

A Symmetric Two-Player All-Pay Contest with Correlated Information¹

Zhiyang Liu and Bo Chen²

April 25, 2016

Abstract

We construct both monotonic and non-monotonic symmetric Bayesian Nash equilibria for a two-player all-pay contest with binary types and correlated information structures. We also employ a class of parametric distributions to illustrate our equilibrium construction explicitly and to derive some comparative statics results.

JEL classifications: C72, D44, D72, D82.

Keywords: All-Pay Auction, Correlated Types, Symmetric Equilibrium.

¹We are grateful to the editor and especially an anonymous referee for very constructive comments and advice. Remaining errors are ours.

²Zhiyang Liu: School of International Business Administration, Shanghai University of Finance and Economics, China (lzy3919@126.com); Bo Chen (Corresponding Author): Department of Economics, Southern Methodist University, USA (bochen@smu.edu).

1 Introduction

Contests, where players invest irreversible resources to win a reward, are abundant and essential in social and economic life. Contest models hence have found various applications in social sciences. Sports competitions, patent races, promotion tournaments in organizations, and election campaigns among politicians are just a few examples that can be fruitfully modeled and analyzed as contests.³

We study a perfectly discriminatory contest (all-pay auction) where two *ex ante* identical players exert effort to compete for a prize. Each player has private information about her effort efficiency, which affects her effort cost and can be either high or low. The players' effort efficiencies are drawn from a symmetric joint distribution with full support. Importantly, the players' effort efficiencies can be statistically (negatively or positively) correlated. To put the model into context, consider a promotion competition between two colleagues. The colleagues have private information about their own abilities. However, due to past interactions and collaborations, such private information may not be statistically independent.

In this contest game, we characterize the symmetric Bayesian Nash equilibrium for *all* symmetric joint distributions with binary and correlated types. We find that the symmetric equilibrium is monotonic when the players' types are mildly (positively or negatively) correlated, while the symmetric equilibrium is non-monotonic when the types are sufficiently (positively or negatively) correlated.⁴ We also use a convenient class of distributions with a parametrized correlation parameter to illustrate our equilibrium construction, the equilibrium strategies, and some comparative statics results.

Our paper is related to the literature on all-pay auctions with incomplete information. Several important papers have studied monotonic equilibria in various settings with discrete/continuous and independent/correlated types (e.g., Amann and Leininger (1996), Krishna and Morgan (1997), Konrad (2004), and Siegel (2014)). While we consider a simple model with binary and correlated types, our focus is on both monotonic and non-monotonic equilibria. Our study is hence more closely related to two recent contributions Lu and Parreiras (2014), who analyze conditions for the existence of monotonic equilibria in a general setting with correlated (continuous) types and interdependent valuations *and* non-monotonic equilibria in a setting with quadratic valuations, and Rentschler and Tur-ocy (2015) who present a useful algorithm for symmetric monotonic *and* non-monotonic equilibria in a setting with discrete signals and interdependent valuations. While our setting is admittedly simpler and more special in comparison, we completely construct symmetric monotonic and non-monotonic equilibria for *all correlation structures* in our setting, and are hence able to delineate the relationship between correlation and the equilibrium type more explicitly.

³Konrad (2009) gives a comprehensive survey of the literature on contests.

⁴Our symmetric equilibrium is always in mixed strategies. Roughly, the equilibrium is monotonic if different types of a contestant randomize on non-overlapping intervals, while the equilibrium is non-monotonic if different types of a contestant randomize on overlapping intervals.

2 Model

Consider an all-pay contest where two risk-neutral players exert effort to compete for a single prize of value $V > 0$. Prior to the competition, each player $i \in \{1, 2\}$ observes a private signal about her ability (or effort efficiency) $\theta_i \in \{H, L\}$ which is referred to as i 's type (with slight abuse of notation) and $H > L > 0$. It is commonly known that the joint probability distribution of the types is $\Pr(\theta_i, \theta_{-i})$ which is symmetric and has full support (conditional probabilities $\Pr(\theta_{-i}|\theta_i)$ are hence well defined and positive):

$$\Pr(H, L) = \Pr(L, H) \text{ and } \Pr(\theta_i, \theta_{-i}) > 0 \text{ for all } (\theta_i, \theta_{-i}) \in \{H, L\}^2. \quad (1)$$

Given (1), we have for example $\Pr(\theta_1 = H|\theta_2 = L) = \Pr(\theta_2 = H|\theta_1 = L)$. We hence use the shorthand notation “ $\Pr(H|H), \Pr(L|L), \Pr(H|L), \Pr(L|H)$ ” hereafter.

