Supplementary Material to the Paper: "Elimination Contests with Collusive Team Players"

Bo Chen and Shanlin Jin

December 30, 2020

We provide details that are omitted from the Appendix of the paper due to space limit.

Proof of Proposition 3.

Denote $\lambda_{31} := b_3/b_1, \lambda_{42} := b_4/b_2$, by dividing the FOCs of $1, 3$ and $2, 4$, we have

$$
\begin{aligned}
\frac{2-r_2}{4} &= \lambda_{31}\left(\frac{1}{1+{\lambda_{42}}^{r_1}}\frac{2+r_2}{4}+\frac{{\lambda_{42}}^{r_1}}{1+{\lambda_{42}}^{r_1}}\frac{2-r_2}{4}\right),\\
\frac{2-r_2}{4} &= \lambda_{42}\left(\frac{1}{1+{\lambda_{31}}^{r_1}}\frac{2+r_2}{4}+\frac{{\lambda_{31}}^{r_1}}{1+{\lambda_{31}}^{r_1}}\frac{2-r_2}{4}\right),
\end{aligned}
$$

which is equivalent to

$$
\lambda_{31}=\frac{\lambda_{42}^{r_1}+1}{\lambda_{42}^{r_1}+\frac{2+r_2}{2-r_2}},\quad \lambda_{42}=\frac{\lambda_{31}^{r_1}+1}{\lambda_{31}^{r_1}+\frac{2+r_2}{2-r_2}}.
$$

By the property of the two sides of $(1-\lambda)(1+\lambda^{r_1})=\frac{2r_2}{2-r_2}\lambda$, $\lambda_{31}<\lambda_{42}$ together with the two equalities implies $\lambda_{31}<\lambda<\lambda_{42}$, which leads to the contradiction.

For $\lambda_{31}=\lambda_{42}=\lambda$, the equilibrium bids can be obtained by multiplying both sides of the FOCs of players by b_j and $b_i=\frac{b_j}{\lambda}$:

$$
\begin{aligned}
b_i &= \frac{r_1\lambda^{r_1-1}}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v,\\
b_j &= \frac{r_1\lambda^{r_1}}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v.
\end{aligned}
$$

The expected payoffs are

$$
\begin{aligned}
U_j &= \frac{\lambda^{r_1}(\lambda^{r_1}+1-r_1)}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v, \\
U_T &= \frac{1}{(\lambda^{r_1}+1)^2}v + \frac{2\lambda^{r_1}}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v - \frac{r_1\lambda^{r_1-1}}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{2}v \\
&= \frac{2+(2-r_2)\lambda^{r_1} - r_1(2-r_2)\lambda^{r_1-1}}{2(\lambda^{r_1}+1)^2}v.
\end{aligned}
$$

The expected revenue of the sponsor is

$$
\begin{aligned}
\Pi_D &= 2b_P^F + 2b_T^F + \left(1 - \frac{1}{(1+\lambda^{r_1})^2}\right)\frac{r_2}{2}v \\
&= \frac{r_2}{2}v + \frac{2r_1\lambda^{r_1-1}(1+\lambda)}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v - \frac{1}{(1+\lambda^{r_1})^2}\frac{r_2}{2}v \\
&= \frac{r_2}{2}v + \frac{r_1(2-r_2)\lambda^{r_1-1}(1+\lambda)-r_2}{2(\lambda^{r_1}+1)^2}v.
\end{aligned}
$$

Next, we demonstrate some details in the steps of verifying optimality of the agents' bids.

Step 1. The comparison is trivial as $v = \frac{r_2}{2}v + 2\frac{2-r_2}{4}v$.

Step 2. We proof that $b_1 \neq b_2$ satisfying the FOCs of the agents was a deviation as local minimum. When $r_1 \leq 1$, the FOCs of b_i are monotonically decreasing in b_i, that there's a unique solution $b_1 = b_2$ thus no other candidates.

When $r_1 > 1$, we define $B_i(b_i)$ by the FOCs of the agents as follows:

$$
\begin{aligned}
\frac{1}{v} &= \frac{r_1 {b_j^F}^{r_1} {b_1}^{r_1-1}}{({b_1}^{r_1}+{b_j}^{r_1})^2}\frac{{b_2}^{r_1}}{{b_2}^{r_1}+{b_j}^{r_1}}\frac{2+r_2}{4} \\
&+ \frac{r_1 {b_j}^{r_1} {b_1}^{r_1-1}}{({b_1}^{r_1}+{b_j}^{r_1})^2}\frac{{b_j}^{r_1}}{{b_2}^{r_1}+{b_j}^{r_1}}\frac{2-r_2}{4} =: B_1(b_1),
\end{aligned}
$$

and $B_2(\cdot)$ is defined symmetrically.

Then we can rewrite the two FOCs together

$$
\begin{aligned}
\frac{B_1(b_1)}{K(b_1,b_2)} &= \frac{{b_1}^{r_1-1}{b_2}^{r_1}}{{b_1}^{r_1}+{b_j}^{r_1}}(2+r_2) + \frac{{b_1}^{r_1-1}{b_j}^{r_1}}{{b_1}^{r_1}+{b_j}^{r_1}}(2-r_2) \\
&= \frac{{b_2}^{r_1-1}{b_1}^{r_1}}{{b_2}^{r_1}+{b_j}^{r_1}}(2+r_2) + \frac{{b_2}^{r_1-1}{b_j}^{r_1}}{{b_2}^{r_1}+{b_j}^{r_1}}(2-r_2) = \frac{B_2(b_2)}{K(b_1,b_2)}. \qquad (1)
\end{aligned}
$$

with

$$K(b_1,b_2):=\frac{r_1{b_j}^{r_1}}{({b_1}^{r_1}+{b_j}^{r_1})({b_2}^{r_1}+{b_j}^{r_1})}$$

The different part from the $r_1 \leq 1$ case is that when $r_1 > 1$, $\frac{B_1(b_1)}{K(b_1,b_2)}$ is not monotonic in b_1, while $\frac{B_2(b_2)}{K(b_1,b_2)}$ is still strictly increasing in b_1. As

$$\begin{aligned}\frac{\partial \frac{{b_1}^{r_1-1}}{{b_1}^{r_1}+{b_j}^{r_1}}}{\partial b_1} &= (r_1-1)\frac{{b_1}^{r_1-2}}{{b_1}^{r_1}+{b_j}^{r_1}}-\frac{r_1{b_1}^{2r_1-2}}{({b_1}^{r_1}+{b_j}^{r_1})^2}\\ &= \frac{{b_1}^{r_1-2}((r_1-1){b_j}^{r_1}-{b_1}^{r_1})}{({b_1}^{r_1}+{b_j}^{r_1})^2},\end{aligned}$$

both terms in $\frac{B_1(b_1)}{K(b_1,b_2)}$ is strictly increasing in b_1 on $[0,(\frac{1}{r_1-1})^{\frac{1}{r_1}}b_j]$. Note that $b_1=b_2$ is still a solution, but there could be another (b_1',b_2') satisfying (1), and which one is the optimal for the team's payoff $u_T(b_1,b_2)$ is not intuitive. We check $\frac{\partial u_T(b_1,b_2^*)}{\partial b_1}=vB_1(b_1)-1$ directly

$$\begin{aligned}B_1'(b_1) &= \frac{r_1{b_j}^{r_1}}{{b_2}^{r_1}+{b_j}^{r_1}}\left({b_2}^{r_1}\frac{2+r_2}{4}+{b_j}^{r_1}\frac{2-r_2}{4}\right)\frac{\partial\frac{{b_1}^{r_1-1}}{({b_1}^{r_1}+{b_j}^{r_1})^2}}{\partial b_1},\\ \frac{\partial\frac{{b_1}^{r_1-1}}{({b_1}^{r_1}+{b_j}^{r_1})^2}}{\partial b_1} &= \frac{(r_1-1){b_1}^{r_1-2}}{({b_1}^{r_1}+{b_j}^{r_1})^2}-\frac{2r_1{b_1}^{2r_1-2}}{({b_1}^{r_1}+{b_j}^{r_1})^3}\\ &= \frac{{b_1}^{r_1-2}}{({b_1}^{r_1}+{b_j}^{r_1})^3}\left((r_1-1){b_j}^{r_1}-(r_1+1){b_1}^{r_1}\right)\end{aligned}$$

