

Article
***The Origins of Psychological Axioms
of Arithmetic and Geometry***

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1. The Problem

The papers by Giaquinto and Wynn concern different aspects of mathematical cognition; Giaquinto examines a cognitive process through which people can acquire novel geometrical knowledge, while Wynn focuses on fundamental arithmetical knowledge present in human infants. This commentary considers an issue that arises with both of these research projects—the evolutionary origin of unlearned mathematical knowledge.

The existence of unlearned mathematical knowledge is explicit in Wynn's discussion, but is present as well as part of the visualization process outlined in Giaquinto's paper. In particular, inferences that lead to geometric insights of the kind described by Giaquinto require some 'visual' knowledge that cannot itself be obtained by a visualization process, but which must be taken as given. In the example he presents, this is the knowledge that the diagonal of a square cuts the square into identical, mirror-image triangles. Without accepting this fact, a person would not know, and presumably would be unable to visualize, that the outer corners fold in to exactly cover the inner square without any overlaps or gaps, and would thus not appreciate the proof that the inner square is half the area of the square that it is contained within. But there is no proof needed—either through visualization or any other means—for the premise that a diagonal of a square cuts it into two equal parts. This aspect of geometrical

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knowledge is somehow privileged, and we can accordingly dub it a 'psychological axiom', one that is presumably unlearned.

Thus we appear to possess psychological axioms of both geometry and arithmetic. But why would humans—and possibly other animals, such as rats and birds—come to possess this sort of knowledge? The axioms are almost certainly not the direct result of selective pressure; it is implausible, for instance, that an animal who believes that the diagonal of a square cuts it into equal parts would have more reproductive success than another animal who lacks such a belief. Given this, the discussion below assumes that there was no selection for knowledge of the specific axioms *per se*; they are instead the by-product of more general mechanisms (*i.e.* the capacity to count and the capacity to visualize) that have themselves evolved as adaptations through the process of natural selection.

2. Number

On the theory advanced by Wynn (this issue), our axiomatic beliefs about number arise from the structure of the outputs yielded by the accumulator mechanism, which has the capacity to measure continuous temporal duration and also to count discrete individuals (see Meck & Church, 1983). The adaptive significance of determining duration is well-known (*e.g.* Gallistel, 1990); the discussion here focuses on why rats, birds, humans, and other animals have evolved the capacity to use this accumulator mechanism to record numerosity (*i.e.* to count) and to make simple basic arithmetical computations on the basis of the output of the accumulator (*i.e.* to add and subtract).

Perhaps the simplest hypothesis is that there was direct selective pressure for the capacity to determine the numerosity of a particular sort of entity and to use this knowledge to govern certain inferences. Gallistel (1990), for instance, reviews evidence that animals can determine rate of return (calculated as number of food encounters per unit of time multiplied by average amount of food observed or obtained per encounter), and he notes that 'the adaptive value of being able to estimate rate of return is obvious'. This suggests that the capacity to count evolved so that animals could count food encounters and thereby calculate rate of food return, which dictates such adaptive behaviors as choice of the optimal habitat. Although the capacity to count has, on this theory, evolved for the purpose of counting a specific sort of entity (food encounters), the more general nature of the accumulator mechanism causes this capacity to be extended to other domains as well; thus rats can count bar presses, tunnels, and discrete sounds.

A different possibility is that the ability to count has evolved because it helps animals to track sets of objects, usually other animals, over time and space. Predators and prey have the unfortunate property of moving, of disappearing behind other objects and re-emerging at a later time.

Multiple objects pose a special problem; if they are perceptually indistinguishable, the capacity to determine their numerosity may play an essential role in tracking them. For instance, imagine being chased by three dogs, who are shifting in position, moving behind trees, and so on. The knowledge that there are three of them governs one's behavior in an obvious way; if there is a moment when there are only two that can be seen, you will know that there still exists the third, temporarily out of sight. In situations such as this, sensitivity to numerosity might be essential. Moving to a more benign example, imagine supervising a large group of children on a field trip. If there are many children, it is virtually impossible to keep track of them (*i.e.* determine that none have wandered off) without counting.

The adaptive role of number in tracking objects should not be overstated, however. In many instances of object tracking, even with multiple objects, sensitivity to number is neither necessary nor sufficient. If one has two cats (Watson and Wolfgang), it is not necessary to count in order to determine if both are in the house; one simply has to determine if Watson is in the house and if Wolfgang is in the house. (As discussed in Wynn's reply to Galloway, activities such as categorizing objects as cats or recognizing individual cats do not require the ability to count.)