To illustrate the distribution in (1), consider the following class of distributions:

	H	L
H	$\alpha^2 + \rho\alpha(1 - \alpha)$	$(1 - \rho)\alpha(1 - \alpha)$
L	$(1 - \rho)\alpha(1 - \alpha)$	$(1 - \alpha)^2 + \rho\alpha(1 - \alpha)$

Figure 1. An Example for the Distribution $\Pr(\theta_i, \theta_{-i})$ in (1).

Here $\alpha \in (0, 1)$ is the ex ante probability of type H , while $\rho \in \left(\max\left\{-\frac{\alpha}{1-\alpha}, -\frac{1-\alpha}{\alpha}\right\}, 1\right)$ is the linear correlation between the types.⁵ The types' interim beliefs are:

$$\begin{aligned} \Pr(H|H) &= \alpha + \rho(1 - \alpha) \text{ and } \Pr(L|H) = 1 - \Pr(H|H), \\ \Pr(H|L) &= \alpha(1 - \rho) \text{ and } \Pr(L|L) = 1 - \Pr(H|L). \end{aligned} \quad (2)$$

After observing their signals, the players simultaneously choose non-negative efforts. The cost of effort $e_i \in [0, +\infty)$ for player i with ability θ_i is $c_i(e) = \frac{e_i}{\theta_i}$ and player i 's payoff from effort profile (e_i, e_{-i}) is hence

$$u_i(e_i, e_{-i}|\theta_i) = \begin{cases} V - \frac{e_i}{\theta_i} & \text{if } e_i > e_{-i}, \\ pV - \frac{e_i}{\theta_i}, p \in [0, 1] & \text{if } e_i = e_{-i}, \\ -\frac{e_i}{\theta_i} & \text{if } e_i < e_{-i}. \end{cases}$$

The above all-pay contest is equivalent to an all-pay auction with two ex ante symmetric bidders, private valuations, and correlated types where e_i is player i 's bid.

3 Analysis

The all-pay contest with the information structure in (1) admits a symmetric Bayesian Nash equilibrium where each type of a player randomizes over a (connected) interval of efforts. We denote the equilibrium as $\sigma^* = (\sigma_1^*(\theta), \sigma_2^*(\theta))$ where $\theta \in \{H, L\}$, $\sigma_1^*(\theta) = \sigma_2^*(\theta)$, and $\sigma_i^*(\theta) = F_\theta(\cdot)$ is player i 's behavioral strategy which is a probability distribution over Borel measurable subsets of $E = [0, HV]$.

⁵The types can be negatively correlated. However, for the probabilities in Figure 1 to be well-defined, we impose the restriction that $\rho > \max\left\{-\frac{\alpha}{1-\alpha}, -\frac{1-\alpha}{\alpha}\right\}$.

Proposition 1 Consider the contest with the correlated information structure in (1). There is a symmetric Bayesian Nash equilibrium where both types play a mixed strategy:

(i) If $\frac{H}{L} \geq \frac{\Pr(L|L)}{\Pr(L|H)}$ and $\frac{H}{L} \geq \frac{\Pr(H|L)}{\Pr(H|H)}$, the symmetric equilibrium is **monotonic**:

$$F_L(e) = \frac{e}{\underline{e}} \text{ for } e \in [0, \underline{e}] \text{ and } F_H(e) = \frac{e - \underline{e}}{\Pr(H|H)HV} \text{ for } e \in [\underline{e}, \bar{e}],$$

$$\underline{e} = \Pr(L|L)LV \text{ and } \bar{e} = \underline{e} + \Pr(H|H)HV.$$