Thus $B_1(b_1)$ is strictly increasing on $\left(0,\left(\frac{r_1-1}{r_1+1}\right)^{\frac{1}{r_1}}b_j\right)$, and strictly decreasing on $\left(\left(\frac{r_1-1}{r_1+1}\right)^{\frac{1}{r_1}}b_j,\infty\right)$. As the equilibrium bid $b_i > b_j > \left(\frac{r_1-1}{r_1+1}\right)^{\frac{1}{r_1}}b_j$, $B_1\left(\left(\frac{r_1-1}{r_1+1}\right)^{\frac{1}{r_1}}b_j\right) > \frac{1}{v}$ that if the equation $B_1(b_1)=\frac{1}{v}$ has another solution b_1^1, then $b_1^1 < \left(\frac{r_1-1}{r_1+1}\right)^{\frac{1}{r_1}}b_j$ is a local minimum.

Step 3. With $b_1=b_2$, the agents' FOC is

$$\frac{1}{v}=\frac{r_1{b_j}^{r_1}{b_T^F}^{2r_1-1}}{({b_T^F}^{r_1}+{b_j}^{r_1})^3}\frac{2+r_2}{4}+\frac{r_1{b_j}^{2r_1}{b_T^F}^{r_1-1}}{({b_T^F}^{r_1}+{b_j}^{r_1})^3}\frac{2-r_2}{4}$$

Then the SOC of the agents is

$$\begin{aligned}
& \frac{r_1 {b_j}^{r_1} b^{r_1-2}}{(b^{r_1}+{b_j}^{r_1})^4}\{(2+r_2)[(2r_1-1){b_j}^{r_1}b^{r_1}-(1+r_1)b^{2r_1}] \\
+ & (2-r_2)[(r_1-1){b_j}^{2r_1}b^{r_1}-(1+2r_1)b^{r_1}{b_j}^{r_1}]\} \\
= & \frac{r_1 {b_j}^{r_1} b^{r_1-2}}{(b^{r_1}+{b_j}^{r_1})^4}\{(2+r_2)[(2r_1-1)yx-(1+r_1)x^2] \\
+ & (2-r_2)[(r_1-1)y^2x-(1+2r_1)xy]\} \\
\propto & \underbrace{(2+r_2)(2r_1-1)yx}_{①}-\underbrace{(2+r_2)(1+r_1)x^2}_{②} \\
+ & \underbrace{(2-r_2)(r_1-1)y^2x}_{③}-\underbrace{(2-r_2)(1+2r_1)xy}_{④} =: f(x,y),
\end{aligned}$$

where $x := b^{r_1}, y := {b_j}^{r_1}$ and the can be simplified to the quadratic function of x as

$$q(x) := (2+r_2)\left(-(r_1+1)x^2+\frac{4(r_1r_2-1)}{2+r_2}yx-\frac{(2-r_2)(1-r_1)}{2+r_2}y^2\right),$$

the peak of which is at $\frac{2(r_1r_2-1)}{(1+r_1)(2+r_2)}y$. Observe that $\frac{2(r_1r_2-1)}{(1+r_1)(2+r_2)}y < \frac{y}{2}$ as $3r_1r_2-r_2-2r_1-6<0$ by $r_2<2$.

However, the expression before simplification is more useful for most conditions as we can derive some sufficient conditions which are intuitively holds.

Notice that $f(x,y) = ① - ② + ③ - ④$, where ②,④ are positive, and the signs of ①,③ depends on $2r_1-1, r_1-1$, respectively. Hence, we can derive several sufficient conditions under which even positive ①,③ will not be large enough to let $f(x,y)\geq 0$.

Case 1. When $r_1 \leq 0.5$, $2r_1-1\leq 0, r_1-1<0$, so $f(x,y)<0$ for non-negative x,y. Thus the utility of the agents is concave in b and the bid satisfying the FOC is optimal.

Case 2. When $r_1 \geq 1$, we refer to the simplified quadratic function $q(x)$. Now $q(0) = -(2-r_2)(1-r_1)y^2 > 0$, that the FOC is always first increasing and then decreasing on $(0,1)$. Hence, the utility of the team has the similar shape as in the benchmark. We show that the solution characterized in the Proposition is actually not the first solution, i.e. local minimum by verifying the sign of FOC:

$$\begin{aligned}
q(x) < q(y) &= -(2+r_2)(r_1+1)y^2+4(r_1r_2-1)y^2-(2-r_2)(1-r_1)y^2 \\
&= 2(r_1r_2-4)y^2 < 0,
\end{aligned}$$

where the first inequality was by $x>y$ from $\lambda<1$, and the property of the quadratic function with $q(0) = -(2-r_2)(1-r_1)y^2 > 0$. Hence the FOC is positive below b_i and negative above b_i in the neighborhood, that b_i is a local maximum. Its also the global maximum as the utility has a single peak.

Case 3. Now the only case left is $r_1 \in (0.5, 1)$. As $q(0) = -(2-r_2)(1-r_1)y^2 < 0$, when $r_1 r_2 \leq 1$, $q(x)$ peak at $\frac{2(r_1r_2-1)}{(1+r_1)(2+r_2)}y \leq 0$ that $q(x) < q(0) < 0$ for all $x > 0$. Hence when $r_1 r_2 \leq 1$, the utility of the team is also concave in b.

When $r_1 r_2 > 1$, there may be 0 or 2 solutions of $SOC = 0$ on (0,y), due to $q(0) < 0$, $\frac{2(r_1r_2-1)}{(1+r_1)(2+r_2)}y \leq 0$ and $\frac{2(r_1r_2-1)}{(1+r_1)(2+r_2)}y < 0.5y < y$. If $SOC = 0$ has 0 solution, the utility is still concave. If it has 2 solutions b_x, b_y, the shape of the FOC is complicated: first decreasing until b_x, increasing on (b_x, b_y) and decreasing on $(b_y, 1)$. We claim that $FOC > 0$ for all $b \in (0, b_y)$, which suffices to show that its positive at the local minimum b_x.[1]

The intuition is that b_x is close to zero that $b_x^{r_1-1}$ is large and leads to $FOC > 0$, while the rigorous proof is technical. Denote the corresponding x at b_x as x_0. A sufficient condition for $FOC > 0$ at b_x is to show that the second term in the FOC along is already greater than 1

$$\frac{y^2 x_0 (2-r_2)}{(x_0+y)^3}\frac{r_1(2-r_2)}{4} > \frac{x_0^{\frac{1}{r_1}}}{v} \Leftrightarrow \frac{y^2 x_0 (2-r_2)}{(x+y)^3}\frac{r_1(2-r_2)}{4} > x_0^{\frac{1}{r_1}}, \tag{2}$$

where we abuse the notation to denote the corresponding x_0, y with $v = 1$ to get rid of v. This is valid as both $\frac{y^2 x}{(x+y)^3}$ and $\frac{x_0^{\frac{1}{r_1}}}{v} = \frac{b_x}{v}$ is invariant of v.