Further, a sensitivity to numerosity is useless without a prior capacity to recognize the kind of entity that the number applies to; knowing that there are three dogs chasing you is useless if you can't tell dogs from non-dogs. And even if one can recognize dogs, and can count how many dogs there are, this still isn't sufficient for tracking *particular* sets of dogs. If one of the three dogs runs behind a tree and a fourth dog appears from another direction, one will be fooled into thinking the set of objects had not changed—unless one has the additional ability to track individual entities (see the discussion of visualization below). Despite these qualifications, though, there may be enough cases where an animal has to track groups of homogeneous objects for there to be selective pressure for a sensitivity to numerosity to evolve.

Assuming that the accumulator mechanism is initially used for measuring and comparing temporal durations or some other continuous property, what further evolution of the brain is required for this mechanism to be used for counting? Interestingly, the answer might be little or none. Given the pre-existing accumulator model, and the capacity to think about and quantify over individuals, the ability to count and to perform arithmetical calculations may be already present.

One can distinguish two modes of cognition; quantification and inference over *individual entities*, and quantification and inference over *non-individuated stuff*. This contrast is expressed in natural language. English, for instance, has a distinction between count nouns (*e.g.* 'dogs') and mass nouns (*e.g.* 'water'); this syntactic distinction corresponds to two radically different ways of conceptualizing entities in the physical world (Bloom, *in press*; Jackendoff, 1991). Even before the acquisition of syntax, 1- and 2-

year-olds are capable of shifting from one construal to another as a function of conceptual context; they will generalize a word referring to a substance to other entities composed of the same stuff and generalize a word referring to an individual object to other individuals of the same kind (Soja, Carey & Spelke, 1991).

The capacity to quantify over individual entities appears to be present in the youngest infants tested. Putting aside for the moment the question of infants' understanding of number, quantification over individuals is essential for such basic activities as tracking a single object over space or time (see below), or making generalizations about the properties of objects (*e.g.* learning that dogs bite).

Consider now that the accumulator mechanism is agnostic as to whether it deals with stuff or individuals. It stores and outputs different quantities of energy, but the reference of the input and output of the mechanism is determined by interpretative processes that are part of the animal's more general conceptual system. For instance, two outputs of the mechanism could correspond to different durations, but could also correspond to different numbers of objects or sounds. Similarly, a single opening and closing of the 'switch' could correspond to the rat determining the total period of time that a light is left on, or it could correspond to the act of counting one food pellet. In general, the mechanism itself behaves identically when it accumulates a continuous quantity of energy (*e.g.* a duration of time that is 200 msec), and when it counts (*e.g.* one food pellet).

Given this, the neural retooling required to use the accumulator mechanism to count might have been quite minimal. Initially, the input to the mechanism and the analysis of the output could have been restricted to quite specific procedures (*e.g.* certain sorts of time comparisons) but since the input and output are connected to conceptual systems, the animal could 'choose' to count and make numerical comparisons without significant modification of the accumulator mechanism.

It should be stressed that this is fully consistent with the hypotheses above concerning the role of activities such as foraging and object tracking in the evolution of the capacity to count. What it suggests, however, is that the capacity was already inherent to the accumulator mechanism itself; any further evolution had to do with (i) using the switch to record individual entities and (ii) analyzing the outputs as corresponding to sets of individuals, not continuous portions. And these are tasks performed primarily by the conceptual system, not the accumulator mechanism.

3. Visualization and Geometry

There are actually two questions that arise when one considers the sort of geometrical reasoning reviewed by Giaquinto. The first concerns the evolutionary pressure for the general capacity to visualize; the second is

how we evolved an understanding of the specific geometrical axioms assumed in the inferential processes that he discusses.

Visualization may play an important role in our abilities to represent objects of different shapes, different parts of objects in varying relations to each other, and, particularly, the relative positions of different objects to each other in a spatial layout. These capacities are of obvious benefit for an animal making its way through the physical world and tracking other objects through space. We illustrate this with some findings from the infant literature.

Human infants not only represent the existence of individual objects, but can also represent the trajectory of a moving object, predict its future location from its current trajectory and speed, and anticipate what its spatial relation will be to other objects (e.g. Baillargeon & DeVos, 1991). They can, furthermore, represent different objects' sizes relative to other objects, even when the objects move out of sight (e.g. Spelke, 1991).

