{x^2 + y^2}}{g \frac{y}{\sqrt{x^2 + y^2}}}}}$$

$$\% \, Gain = \frac{\sqrt{\frac{2(x^2 + y^2)}{yg}} - \sqrt{\frac{y}{g}} t}{\sqrt{\frac{2(x^2 + y^2)}{yg}}}$$

$$\% \, Gain = \frac{\sqrt{\frac{2(x^2+y^2)}{y}} - t\sqrt{y}}{\sqrt{\frac{2(x^2+y^2)}{y}}}$$

$$\% \, Gain = \frac{\sqrt{\frac{2(x^2+y^2)-t^2y^2}{y}}}{\sqrt{\frac{2(x^2+y^2)}{y}}}$$

$$\% \, Gain = \sqrt{1 - \frac{t^2y^2}{2(x^2+y^2)}}$$

Interestingly, it seems like regardless of the degree to which an angle is scaled to, the percent gain in time of a cycloid versus a linear line will be a fixed percent. In addition, the gravitational constant will not affect this value either.

A limitation for this solution is, as stated earlier, $y_2 \leq y_1$, in order to fulfill conservation of energy. Because the kinetic energy needed to move the bead comes from the difference in potential energy between the starting point and the ending point, there will only be enough energy in the system if $y_1 - y_2 \leq 0, \therefore y_1 \leq y_2$.

7.0 Conclusion

Fundamental truths are not discovered by scientists lounging around, trying to think of something to discover. Rather, it is through the application of challenging and interesting problems that create new lands of knowledge. Some mathematicians do believe it worthwhile to pursue single fields wholeheartedly, such as Hilbert, who in his namesake 23 problems, lists the 23rd to simply be "Further development of the methods of the calculus of variations[8]." However, I must digress: Further mathematics will be discovered by the continued appetite for curiosity that students of the discipline have, not because of a proverbial carrot in the form of recognition of fame.

[8] Hilbert, David.

What has this investigation taught me of the nature of mathematics? In order to seriously improve my skills, I should not entirely depend on what a spoon-fed textbook tells me, but rather, find and contemplate problems elsewhere. Through my "simple" exploration, discovered while browsing Wikipedia late one night, I discovered numerous Mathematical Association of America articles, books, lecture notes, physics applications, and other, non-traditional, sources of knowledge, such as my father's brilliant deductive skills and my math teacher's perseverance to develop an excellent project. I not only got my first taste of the calculus of variations, a field generally taught in the college level, but also now understand more about partial derivatives and parametric equation, as well as the branch between analytic calculus and the world of geometry. Jarringly, I realized through this investigation my own shortfalls in traditional calculus, as misremembering the product rule, the quotient rule, and the chain rule (in that order!) led to a week of frustrated head scratching and puzzlement at my notes.

In the future, when I want to train and practice my skills, I will certainly turn to my trusted mathematics textbooks, but when I am ready to discover something for myself … I hear that one of the fascinating properties of the cycloid is that it also satisfies the tautochrone curve. Would anyone like to explore with me?

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