Observe that on the LHS of (2), $\frac{x}{(x+y)^3}$ is increasing and concave on $(0, \frac{y}{2})$ and on the RHS, $x^{\frac{1}{r_1}} < x$ by $x < 1$ and $r_1 < 1$.[2] On $(0, \frac{y}{2})$, as $\frac{x}{(x+y)^3}$ in concave and x is linear, for (2) we only need to show that (2) holds at $x = \frac{y}{2}$

$$\begin{aligned} &\frac{8}{27}\frac{1}{2}\frac{r_1(2-r_2)}{4} > \frac{r_1\lambda^{r_1}}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}\frac{1}{2^{\frac{1}{r_1}}} \\ \Leftrightarrow\ &\frac{4\cdot 2^{\frac{1}{r_1}}}{27} > \frac{\lambda^{r_1}}{(\lambda^{r_1}+1)^2}. \end{aligned}$$

As $r_1 < 1$, $2^{\frac{1}{r_1}} > 2$. As $\frac{s}{(1+s)^2}$ is increasing in s on $(0,1)$, $\frac{\lambda^{r_1}}{(\lambda^{r_1}+1)^2} < \frac{1}{(1+1)^2} < \frac{8}{27} < \frac{4\cdot 2^{\frac{1}{r_1}}}{27}$, that (2) holds. Hence, $FOC > 0$ at b_x, that even when the SOC has 2 solutions, the utility of the team is still increasing and has a single peak at b_i.

In conclusion, the bids we specified are indeed optimal for all admissible r_1, r_2 □

[1] Note that the FOC tends to $+\infty$ as $x \to 0^+$ by $r_1 < 1$.

[2] Note that $x_0 < \frac{y}{2}$ since $SOC = 0$ at x_0 and $\frac{2(r_1r_2-1)}{(1+r_1)(2+r_2)}y < \frac{y}{2}$.

Proof of Proposition 4. We demonstrate the details of establishing the condition

$$2r_1(3-2r_1) < (2-r_2)(r_1+1)\lambda^{r_1} + (2+r_2), \tag{3}$$

which is sufficient for $\frac{df(r_1,r_2)}{dr_2} < 0$ when $r_1 < 1$.

For $r_1 \in (0, 0.5]$, $2r_1(3-2r_1) \le 2 < 2+r_2$ thus $\frac{df(r_1,r_2)}{dr_2} < 0$. As $2r_1(3-2r_1) < 1.5(3-1.5) = 2.25$, $\frac{df(r_1,r_2)}{dr_2} < 0$ for $r_2 \in [0.25, 2)$

Observe that for $\{(r_1,r_2)|r_1 \in (0.5,1), r_2 \in (0,0.25)\}$,

$$2r_1(3-2r_1) < (2-r_2)(r_1+1)\lambda^{r_1} \quad \Leftrightarrow \quad \frac{2r_1(3-2r_1)}{r_1+1} < (2-r_2)\lambda^{r_1}.$$

By $r_1 \in (0.5,1), r_2 \in (0,0.25)$ and λ^{r_1} is decreasing in r_1, r_2, $(2-r_2)\lambda^{r_1} > \frac{7}{4}\lambda_{(1,0.25)} \approx 1.52$. Meanwhile, as the derivative of $\frac{2r_1(3-2r_1)}{r_1+1}$ is $\frac{10}{(r_1+1)^2} - 4$, $\frac{2r_1(3-2r_1)}{r_1+1}$ is increasing on $(0.5, \frac{\sqrt{10}-2}{2})$ and decreasing on $(\frac{\sqrt{10}-2}{2}, 1)$, that at $r_1 = \frac{\sqrt{10}-2}{2}$ it takes the maximum $14 - 4\sqrt{10} \approx 1.35 < 1.52$. Thus

$$\frac{2r_1(3-2r_1)}{r_1+1} < 1.36 < 1.51 < \frac{7}{4}\lambda_{(1,0.25)} < (2-r_2)\lambda^{r_1},$$

that $\frac{df(r_1,r_2)}{dr_2} < 0$ also holds on $r_1 \in (0.5,1), r_2 \in (0,0.25)$.

Now we proof the claim $\lambda^{r_1-1} < \frac{2}{2-r_2}$. For $\lambda_0 = (\frac{2-r_2}{2})^{\frac{1}{1-r_1}}$, recall that λ is the solution of $(1-\lambda)(1+\lambda^{r_1}) = \frac{2r_2}{2-r_2}\lambda$. Equivalently,

$$\begin{aligned}
\lambda > \lambda_0 \quad &\Leftrightarrow \quad (1-\lambda_0)(1+\lambda_0^{r_1}) > \frac{2r_2}{2-r_2}\lambda_0 = r_2\lambda_0^{r_1} \\
&\Leftrightarrow \quad 1-\lambda_0 > r_2\frac{\lambda_0^{r_1}}{1+\lambda_0^{r_1}} \\
&\Leftrightarrow \quad 1-r_2-\lambda_0 + r_2\frac{1}{1+\lambda_0^{r_1}} > 0,
\end{aligned}$$

where the first equivalence was obtained by the property of $(1-\lambda)(1+\lambda^{r_1}) = \frac{2r_2}{2-r_2}\lambda$ as shown in the paper.

Consider μ as the solution of $1 - x = r_2\frac{x^{r_1}}{1+x^{r_1}}$. As $1-x$ is strictly decreasing in x and $r_2\frac{x^{r_1}}{1+x^{r_1}}$ is increasing in x, $1-\lambda_0 > r_2\frac{\lambda_0^{r_1}}{1+\lambda_0^{r_1}} \Leftrightarrow \lambda_0 < \mu$. By $1-0>0$ and $1-1 < \frac{r_2}{2}$, $\mu \in (0,1)$. Then we have $\lambda_0 < \mu$ as

$$\mu = 1 - r_2 + r_2\frac{1}{1+\mu^{r_1}} > 1 - r_2 + r_2\frac{1}{1+1} = \frac{2-r_2}{2} > (\frac{2-r_2}{2})^{\frac{1}{1-r_1}} = \lambda_0,$$

where the first inequality was by $\mu < 1, r_1 > 0$ and the last inequality is by $\frac{1}{1-r_1} > 1$.

Thus $\lambda > \lambda_0$ that the claim is valid. $\square$

Proof of Proposition 5. Here we verify the sufficient condition for $\Pi_D < \Pi_N$ on $r_1 \in (0.5, 1], r_2 \to 0, 0.25$ by brute force

$$\lambda^{r_1}(2{r_1}^2\lambda^{r_1} - (r_1 + 1)) + \frac{2r_1(3 - 2r_1) - 2}{2 - 0.25} \leq 0. \tag{4}$$

As $r_2 \to 0$, $\lambda \to 1$ that (4) is now

$$\begin{aligned}
& 1^{r_1}(2{r_1}^2 1^{r_1} - (r_1 + 1)) + \frac{2r_1(3 - 2r_1) - 2}{2 - 0.25} \leq 0 \\
\Leftrightarrow \quad & -2{r_1}^2 + 17r_1 - 15 \leq 0 \\
\Leftrightarrow \quad & -(2r_1 - 15)(r_1 - 1) \leq 0.
\end{aligned}$$

which holds as $r_1 \leq 1$.

At $r_2 = 0.25$, the proof is more complicated. As $\lambda^{r_1}(2{r_1}^2\lambda^{r_1} - (r_1 + 1)) < 0$ and λ^{r_1} is decreasing in r_1,

$$\begin{aligned}
\lambda^{r_1}(2{r_1}^2\lambda^{r_1} - (r_1 + 1)) &< {\lambda_{(1,0.25)}}^1(2{r_1}^2{\lambda_{(0.5,0.25)}}^{0.5} - (r_1 + 1)) \\
&< \frac{6}{7}(2{r_1}^2 - (r_1 + 1)),
\end{aligned}$$

as $\lambda_{(1,0.25)} \approx 0.867 > \frac{6}{7}$ and ${\lambda_{(0.5,0.25)}}^{0.5} < 1$. The subscripts $(1, 0.25)$ denotes that this λ is the value at $r_1 = 1, r_2 = 0.25$.