For example, in one study (Baillargeon & DeVos, 1991) infants were shown a toy train track that passed in front of the infant and then went up a small hill. At the top of the hill was a train car, ready to go down along the track. By the flat part of the track passing in front of the infant, a Mickey Mouse doll was standing either directly on the track, or else in front of or behind the track. Just before the train car began moving, a screen was raised to occlude the infant's view of the Mickey Mouse doll and the corresponding portion of track. The train was then released, rolled down the hill along the track, went behind the screen, and then re-emerged at the other side of the screen, continuing its journey down the track. Infants showed significant surprise at this event when the Mickey Mouse had been seen to be standing directly on the track, but not when it had been standing to one or the other side of the track.

In order to react in this manner, infants had to be representing the continued existence of the Mickey Mouse doll and its precise location; they had to be able to anticipate the train's future pathway through space, even when it went out of sight. They had to notice that a portion of the future path of the train coincided with the location of the occluded doll, and expect that this should cause some change in the train's speed or trajectory.

This finding suggests that infants have an impressive ability to represent changing spatial relationships between different objects in dynamic situations, a capacity of obvious evolutionary benefit. The underpinnings of this ability are likely to lie in the possession of visualization processes and thus we could speculate that the capacity to visualize has evolved to perform these functions.

But why would this capacity provide us with geometric axioms? Put differently, how does our ability to visualize actual physical objects and their trajectories relate to unlearned axioms concerning properties of a square—an ideal figure that has no true physical instantiation?

One solution is based on two theories of vision and visualization. The

first is that the two processes are intimately related; there is extensive evidence that the neural and computational mechanisms used in mental imagery are those normally dedicated to visual perception (*e.g.* Farah, 1985). As such, visualization is constrained by the representations and processes that apply at some intermediate level of perception. Second, it is often argued that the procedures involved in the recognition of objects involve some sort of componential process; in particular, our mental representations of objects are composed of primitives that interact in certain fixed ways (*e.g.* Biederman, 1987; Marr & Nishihara, 1978). One proposal about the nature of these primitives is that they are simple, volumetric forms, such as cones, wedges, cylinder, and bricks (dubbed 'geons' by Biederman) that are themselves derived from features such as straight and curved lines.

The explanation offered by Wynn (this issue) for the psychological axioms of arithmetic can thus find a parallel in the visual domain. Axioms of arithmetic are posited to involve relations inherent in the output of the accumulator mechanism. For instance, when the output corresponds to two entities, it must contain as a proper subpart the output corresponding to one entity twice over, and thus $1 + 1 = 2$ is construed as psychologically necessary. Similar subset relations might exist within the primitives of the visual system. If there are two primitives X and Y, and if X is a proper subpart of Y then the fact 'Y contains X' might thus be a psychological axiom of geometry. More subtle aspects of the interrelation between such primitives might explain the axioms inherent in Giaquinto's example.

4. Conclusion

The approach taken here is that the arithmetical and geometrical psychological axioms, although innate, are not themselves the direct object of natural selection; instead they are by-products of more general adaptations. In the case of arithmetic, these are the accumulator mechanism and the capacity to think about individual entities; in the case of geometry, these are the capacity to visualize combined with the combinatorial nature of object representations.

One question that immediately arises is how these sets of self-evident psychological axioms relate to their respective formal systems. How, for example, does our initial numerical understanding bear on the abstract system of arithmetic that has been developed over the course of thousands of years? How does our intuitive acceptance of certain geometric principles relate to different systems of geometry? These psychological axioms of geometry cannot exactly coincide with the axioms of all formal systems; for instance, the axioms of Euclidean and non-Euclidean geometries conflict. But is not implausible that there may be *some* correspondence between our innate geometrical axioms and those of Euclidean geometry, or between our innate numerical knowledge and the formal axioms of arith-

metic. We refer the reader to Kitcher (1984) and Maddy (1990) for some discussion of these issues.

If the above explanation of the origin of arithmetical and geometrical knowledge is correct, then the study of psychological axioms can provide useful clues to cognitive structure. We can work backwards and attempt to use what people intuitively view as necessary truths as a way to study the nature of the accumulator mechanism and the computational mechanisms of vision. It should be noted that exactly this research program has been carried out by Macnamara (1986) in the domain of deductive logic, where intuitions of logical necessity are applied to develop an account of the representations underlying language learning and reasoning. It might be revealing to extend this research program to the domains of number and geometry.

A different issue concerns the psychological processes that act upon these axioms. For instance, how is it that we are conscious of these axioms? How can we learn the language to express them? And most notably, how is it that these axioms can play a role in the discovery of genuinely new knowledge? Giaquinto's article is one of the few careful attempts to address this final question; as such, it is an important and stimulating contribution.

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