(ii) If $\frac{\Pr(L|L)}{\Pr(L|H)} > \frac{H}{L}$, the symmetric equilibrium is **non-monotonic**:

$$F_H(e) = \begin{cases} \frac{(\Pr(L|L)L - \Pr(L|H)H)e}{(\Pr(L|L) - \Pr(L|H))HLV} & \text{for } e \in [0, \underline{e}] \\ \frac{e - \Pr(L|H)HV}{\Pr(H|H)HV} & \text{for } e \in [\underline{e}, \bar{e}] \end{cases},$$

$$F_L(e) = \frac{(\Pr(H|H)H - \Pr(H|L)L)e}{(\Pr(L|L) - \Pr(L|H))HLV} \text{ for } e \in [0, \underline{e}],$$

$$\underline{e} = LV \frac{(\Pr(L|L) - \Pr(L|H))H}{\Pr(H|H)H - \Pr(H|L)L} \text{ and } \bar{e} = HV.$$

(iii) If $\frac{H}{L} < \frac{\Pr(H|L)}{\Pr(H|H)}$, the symmetric equilibrium is **non-monotonic**:

$$F_H(e) = \frac{(\Pr(L|H)H - \Pr(L|L)L)e - \Pr(L|L)(H - L)\bar{e}}{(\Pr(L|H) - \Pr(L|L))H\bar{e}}, \text{ for } e \in [\underline{e}, \bar{e}],$$

$$F_L(e) = \begin{cases} \frac{e}{\Pr(L|L)\bar{e}}, & \text{for } e \in [0, \underline{e}] \\ \frac{(\Pr(H|L)L - \Pr(H|H)H)e + \Pr(H|L)(H - L)\bar{e}}{(\Pr(H|L) - \Pr(H|H))H\bar{e}}, & \text{for } e \in [\underline{e}, \bar{e}] \end{cases},$$

$$\underline{e} = \frac{\Pr(L|L)(H - L)\bar{e}}{\Pr(L|H)H - \Pr(L|L)L} \text{ and } \bar{e} = LV.$$

Notice that given the symmetry and full support in (1), we have

$$\frac{\Pr(L|L)}{\Pr(L|H)} = \frac{\Pr(H|L)}{\Pr(H|H)} = 1 \text{ if and only if } \theta_1 \text{ and } \theta_2 \text{ are independently distributed.}$$

Given $H > L$, case (ii) (resp., case (iii)) in Proposition 1 hence arises when θ_1 and θ_2 are sufficiently positively (resp., negatively) correlated, while case (i) arises when θ_1 and θ_2 are independent or mildly correlated.⁶ For further illustration, using our parametric distribution in Figure 1, we have (see (2)):

$$\frac{H}{L} \geq \frac{\Pr(L|L)}{\Pr(L|H)} \Rightarrow \rho \leq \frac{(1 - \alpha)(H - L)}{H - \alpha(H - L)}; \quad \frac{H}{L} \geq \frac{\Pr(H|L)}{\Pr(H|H)} \Rightarrow \rho \geq \frac{-\alpha(H - L)}{H - \alpha(H - L)}.$$

Hence, a monotonic symmetric equilibrium exists here if and only if the players' types are neither very positively nor very negatively correlated. The monotonic equilibrium is separating in that the players' types are (almost always) fully revealed after

⁶Strictly speaking, the magnitudes of effort efficiencies (H and L) are also important. We follow the literature to base our discussion here mainly on the degree of correlation.

the players playing their equilibrium strategies. On the other hand, if the players' types are sufficiently (positively or negatively) correlated, then the symmetric equilibrium is non-monotonic or partially pooling where the types randomize on overlapping intervals and the players' types are not necessarily fully revealed in equilibrium.

As mentioned, our results differ from most of the previous literature in that we also characterize non-monotonic equilibria. To illustrate, if $\frac{\Pr(L|L)}{\Pr(L|H)} > \frac{H}{L}$, the non-monotonic equilibrium in Proposition 1 features that type L randomizes uniformly on $[0, \underline{e}]$, while type H randomizes uniformly on $[0, \underline{e}]$ and also uniformly on $[\underline{e}, \bar{e}]$, i.e., both types randomize on the same set of bids ($[0, \underline{e}]$) with positive measure. Our non-monotonic equilibrium construction employs a heavy dose of “guess and verify”: Using equilibria for the complete information all-pay contest (i.e., our setting with limiting correlation) as a useful guide, we first propose an interval of efforts that is commonly shared in both types' strategies. This enables us to obtain two simultaneous equations on and further explicitly *solve* (part of) $F_H(e)$ and $F_L(e)$ from the two types' indifferences among all efforts in the interval. The construction is then completed using the types' (other) incentive constraints and that $F_H(e)$ and $F_L(e)$ are distribution functions. We illustrate our equilibrium construction further in the analysis after Figure 2 (and in the proof for case (ii) of Proposition 1). We also note here that our result is consistent with the characterizations in Rentschler and Turocy (2015).