Then a sufficient condition for (4) holding at $r_2 = 0.25$ is

$$\begin{aligned}
& \frac{6}{7}(2{r_1}^2 - (r_1 + 1)) + \frac{2r_1(3 - 2r_1) - 2}{2 - 0.25} \leq 0 \\
\Leftrightarrow \quad & -4r_1^2 + 16r_1 - 15 \leq 0,
\end{aligned}$$

which holds as $r_1 \leq 1$ and $-4 + 16 - 15 = -3 < 0$.

In conclusion, for $r_1 \in (0.5, 1]$, (4) holds at the two ends $r_2 \to 0, 0.25$. As the LHS of (4) is a quadratic function of λ^{r_1} with $2{r_1}^2 > 0$, it holds for all $r_2 \in (0, 0.25]$. $\square$

Proof of Proposition 4 when $r_1 = r_2 = r$. The proof follows four steps: (i) $\frac{d\lambda}{dr} < 0$, (ii) $\Pi_D < \Pi_S$ on $(1, 2]$, (iii) $\Pi_D > \Pi_S$ on $(0, 0.8]$ and (iv) Π_D, Π_S has single crossing on $(0.8, 1)$. Steps (ii)–(iv) are based on an equivalent function $f(r)$ with $\Pi_D > \Pi_S \Leftrightarrow f(r) > 0$.

Step 1. As we restrict $r_1 = r_2 = r$, for $r < 2$,

$$\lambda^{r+1} - \lambda^r + \frac{2+r}{2-r}\lambda - 1 = 0 \quad \Leftrightarrow \quad (1-\lambda)(1+\lambda^r) = \frac{2r}{2-r}\lambda.$$

The derivative itself was too complicated for analysis:

$$\lambda^{r+1}\ln\lambda dr - \lambda^r \ln\lambda dr + \frac{4}{(2-r)^2}\lambda dr + (r+1)\lambda^r d\lambda - r\lambda^{r-1}d\lambda + \frac{2+r}{2-r}d\lambda = 0$$

$$\Rightarrow \quad \frac{d\lambda}{dr} = \frac{(1-\lambda)\lambda^r \ln\lambda - \frac{4\lambda}{(2-r)^2}}{((r+1)\lambda - r)\lambda^{r-1} + \frac{2+r}{2-r}}.$$

We claim that given any $r \in (0,2)$, there exists a unique solution $\lambda \in (0,1)$, around which the expression $(1-\lambda)(1+\lambda^r)$ is decreasing in λ and $\frac{2r}{2-r}\lambda$ is increasing in λ. Following this claim, $\frac{d\lambda}{dr} < 0$ is trivial: Since $(1-\lambda)(1+\lambda^r)$ is decreasing in r by $\lambda \in (0,1)$ and $\frac{2r}{2-r}\lambda$ is increasing in r, any $r' = r+\epsilon > r$ with $\epsilon \to 0$ implies $(1-\lambda)(1+\lambda^{r'}) < (1-\lambda)(1+\lambda^r) = \frac{2r}{2-r}\lambda < \frac{2r'}{2-r'}\lambda$, that the new solution $\lambda' < \lambda$.

Now we prove the claim by monotonicity. As $r \in [0,2)$, $\frac{2r}{2-r} > 0$ that $\frac{2r}{2-r}\lambda$ is increasing in λ. For $(1-\lambda)(1+\lambda^r)$,

$$\begin{aligned}
\frac{d(1-\lambda)(1+\lambda^r)}{d\lambda} &= r\lambda^{r-1} - 1 - (r+1)\lambda^r \\
&= \lambda^{r-1}(r - (r+1)\lambda) - 1, \\
\frac{d^2(1-\lambda)(1+\lambda^r)}{d\lambda^2} &= r(r-1)\lambda^{r-2} - r(r+1)\lambda^{r-1} \\
&= r\lambda^{r-2}(r - 1 - (r+1)\lambda).
\end{aligned}$$

① For $r > 1$, $\frac{d(1-\lambda)(1+\lambda^r)}{d\lambda}$ is increasing on $(0, \frac{r-1}{r+1})$ and decreasing on $(\frac{r-1}{r+1}, 1)$. As $\frac{d(1-\lambda)(1+\lambda^r)}{d\lambda}|_{\lambda=0} = -1$ and $\frac{d(1-\lambda)(1+\lambda^r)}{d\lambda}|_{\lambda=1} = -2$, $\frac{d(1-\lambda)(1+\lambda^r)}{d\lambda} < 0$ on $(0,1)$ that $(1-\lambda)(1+\lambda^r)$ is strictly decreasing in λ .

Now we only need to prove that $\exists! \lambda \in (0,1)$. Observe that there's no solution on $[1, +\infty)$ since $(1-\lambda)(1+\lambda^r) \le 0 < \frac{2r}{2-r}\lambda$. As $(1-\lambda)(1+\lambda^r)$ is strictly decreasing in λ and $\frac{2r}{2-r}\lambda$ is strictly increasing in λ, the existence and uniqueness is trivial since $(1-0)(1+0^r) = 1 > 0 = \frac{2r}{2-r}\cdot 0$ and $(1-1)(1+1^r) = 0 < \frac{2r}{2-r}\cdot 1$.

Note that when $r = 1$, $(1-\lambda)(1+\lambda^r) = 1-\lambda^2$ is decreasing on $(0,1)$, thus there also exists a unique λ.

② For $r < 1$, $r - 1 < 0$ that $\frac{d^2(1-\lambda)(1+\lambda^r)}{d\lambda^2} < 0$ on $\lambda \in (0,1)$. Then $\frac{d(1-\lambda)(1+\lambda^r)}{d\lambda}$ is decreasing in λ on $(0,1)$. Since $\frac{d(1-\lambda)(1+\lambda^r)}{d\lambda}|_{\lambda\to 0+} = +\infty$ and $\frac{d(1-\lambda)(1+\lambda^r)}{d\lambda}|_{\lambda=1} = -2 < 0$, $(1-\lambda)(1+\lambda^r)$ is concave, and increasing on $(0,\bar{\lambda})$, decreasing on $(\bar{\lambda}, 1)$ for some $\bar{\lambda}$ as the solution of $\lambda^{r-1}(r-(r+1)\lambda)=1$, which is equivalent to $(r-(r+1)\lambda)=\lambda^{1-r}$. As $1^{1-r}=1>-1=(r-(r+1)\cdot 1)$, $0^{1-r}=0<r=(r-(r+1)\cdot 0)$, and $\lambda^{1-r}>\lambda$ on $(0,1)$, $\bar{\lambda}<\frac{r}{r+2}$, which is the solution of $(r-(r+1)\lambda)=\lambda$.

Observe that at $\frac{r}{r+2}$, $(1-\lambda)(1+\lambda^r)>\frac{2r}{2-r}\lambda$ since

$$
\begin{aligned}
(1-\lambda)(1+\lambda^r) &= \frac{2}{r+2}\left(1+\left(\frac{r}{r+2}\right)^r\right) > \frac{2r}{2-r}\frac{r}{r+2} \\
\Leftrightarrow\quad 1+\left(\frac{r}{r+2}\right)^r &> \frac{r^2}{2-r},
\end{aligned}
$$

which holds as $1+\left(\frac{r}{r+2}\right)^r>1>\frac{r^2}{1}$ by $r\in(0,1)$. Then as $(1-\lambda)(1+\lambda^r)$ is increasing on $(0,\bar{\lambda})$ and decreasing on $(\bar{\lambda},1)$, $(1-\lambda)(1+\lambda^r)>\frac{2r}{2-r}\lambda$ on $(0,\frac{r}{r+2})$, that any solution of $(1-\lambda)(1+\lambda^r)=\frac{2r}{2-r}\lambda$ lies in $(\frac{r}{r+2},1)$, on which $(1-\lambda)(1+\lambda^r)$ is strictly decreasing.

Similarly, the existence and uniqueness directly follows from $(1-\lambda)(1+\lambda^r)>\frac{2r}{2-r}\lambda$ at $\frac{r}{r+2}$, and $(1-1)(1+1^r)=0<\frac{2r}{2-r}\cdot 1$.

