All answers should be given with proof. Proofs should be written in complete sentences and include justifications of each step. The word show is synonymous with prove. This assignment has five problems and two pages.

(1) Transposes. Let $T : V \to W$ be a linear map between finite-dimensional vector spaces.
   (a) Show that $T$ is injective if and only if $T^*$ is surjective.
   (b) Show that $T$ is surjective if and only if $T^*$ is injective.

(2) Pairings. Let $V$ and $W$ be vector spaces. A pairing of $V$ with $W$ is a function $f : V \times W \to \mathbb{F}$ that is bilinear, i.e., linear in each variable separately. We say that $f$ is left non-degenerate if $f(v, w) = 0, \forall w \in W \iff v = 0$.
   (a) Show that the assignment $v \mapsto f_v$ defines a linear map $V \to W^*$, where $f_v(w) = f(v, w)$.
   (b) Show that, if $f$ is left non-degenerate, then the map defined in (2a) is injective.
   (c) Formulate a definition of right non-degenerate, state the analogues of (2a) and (2b), and explain why they are true.

Let $\sigma : \mathbb{F} \to \mathbb{F}$ be a field automorphism. Given a vector space $V$, we obtain a new vector space $V^\sigma = \{v^\sigma : v \in V\}$ by declaring that $v_1^\sigma + v_2^\sigma = (v_1 + v_2)^\sigma$ and $av^\sigma = (\sigma(a)v)^\sigma$.

A $\sigma$-sesquilinear pairing is a pairing of $V$ with $V^\sigma$.

(d) Assume that $\mathbb{F} \subseteq \mathbb{C}$. Show that an inner product on $V$ induces a left and right non-degenerate $\sigma$-sesquilinear pairing on $V$ for a certain field automorphism $\sigma$.

(3) Double dual. Let $V$ be a vector space, and define $V^{**} = (V^*)^*$.
   (a) Define a linear map $V \to V^{**}$, and show that it is injective. This map should be canonical, i.e., it should involve no choices.
   (b) Show that, if $V$ is finite-dimensional, then the map defined in (3a) is an isomorphism. In fact, the converse to (3b) is also true.

(4) Isomorphism with the dual.
   (a) Show that, if $V$ and $W$ are finite-dimensional and $f : V \times W \to \mathbb{F}$ is a left and right non-degenerate pairing, then the induced map $V \to W^*$ is an isomorphism (hint: compare dimensions using (1), (2b), and (3b)).
   (b) Show that the function $f : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ defined by $f((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \sum_{i=1}^{n} a_i b_i$

is a left and right non-degenerate pairing.
(c) Use (4b) to show that any finite-dimensional vector space $V$ is isomorphic to its dual (non-canonically!).

(5) Dual bases. By the previous problem, given a left and right non-degenerate pairing on a finite-dimensional vector space, a choice of basis gives rise to a dual basis for the dual space. For simplicity, we will work with $\mathbb{F}^n$, the standard pairing of (4b), and the standard basis.

(a) Show that the dual basis to the standard basis is the set of functionals $\{\varphi_1, \ldots, \varphi_n\}$ given by

$$\varphi_i(e_j) = \delta_{ij},$$

where $e_j$ is the $j$th standard basis vector. Conclude that $\varphi_i((a_1, \ldots, a_n)) = a_i$.

(b) Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation given by the matrix $A \in M_{m,n}(\mathbb{F})$. Show that the matrix of $T^*$ with respect to the dual standard bases is the transpose matrix $A^T$ defined by $(A^T)_{ij} = A_{ji}$. 