Finally, in Proposition 1, both types' (and hence both players') equilibrium payoffs are identically zero when $\frac{\Pr(L|L)}{\Pr(L|H)} > \frac{H}{L}$ (case (ii)). This latter condition is satisfied when either the bidders' types are sufficiently positively correlated (ρ close to 1), or the valuations of the two types are sufficiently identical (H is close to L) — This is intuitive: When either ρ is sufficiently close to 1 or H is close to L , the competition between the two types is sufficiently fierce, driving the bidders' equilibrium payoffs to be zero.

For illustration, consider the parametric class of distributions in Figure 1. Figure 2 shows the equilibrium strategies $\sigma_i^*(H)$ and $\sigma_i^*(L)$ in Proposition 1, using the parameters $H = V = 2$, $L = 1$ and $\alpha = \frac{1}{2}$ (hence $\rho \in (-1, 1)$).⁷

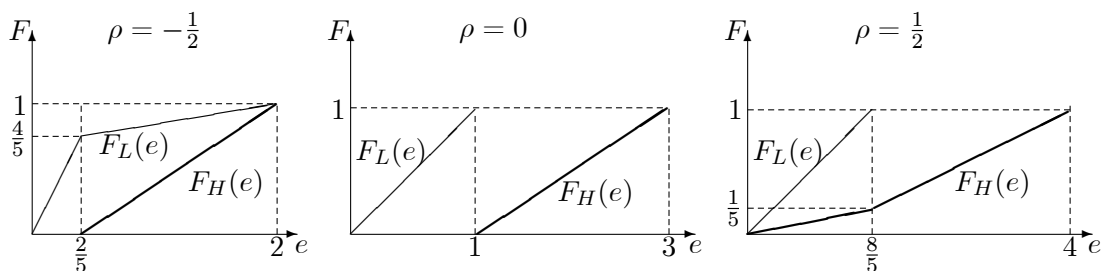


Figure 2. An Illustration of Equilibrium Strategies in Proposition 1.

To understand the graphs in Figure 2, consider for example the non-monotonic equilibria where effort level e is in the support of both types' randomizations (i.e., $e \in [\frac{2}{5}, 2]$ for $\rho = -\frac{1}{2}$ and $e \in [0, \frac{8}{5}]$ for $\rho = \frac{1}{2}$). The types' expected payoffs at e are

$$\begin{aligned} u_i(e, \sigma_{-i}^*|H) &= [\Pr(L|H) F_L(e) + \Pr(H|H) F_H(e)] V - \frac{e}{H}, \\ u_i(e, \sigma_{-i}^*|L) &= [\Pr(L|L) F_L(e) + \Pr(H|L) F_H(e)] V - \frac{e}{L}. \end{aligned}$$

⁷See Appendix for explicit formulae of the equilibrium strategies.

The first-order conditions given the parameterization in Figure 1 are

$$\begin{aligned}\frac{du_i(e, \sigma_{-i}^* | H)}{de} &= [(1 - \alpha + \alpha\rho - \rho) F'_L(e) + (\alpha - \alpha\rho + \rho) F'_H(e)] V = \frac{1}{H}, \\ \frac{du_i(e, \sigma_{-i}^* | L)}{de} &= [(1 - \alpha + \alpha\rho) F'_L(e) + (\alpha - \alpha\rho) F'_H(e)] V = \frac{1}{L}.\end{aligned}$$

And a little algebra implies that

$$\rho (F'_H(e) - F'_L(e)) = \frac{1}{H} - \frac{1}{L},$$

or $F_H(e)$ is steeper for $\rho < 0$ (i.e., $F'_H(e) > F'_L(e)$) and $F_L(e)$ is steeper for $\rho > 0$ (i.e., $F'_H(e) < F'_L(e)$) — recall that $H > L$.⁸ In particular, since $F_H(e)$ and $F_L(e)$ are probability distributions, this further implies that type L randomizes over a larger set of efforts when ρ is close to -1 , and type H randomizes over a larger set of efforts when ρ is close to 1 (here “larger” in the sense of set inclusion). The above heuristic analysis hence provides useful information toward constructing the non-monotonic equilibria.⁹