In conclusion, ① and ② prove the claim, which implies (i).

Step 2. Recall that the expected revenue of the sponsor under different seedings are

$$
\begin{aligned}
\Pi_B &= \frac{(4-r)r}{4}v, \quad \Pi_S=\frac{(6-r)r}{8}v, \\
\Pi_D &= \frac{r}{2}v+\frac{r(2-r)\lambda^{r-1}(1+\lambda)-r}{2(\lambda^r+1)^2}v.
\end{aligned}
$$

Observe that for $r\in(0,2)$ satisfying Assumption 2,

$$
\begin{aligned}
\Pi_D>\Pi_S \quad &\Leftrightarrow\quad \frac{r}{2}v+\frac{r(2-r)\lambda^{r-1}(1+\lambda)-r}{2(\lambda^r+1)^2}v>\frac{(6-r)r}{8}v \\
&\Leftrightarrow\quad f(r)=4\lambda^{r-1}-(1-\lambda^r)^2-\frac{4}{2-r}>0.
\end{aligned}
$$

At $r=1$, the solution of $\lambda^2-\lambda+\frac{2+1}{2-1}\lambda-1=\lambda^2+2\lambda-1=0$ is $\lambda=\sqrt{2}-1$, and $\Pi_D=\frac{\sqrt{2}+1}{4}v<\frac{5}{8}=\Pi_S$, then for $\Pi_D<\Pi_S$ to hold on $(1,2)$, we only need to show that $f'(r)<0$ on $(1,2)$.

$$
\begin{aligned}
f'(r) &= -2\lambda^{2r}\ln\lambda+2\lambda^r\ln\lambda+4\lambda^{r-1}\ln\lambda \\
&+ \left(-2r\lambda^{2r-1}+2r\lambda^{r-1}+4(r-1)\lambda^{r-2}\right)\frac{d\lambda}{dr}-\frac{4}{(2-r)^2}.
\end{aligned}
$$

For $r\in(1,2)$, as $\lambda\in(0,1)$ and $\frac{d\lambda}{dr}<0$, $-2\lambda^{2r}\ln\lambda+2\lambda^r\ln\lambda<0, 4\lambda^{r-1}\ln\lambda<$

$0, -\frac{4}{(2-r)^2} < 0$, $\left(-2r\lambda^{2r-1} + 2r\lambda^{r-1}\right)\frac{d\lambda}{dr} < 0$ and $4(r-1)\lambda^{r-2}\frac{d\lambda}{dr} < 0$, thus $f'(r) < 0$, i.e. $\Pi_D < \Pi_S$ holds on $(1, 2)$.

Step 3.

$$\begin{aligned} & f(r) = 4\lambda^{r-1} - (1-\lambda^r)^2 - \frac{4}{2-r} = 0 \\ \Leftrightarrow \quad & 4\lambda^{r-1} = (1-\lambda^r)^2 + \frac{4}{2-r}. \end{aligned} \tag{5}$$

As λ^r is decreasing in r, $(1-\lambda^r)^2$ is increasing in r. As $\frac{4}{2-r}$ is also strictly increasing in r, the RHS $(1-\lambda^r)^2 + \frac{4}{2-r}$ is strictly increasing in r. Notice that $4\lambda^{r-1} \geq 4$. By $(1-\lambda^r)^2 + \frac{4}{2-r} \approx 3.48 < 4$ at $r = 0.8$, $\forall r \in (0, 0.8]$, $4\lambda^{r-1} \geq 4 > (1-\lambda^r)^2 + \frac{4}{2-r}$, i.e. $f(r) > 0$ that there's no zero point in $(0, 0.8]$.

Step 4. We show that there exists a unique zero point by showing the monotonicity of each side of equation (5). We already know the RHS is strictly increasing in r as stated in Step 3. As $4\lambda^{r-1} > (1-\lambda^r)^2 + \frac{4}{2-r}$ at $r = 0.8$ and $4 < (1-(\sqrt{2}-1)^1)^2 + \frac{4}{2-1}$ at $r = 1$, we only need to show that λ^{r-1} is strictly decreasing on $(0.8, 1)$. The first derivative is

$$\frac{d\lambda^{r-1}}{dr} = \lambda^{r-1}\ln\lambda + (r-1)\lambda^{r-2}\frac{d\lambda}{dr} = \lambda^{r-2}\left(\lambda\ln\lambda - (1-r)\frac{d\lambda}{dr}\right),$$

For general $r \in (0.8, 1)$, plug in $\frac{d\lambda}{dr}$:

$$\begin{aligned} \frac{d\lambda^{r-1}}{dr} < 0 \quad \Leftrightarrow \quad & \lambda\ln\lambda - (1-r)\frac{d\lambda}{dr} < 0 \\ \Leftrightarrow \quad & \frac{\lambda\ln\lambda}{(1-r)} < \frac{d\lambda}{dr} = \frac{(1-\lambda)\lambda^r\ln\lambda - \frac{4\lambda}{(2-r)^2}}{((r+1)\lambda - r)\lambda^{r-1} + \frac{2+r}{2-r}} \\ \Leftrightarrow \quad & -\frac{4\lambda}{(2-r)^2\ln\lambda} < \frac{\lambda}{1-r}((r+1)\lambda - r)\lambda^{r-1} + \frac{\lambda}{1-r}\frac{2+r}{2-r} - (1-\lambda)\lambda^r \\ \Leftrightarrow \quad & -\frac{4\lambda}{(2-r)^2\ln\lambda} < \frac{2}{1-r}\lambda^{r+1} - \frac{1}{1-r}\lambda^r + \frac{\lambda}{1-r}\frac{2+r}{2-r} \\ \Leftrightarrow \quad & \frac{1-r}{(2-r)^2}\frac{4\lambda}{-\ln\lambda} < \lambda^{r+1} + 1, \end{aligned}$$

where the last equivalence was by $\lambda^{r+1} - \lambda^r + \frac{2+r}{2-r}\lambda - 1 = 0$.

The RHS $\lambda^{r+1} + 1$ is decreasing in r as $r + 1 > 0$ and $\lambda \in (0, 1)$, that $\lambda^{r+1} + 1$ is greater than the value at $r = 1$, i.e. $(\sqrt{2}-1)^2 + 1 = 4 - 2\sqrt{2} \approx 1.17$. We can also show that both

$\frac{1-r}{(2-r)^2}$ and $\frac{4\lambda}{-\ln\lambda}$ are strictly decreasing in r,[3] that $\frac{1-r}{(2-r)^2}\frac{4\lambda}{-\ln\lambda}$ is strictly decreasing. Then for $r \in (0.8, 1)$ $\frac{1-r}{(2-r)^2}\frac{4\lambda}{-\ln\lambda}$ is lower than the value at $r = 0.8$, which is $\frac{5}{36}\frac{4\lambda}{-\ln\lambda} \approx 0.51$, which is much smaller than $4-2\sqrt{2}$. Hence, for $r \in (0.8, 1)$, $\frac{1-r}{(2-r)^2}\frac{4\lambda}{-\ln\lambda} < 0.51 < 4-2\sqrt{2} < \lambda^{r+1}+1$, that $\frac{d\lambda^{r-1}}{dr} < 0$.

Now we have the monotonicity of both $4\lambda^{r-1}$ and $(1-\lambda^r)^2 + \frac{4}{2-r}$ on $(0.8, 1)$. Then by $f(0.8) > 0$ and $f(1) < 0$, $f(r) = 0$ has a unique solution $\bar{r}$ in $(0.8, 1)$ with $f(r) > 0$ on $(0.8, \bar{r})$ and $f(r) < 0$ on $(\bar{r}, 1)$. As $f(r) > 0$ in $(0, 0.8)$ and $\Pi_D > \Pi_S \Leftrightarrow f(r) > 0$, Proposition 4 also holds when we exert the restriction $r_1 = r_2 = r$, with the cutoff $\bar{r} \in (0.8, 1)$. $\square$

Proof of $\lambda_1 \geq k$ and $\lambda \geq k$. As

$$\lambda_1 = \frac{k(1+k^{r_1})^2}{\frac{2+r_1}{4}k(1+k^{r_1})^2 + 1 + (1-r_1)k^{r_1}},$$

$$\begin{aligned}\lambda_1 \geq k \quad &\Leftrightarrow \quad \frac{k(1+k^{r_1})^2}{\frac{2+r_1}{4}k(1+k^{r_1})^2 + 1 + (1-r_1)k^{r_1}} \geq k \\ &\Leftrightarrow \quad h(k) := (1+r_1)k^{r_1-1} + k^{2r_1-1} - \frac{2+r_1}{4}\left(1 + 2k^{r_1} + k^{2r_1}\right) \geq 0.\end{aligned}$$

The simplest case is $r_1 = 1$, where

$$h(k) = 2 + k - \frac{3}{4}\left(1 + 2k + k^2\right) \geq 0 \quad \Leftrightarrow \quad \left(k + \frac{1}{3}\right)^2 \leq \left(\frac{4}{3}\right)^2,$$

which always holds for $k + \frac{1}{3} \in [\frac{1}{3}, \frac{4}{3}]$ by $k \in [0, 1]$.

For $r_1 \neq 1$, $h(1) = 0$,

$$\begin{aligned}h'(k) \quad &= \quad ({r_1}^2 - 1)k^{r_1-2} + (2r_1 - 1)k^{2r_1-2} - \frac{2+r_1}{4}\left(2r_1 k^{r_1-1} + 2r_1 k^{2r_1-1}\right) \\ &= \quad k^{r_1-2}\left({r_1}^2 - 1 + (2r_1-1)k^{r_1} - \frac{2+r_1}{2}r_1\left(k + k^{r_1+1}\right)\right) \\ &\propto \quad {r_1}^2 - 1 + (2r_1-1)k^{r_1} - \frac{2+r_1}{2}r_1\left(k + k^{r_1+1}\right) =: h_1(k).\end{aligned}$$

If $r_1 \leq \frac{1}{2}$, ${r_1}^2 < 1, 2r_1 - 1 \leq 0$, that $h'(k) < 0$, $h(k) \geq h(1) = 0$ on $[0, 1]$, that $\lambda_1 \geq k$ always holds.

[3] The first derivatives are $\frac{d\frac{1-r}{(2-r)^2}}{dr} = \frac{r}{(r-2)^3} < 0$ and $\frac{d\frac{4\lambda}{-\ln\lambda}}{dr} = \frac{4}{(-\ln\lambda)^2}\frac{d\lambda}{dr}(1-\ln\lambda) < 0$.

If $\frac{1}{2} < r_1 \leq 1$,

$$\begin{aligned} h(k) &= (1+r_1)k^{r_1-1} + k^{2r_1-1} - \frac{2+r_1}{4}\left(1+2k^{r_1}+k^{2r_1}\right) \\ &\geq 1 + r_1 + k^{2r_1} - \frac{2+r_1}{4} - \frac{2+r_1}{4}\left(2k^{r_1}+k^{2r_1}\right) \\ &= \frac{2+3r_1}{4} - \frac{(2+r_1)^2}{4(2-r_1)} + \frac{2-r_1}{4}\left(k^{r_1} - \frac{2+r_1}{2-r_1}\right)^2 \\ &\geq 1 + r_1 + 1 - \frac{2+r_1}{4} - 3\frac{2+r_1}{4} \\ &= 0, \end{aligned}$$

where the first inequality is by $r_1 \in (\frac{1}{2}, 1]$ and $k \in [0,1]$, and the last inequality is by $k^{r_1} \in [0,1]$ and $\frac{2+r_1}{2-r_1} \geq 1$. So $\lambda_1 \geq k$ also holds.

For λ, for all $r_2 \in (0,2)$, it always holds:

$$\lambda > k \Leftrightarrow (1-k)(1+k^{r_1}) > \frac{2r_2}{2-r_2}k \Leftrightarrow 1 + k^{r_1} > 1. \qquad \square$$

Proof of Proposition 8. We discuss why there are three candidates for the equilibrium when $r > 1$ and show that candidates (2) and (3) are suboptimal. Recall that the FOC of any player i given bids (b_j, b_k) from opponents j, k is

$$rb_i^{r-1}\frac{{b_j}^r + {b_k}^r}{({b_i}^r + {b_j}^r + {b_k}^r)^2}v = 1,$$

and the corresponding SOC is

$$-rb_i^{r-2}\frac{({b_j}^r + {b_k}^r)((r+1)b_i^r - (r-1)({b_j}^r + {b_k}^r))}{({b_i}^r + {b_j}^r + {b_k}^r)^3}v.$$

From the FOCs, we obtain

$$\lambda_{42}^{r-1} = \frac{\lambda_{32}^r + \lambda_{42}^r}{1+\lambda_{32}^r} \quad \Rightarrow \quad \lambda_{42}^r = (1+\lambda_{32}^r)\lambda_{42}^{r-1} - \lambda_{32}^r, \tag{6}$$

$$\lambda_{32}^{r-1} = \frac{\lambda_{32}^r + \lambda_{42}^r}{1+\lambda_{42}^r} \quad \Rightarrow \quad \lambda_{42}^r = \frac{\lambda_{32}^{r-1}(1-\lambda_{32})}{(1-\lambda_{32}^{r-1})} \text{ or } \lambda_{32} = 1. \tag{7}$$

Compared with the $r \leq 1$ case, the discussion of $r > 1$ is more complicated as the equation $\lambda^2 - 2\lambda + 1$ have two solutions: 1 and $\eta > 1$. If $\lambda_{32} = \lambda_{42}$, we get the candidates (1) and (2). If $\lambda_{32} = 1 < \lambda_{42}$, by (6) $\lambda_{42} = \epsilon$ which is the solution of $\lambda^r = 2\lambda^{r-1} - 1$. If

$r \in (1,2)$, the solution is smaller than 1, which violates $\lambda_{32} = 1 < \lambda_{42}$, while we have the candidate (3) by similarly considering $\lambda_{32} < 1 = \lambda_{42}$. Note that when $\lambda_{32} = \epsilon$, by (7) λ_{42} can only be 1. If $r = 2$, $\lambda_{32} = \lambda_{42} = 1$ is the unique candidate, which lead to negative payoffs which is suboptimal. If $r > 2$, we have a candidate $\lambda_{32} = 1 < \lambda_{42} = \epsilon$, but the payoff of player 3 would be $\frac{1+2(1-r)\epsilon^{r-1}}{(2+\epsilon^r)^2} < 0$ which is still suboptimal. Note that then the candidate (1) results in $u_j = \frac{3-2r}{9} < 0$, while (2) results in $u_T = \frac{2(2-r)\lambda^{r-1}-1}{(1+2\lambda^r)^2} < 0$, both of which are suboptimal. Actually, there's no pure strategy equilibrium when $r \geq 2$ using the discussion which we will go through in detail for $r \in (1,2)$ utilizing $\lambda_{32}^r < \lambda_{42}^r$ and (7):

$$\lambda_{32} < 1 \quad \Rightarrow \quad \lambda_{32}^r - 2\lambda_{32} + 1 > 0, U_3 = \frac{\lambda_{32}^r(1 + \lambda_{32}^r - r) - (r-1)\lambda_{32}^r\lambda_{42}^r}{(1 + \lambda_{32}^r + \lambda_{42}^r)^2} < 0,$$

$$\lambda_{32} > 1 \quad \Rightarrow \quad \lambda_{32} \in (1, \eta), U_T = \frac{1 + \lambda_{32}^r + \lambda_{42}^r - r\lambda_{32}^r - r\lambda_{42}^r}{(1 + \lambda_{32}^r + \lambda_{42}^r)^2} < 0.$$