We now derive some comparative statics results for the monotonic equilibrium using the parametric distribution in Figure 1:

Proposition 2 *Consider the all-pay contest with correlated information in Figure 1. In the unique symmetric and monotonic equilibrium (σ_1^*, σ_2^*) characterized in Proposition 1 where $\frac{-\alpha(H-L)}{H-\alpha(H-L)} \leq \rho \leq \frac{(1-\alpha)(H-L)}{H-\alpha(H-L)}$, we have:*

(i) *the equilibrium payoffs are*

$$\mathbb{E}u_i(\sigma_1^*, \sigma_2^* | H) = (1 - \alpha)(1 - \rho)V - \frac{(1 - \alpha(1 - \rho))LV}{H} \text{ and } \mathbb{E}u_i(\sigma_1^*, \sigma_2^* | L) = 0.$$

(ii) *the expected total equilibrium effort $\mathbb{E}\mathcal{E}$ is*

$$\mathbb{E}\mathcal{E} = \underline{e} + \alpha\bar{e} = (1 + \alpha)(1 - \alpha(1 - \rho))LV + (\alpha + \rho - \alpha\rho)HV.$$

(iii) *the comparative statics results*

$$\frac{\partial \mathbb{E}u_i(\sigma_1^*, \sigma_2^* | H)}{\partial \rho} < 0 \text{ and } \frac{\partial \mathbb{E}\mathcal{E}}{\partial \rho} > 0.$$

Proposition 2 provides closed-form representations for the types’ equilibrium payoffs and for the expected total effort in the contest (or equivalently the expected total revenue in the all-pay auction). Moreover, the equilibrium payoff for an H type is strictly decreasing in the correlation parameter, implying that with correlated information, a higher type is not necessarily good news. The expected total equilibrium effort/revenue is however strictly increasing in the correlation parameter. This is a direct result of more fierce competition between the contestants.

⁸A similar illustration can be made analogously for the monotonic equilibrium in Proposition 1.

⁹Notice that the uniqueness of equilibrium (in Proposition 1(i)) is guaranteed when $\frac{H}{L} > \max\{\frac{\Pr(L|L)}{\Pr(L|H)}, \frac{\Pr(H|L)}{\Pr(H|H)}\}$, which is equivalent to Condition M in Siegel (2014), leading to uniqueness according to Siegel (2014). Our guess-and-verify approach however makes it difficult to establish uniqueness for the non-monotonic equilibria. Nevertheless, we conjecture that the (monotonic and non-monotonic) equilibrium characterized in Proposition 1 is unique — our above heuristic discussion of our equilibrium construction (after Figure 2) indeed substantiates this conjecture.

4 Conclusion

We have characterized both monotonic and non-monotonic symmetric equilibria for a two-player all-pay contest with general binary and symmetric information structures. We have also used a convenient class of parametric distributions to demonstrate the equilibrium construction, as well as some intuitive comparative statics results. Our results contribute to the literature on equilibrium analysis for all-pay auctions with discrete, correlated private information by allowing for all correlated information structures in our setting. Our results can hence shed some light on contest design and in particular the construction and occurrence of non-monotonic equilibria for more general settings.

References

- [1] Amann, E. and W. Leininger, 1996. Asymmetric All-Pay Auctions with Incomplete Information: The Two Player Case. **Games and Economic Behavior** 14, 1–18.
- [2] Konrad, K.A., 2004. Altruism and Envy in Contests: An Evolutionary Stable Symbiosis. **Social Choice and Welfare**, 22, 479-490.
- [3] Konrad, K.A., 2009. Strategy and Dynamics in Contests. **Oxford University Press**.
- [4] Krishna, V. and J. Morgan, 1997. An Analysis of the War of Attrition and the All-Pay Auction. **Journal of Economic Theory** 72, 343-362.
- [5] Lu, J. and S.O. Parreiras, 2014. Two-Bidder All-Pay Auctions: Tying Functions, Monotone and Non-Monotone Equilibria. **Mimeo**.
- [6] Rentschler, L. and T. L. Turocy, 2015. Two Bidder All-Pay Auctions with Interdependent Valuations, including the Highly Competitive Case. Forthcoming at **Journal of Economic Theory**.
- [7] Siegel, R., 2014. Asymmetric All-Pay Auctions with Interdependent Valuations. **Journal of Economic Theory** 153, 684—702.