For $r \in (1,2)$, we show that there's no candidate with $\lambda_{32} < \lambda_{42}$ satisfying $\lambda_{32}, \lambda_{42} \notin \{1, \eta\}$. As $r > 0$, $\lambda_{32} < \lambda_{42} \Rightarrow \lambda_{32}^r < \lambda_{42}^r$, then using (7),

$$\lambda_{42}^r = \frac{\lambda_{32}^{r-1}(1 - \lambda_{32})}{(1 - \lambda_{32}^{r-1})} > \lambda_{32}^r \quad \Rightarrow \quad \frac{1 - \lambda_{32}}{1 - \lambda_{32}^{r-1}} > \lambda_{32}, \tag{8}$$

and symmetrically

$$\lambda_{32}^r = \frac{\lambda_{42}^{r-1}(1 - \lambda_{42})}{(1 - \lambda_{42}^{r-1})} < \lambda_{42}^r \quad \Rightarrow \quad \frac{1 - \lambda_{42}}{1 - \lambda_{42}^{r-1}} < \lambda_{42}. \tag{9}$$

Note that the λ^r and $2\lambda - 1$ cut the $\mathbb{R}_{++}$ into three regions: $(0,1)$ and $(\eta, +\infty)$ in which $\lambda^r > 2\lambda - 1$, and $(1, \eta)$ in which $\lambda^r < 2\lambda - 1$. We use this property to rule out cases of $\lambda_3 2, \lambda_4 2$ violating (8) or (9).

If $\lambda_{32} < 1$, by $r \in (1,2)$, $1 - \lambda_{32}^{r-1} < 0$ and $\lambda_{32}^r - 2\lambda_{32} + 1 > 0$ holds in (8). For λ_{42}, (9) requires that either $\lambda_{42} < 1$ and $\lambda_{32}^r - 2\lambda_{32} + 1 < 0$ which is a contradiction, or $\lambda_{42} > 1$ and $\lambda_{42}^r - 2\lambda_{42} + 1 > 0$ which implies $\lambda_{42} > \eta$.

Similarly, if $\lambda_{32} > 1$, we obtain $1 < \lambda_{32} < \eta < \lambda_{42}$.

However, $\lambda_{42} > \eta > 1$ result in $U_T < 0$ as $1 + \lambda_{42}^r - r\lambda_{42}^r < 0$: given any $\lambda_{32} > 0$, in (6), $\lambda_{42}^r = (1 + \lambda_{32}^r)\lambda_{42}^{r-1} - \lambda_{32}^r$ has two solutions 1 and t, the relative size of which depends on the first derivatives at 1:

$$\lambda_{42} > 1 \quad \Leftrightarrow \quad r\lambda_{42}^{r-1}\Big|_{\lambda_{42}=1} < (r-1)(1+\lambda_{32}^r)\lambda_{42}^{r-2}\Big|_{\lambda_{42}=1}$$
$$\Leftrightarrow \quad \lambda_{32} > \left(\frac{r}{r-1}\right)^{\frac{1}{r}}.$$

As $\lambda_{32} < \lambda_{42}$, $\lambda_{42} > \left(\frac{r}{r-1}\right)^{\frac{1}{r}} > \left(\frac{1}{r-1}\right)^{\frac{1}{r}}$, i.e. $1 + \lambda_{42}^r - r\lambda_{42}^r < 0$.

Hence, (1)-(3) are the only candidates. For (2) where $\lambda_{32} = \lambda_{42} = \eta$, the expected payoff of the team is negative as

$$U_T = \frac{1+2\eta^r - 2r\eta^r}{(1+\lambda_{32}^r+\lambda_{42}^r)^2} < 0 \quad \Leftrightarrow \quad 1+2\eta^r - 2r\eta^r < 0$$
$$\Leftrightarrow \quad \eta > \left(\frac{1}{2r-2}\right)^{\frac{1}{r}}$$
$$\Leftrightarrow \quad \frac{1}{2r-2} < 2\left(\frac{1}{2r-2}\right)^{\frac{1}{r}} - 1$$
$$\Leftrightarrow \quad r - \frac{1}{2} < (2r-2)^{1-\frac{1}{r}},$$

which holds for all $r \in (1, 1.5)$. For $r \in (1.5, 2)$, we use $\eta^r - 2\eta + 1 = 0$ to get another equivalent condition $\eta > \frac{2r-1}{4(r-1)}$ which is trivial as $\eta > 1$ and $2r - 1 < 4(r-1) \Leftrightarrow 3 < 2r$. $r - \frac{1}{2} < (2r-2)^{1-\frac{1}{r}}$ for $r \in (1, 1.5)$ is trickier to proof, but one can verify that this inequality is binding at $r = 1.5$, and $(2r-2)^{1-\frac{1}{r}}$ is first decreasing on $(1, r_t)$ and then increasing at a rate smaller than 1 on $(r_t, 1.5)$, where r_t solves $r + \ln(2r-2) = 0$. Thus this inequality holds on $(1, 1.5)$. In conclusion, for $r \in (1, 2)$ candidate (2) results in $U_T < 0$, thus suboptimal.

For candidate (3), $U_3 < 0 \Leftrightarrow \epsilon < (\frac{r-1}{2-r})^{\frac{1}{r}}$ is still trivial for $r \in (1.5, 2)$ as $r - 1 > 2 - r \Leftrightarrow 2r > 3$. For $r \in (1, 1.5)$, one can verify that the SOC of player 3 is positive thus the bid is suboptimal

$$-r\frac{\epsilon^r}{b_3^2}\frac{2((r+1)\epsilon^r - 2(r-1))}{(\epsilon^r+2)^3}v > 0 \quad \Leftrightarrow \quad \epsilon^r < \frac{2(r-1)}{r+1}$$
$$\Leftrightarrow \quad \frac{2(r-1)}{r+1} < 2\left(\frac{2(r-1)}{r+1}\right)^{\frac{r-1}{r}} - 1,$$

which is more complicated, but still has a familiar shape: the RHS is first decreasing and then increasing at a rate smaller than the LHS while $RHS = 0.47 > 0.4 = LHS$ at $r = \frac{3}{2}$.

Lastly, candidate (1) $b_2 = b_3 = b_4 = b = \frac{2r}{9}v$ is optimal, as the SOC is negative

$$-rb^{r-2}\frac{(b^r + b^r)b^r((r+1) - 2(r-1))}{9b^{2r}}v < 0,$$

by $(r+1) > 2(r-1) \Leftrightarrow r < 3$. And non-negative payoff requires $\frac{3-2r}{9} \geq 0 \Leftrightarrow r \in (0, \frac{3}{2}]$.

Now we demonstrate the details of result (ii) on $r \in (0,1)$ following three steps:

1. Π_C and Π_S has a single crossing r^{**} s.t. $\Pi_C < \Pi_S$.

2. $r^{**} < 0.8 < \bar{r}$ that $\Pi_C < \Pi_S < \Pi_D$ on $(0, r^{**})$ and seeding $\{(1,2),(3,4)\}$ is not optimal on $(r^{**}, 1)$.

3. $\frac{\Pi_D}{rv}$ is decreasing and $\frac{\Pi_C}{rv}$ is increasing. Therefore $\exists r^* \in (r^{**}, \bar{r})$ such that for $r \in (r^{**}, \bar{r})$, $\frac{\Pi_D}{rv} > \frac{\Pi_C}{rv} \Leftrightarrow r \in (r^{**}, r^*)$.

Recall

$$\Pi_S = \frac{(6-r)rv}{8},\ \Pi_D = \frac{r}{2}v + \frac{r(2-r)\lambda^{r-1}(1+\lambda) - r}{2(\lambda^r+1)^2}v,\ \Pi_C = \begin{cases} \dfrac{r\xi^r(1+\xi)}{(1+\xi^r)^2}v, & \text{if } r < 1 \\ \dfrac{2rv}{3}, & \text{if } r \geq 1 \end{cases}.$$

For $r \in (0,1)$, from ξ solves $\lambda^r = 2\lambda - 2$ we obtain $\xi \in (1.5, 2)$ and the derivatives as

$$\frac{d\xi}{dr} = \frac{\xi^r \ln \xi}{2 - r\xi^{r-1}} > 0.$$

Note that $\frac{d^2\xi}{dr^2} > 0$ since the numerator of $\frac{d\xi}{dr}$ is increasing in r and the denominator is decreasing in r.

Step 1. For $r \in (0,1)$, with $\xi^r = 2\xi - 2$,

$$\begin{aligned} \Pi_C > \Pi_S \quad &\Leftrightarrow \quad \frac{\xi^r(1+\xi)}{(1+\xi^r)^2} > \frac{6-r}{8} \\ &\Leftrightarrow \quad \frac{2(\xi^2-1)}{(2\xi-1)^2} > \frac{6-r}{8} \\ &\Leftrightarrow \quad m(\xi) := -4(2-r)\xi^2 + 4(6-r)\xi - 22 + r > 0. \end{aligned}$$

We obtain the single crossing of Π_C and Π_S by verifying the monotonicity of $m(\xi)$

$$\frac{dm(\xi)}{dr} \quad = \quad 4((6-r) - 2(2-r)\xi)\frac{d\xi}{dr} + (2\xi - 1)^2.$$

As $\frac{d\xi}{dr} > 0$ and $(2\xi - 1)^2 > 0$, a sufficient condition for $\frac{dm(\xi)}{dr} > 0$ is

$$\begin{aligned}(6-r) - 2(2-r)\xi \geq 0 \;&\Leftrightarrow\; \xi \leq \frac{6-r}{4-2r} \\ &\Leftarrow\; \xi \leq \frac{3+r}{2} < \frac{6-r}{4-2r}.\end{aligned}$$

In the last inequality, $\xi \leq \frac{3+r}{2}$ holds since ξ is increasing and convex, and $\xi = 2 = \frac{3+1}{2}$ at $r = 1$. The other side also holds as $\frac{3+r}{2} < \frac{6-r}{4-2r} \Leftrightarrow 12 - 2r > 12 - 2r - 2r^2$. Hence, $\frac{dm(\xi)}{dr} > 0$ and the single crossing with cutoff $r^{**} < 0.8$ is obtained by $m(1.5) = -4(2 - 0)1.5^2 + 4(6-0)1.5 - 22 + 0 = -4 < 0$ and $m(\xi)|_{r=0.8} \approx 0.69 > 0$.

Step 2. As we showed $0.8 < \bar{r}$ in the Proof of Proposition 4 when $r_1 = r_2 = r$, we have $r^{**} < 0.8 < \bar{r}$ that $\Pi_C < \Pi_S < \Pi_D$ on $(0, r_C)$ and seeding $\{(1,2),(3,4)\}$ is not optimal on $(r_C, 1)$. By Steps 1 and 2, for result (ii) on $r \in (0,1)$ we only need to show that Π_C and Π_D has a single crossing r^* on $(r^{**}, 1)$. Note that numerical simulation suggests $r^* < \bar{r}$, which is not essential as $\Pi_C > \max\{\Pi_S, \Pi_D\}$ on $(r^*, 1)$ by the single crossing proved in Step 1.

Step 3. Dividing the revenue of the sponsor by rv yields

$$\begin{aligned}\frac{\Pi_D}{rv} &= \frac{1}{2} + \frac{(2-r)\lambda^{r-1}(1+\lambda) - 1}{2(\lambda^r + 1)^2}, \\ \frac{\Pi_C}{rv} &= \frac{\xi^r(1+\xi)}{(1+\xi^r)^2} = \frac{2(\xi^2 - 1)}{(2\xi - 1)^2}.\end{aligned}$$

It is easier to verify that $\frac{\Pi_C}{rv}$ is increasing in ξ thus also increasing in r on $(0,1)$:

$$\frac{d\frac{\Pi_C}{rv}}{d\xi} = \frac{4(2-\xi)}{(2\xi-1)^3} > 0,$$

as $\xi \in (1.5, 2)$.

On the other hand, the monotonicity of $\frac{\Pi_D}{rv}$ is complicated as we cannot do a similar simplification to λ^r. The idea is that $\frac{d\Pi_D/rv}{dr} = \frac{(2-r)\lambda^{r-1}}{2(\lambda^r+1)^2}(A(\lambda)\frac{d\lambda}{dr} + B(\lambda)) < \frac{(2-r)\lambda^{r-1}}{2(\lambda^r+1)^2}(B(\lambda) - A(\lambda))$ as $\frac{d\lambda}{dr} > -1$ which can be verified by checking the second derivative, where

$$\begin{aligned}A(\lambda) &:= \frac{r-1}{\lambda} + r - 2r(1+\lambda)\frac{\lambda^{r-1}}{1+\lambda^r} + \frac{1}{1+\lambda^r}\frac{2r}{2-r}, \\ B(\lambda) &:= -\frac{1+\lambda}{2-r} + (1 + \lambda - 2(1+\lambda)\frac{\lambda^r}{1+\lambda^r} + \frac{1}{1+\lambda^r}\frac{2}{2-r}\lambda)\ln\lambda.\end{aligned}$$

Note that $\lim_{r\to 0} A(\lambda) = \lim_{r\to 0} B(\lambda) = -1$, and $A(\lambda) > -1$ as

$$\begin{aligned} A(\lambda) > -1 \quad &\Leftrightarrow \quad -(r-1)\lambda^r - (r+1)\lambda^{r-1} + \frac{r-1}{\lambda} + r + 1 + \frac{2r}{2-r} > 0 \\ &\Leftrightarrow \quad -2r\lambda^{r-1} + \frac{4r}{2-r}, \end{aligned}$$

which holds by the claim $\lambda^{r-1} < \frac{2}{2-r}$ on $r \in (0,1)$. Note that the first equivalence was by multiplying $1+\lambda^r$ on both sides and the second was by $\lambda^{r+1} - \lambda^r + \frac{2+r}{2-r}\lambda - 1 = 0$.

Meanwhile, $B(\lambda) < -1$ as both

$$\begin{aligned} -\frac{1+\lambda}{2-r} < -1 \quad &\Leftrightarrow \quad \lambda > 1-r \\ &\Leftrightarrow \quad (1-1+r)(1+(1-r)^r) > \frac{2r}{2-r}(1-r) \\ &\Leftrightarrow \quad 1+(1-r)^r > \frac{2(1-r)}{2-r} \\ &\Leftarrow \quad 1+(1-r)^r > 1 > \frac{2(1-r)}{2-r}, \end{aligned}$$

which always holds, and $1+\lambda-2(1+\lambda)\frac{\lambda^r}{1+\lambda^r} + \frac{1}{1+\lambda^r}\frac{2}{2-r}\lambda > 0$ by

$$1+\lambda-2(1+\lambda)\frac{\lambda^r}{1+\lambda^r} \quad = \quad (1+\lambda)(1-\frac{2\lambda^r}{1+\lambda^r}) > 0,$$

as $\lambda^r < 1$.

In conclusion, by $A(\lambda) > -1, B(\lambda) < -1$, we obtain $B(\lambda) - A(\lambda) < 0$, i.e. $\frac{d\frac{\Pi_D}{rv}}{dr} < 0$.

For $r \in (0,1)$, as $\frac{\Pi_D}{rv}$ is decreasing and $\frac{\Pi_C}{rv}$ is increasing, there exists a cutoff $r^* \in (0.8, 1)$ since $\frac{\Pi_D}{rv} \approx 0.55v < \frac{\Pi_C}{rv} \approx 0.66v$ at $r = 0.8$ and $\frac{\Pi_D}{rv} \approx 0.60v < \frac{\Pi_C}{rv} = \frac{2}{3}v$ at $r = 1$.

In conclusion, we have the revenue comparison result (ii). □