Appendix

Proof of Proposition 1. Case (i): $\frac{H}{L} \geq \frac{\Pr(L|L)}{\Pr(L|H)}$ and $\frac{H}{L} \geq \frac{\Pr(H|L)}{\Pr(H|H)}$. For $e \in [0, \underline{e}]$ and $\hat{e} \in [\underline{e}, \bar{e}]$, type θ 's equilibrium payoff can be calculated as:

$$\begin{aligned}
 u_i(e, \sigma_{-i}^*(\theta) | L) &= \Pr(L|L) F_L(e) V - \frac{e}{L} = \frac{\Pr(L|L)e}{\Pr(L|L)LV} V - \frac{e}{L} = 0, \\
 u_i(\hat{e}, \sigma_{-i}^*(\theta) | H) &= [\Pr(L|H) + \Pr(H|H) F_H(\hat{e})] V - \frac{\hat{e}}{H} = \frac{\Pr(L|H)HV - \Pr(L|L)LV}{H}.
 \end{aligned}$$

Hence both types are willing to randomize according to $F_L(e)$ and $F_H(e)$ respectively. Next, an effort $e' \in [0, \underline{e}]$ from type H yields — using $H \Pr(L|H) \geq L \Pr(L|L)$:

$$\begin{aligned} u_i(e', \sigma_{-i}^*(\theta) | H) &= \Pr(L|H) F_L(e') V - \frac{e'}{H} = \frac{\Pr(L|H) e'}{\Pr(L|L) L} - \frac{e'}{H} \\ &\leq \frac{\Pr(L|H) \underline{e}}{\Pr(L|L) L} - \frac{\underline{e}}{H} = \frac{\Pr(L|H) HV - \Pr(L|L) LV}{H}. \end{aligned}$$

while an effort $e'' \in [\underline{e}, \bar{e}]$ from type L yields

$$\begin{aligned} u_i(e'', \sigma_{-i}^*(\theta) | L) &= [\Pr(L|L) + \Pr(H|L) F_H(e'')] V - \frac{e''}{L} \\ &\leq u_i(\underline{e}, \sigma_{-i}^*(\theta) | L) = \Pr(L|L) V - \frac{\underline{e}}{L} = 0. \end{aligned}$$

where the inequality comes from $\frac{\partial u_i(e'', \sigma_{-i}^*(\theta) | L)}{\partial e''} \geq 0$ given $\frac{H}{L} \leq \frac{\Pr(H|L)}{\Pr(H|H)}$. Hence, no type will deviate to any effort choice of the other type, establishing Case (i).

Case (ii): $\frac{H}{L} < \frac{\Pr(L|L)}{\Pr(L|H)}$. We shall illustrate our construction for the non-monotonic equilibrium here. Given the non-existence of a monotonic equilibrium here by **Case (i)**, we conjecture that both types “pool” in the interval $[0, \underline{e}]$. For $e \in [0, \underline{e}]$, we have

$$\begin{aligned} u_i(e, \sigma_{-i}^*(\theta) | L) &= \Pr(L|L) F_L(e) V + \Pr(H|L) F_H(e) V - \frac{e}{L} = 0, \\ u_i(e, \sigma_{-i}^*(\theta) | H) &= \Pr(L|H) F_L(e) V + \Pr(H|H) F_H(e) V - \frac{e}{H} = 0. \end{aligned}$$

Solving $F_L(e)$ and $F_H(e)$ from the above simultaneous equations gives:¹⁰

$$F_H(e) = \frac{[\Pr(L|L) L - \Pr(L|H) H] e}{[\Pr(L|L) - \Pr(L|H)] HL V}, F_L(e) = \frac{[\Pr(H|H) H - \Pr(H|L) L] e}{[\Pr(L|L) - \Pr(L|H)] HL V}.$$

This also implies that both types are willing to randomize in $[0, \underline{e}]$.

Next, $F_L(\underline{e}) = 1$ yields that $\underline{e} = \frac{[\Pr(L|L) - \Pr(L|H)] HL V}{\Pr(H|H) H - \Pr(H|L) L}$. Hence $F_H(\underline{e}) = \frac{\Pr(L|L) L - \Pr(L|H) H}{\Pr(H|H) H - \Pr(H|L) L} < 1$, where the inequality is due to $\Pr(H|H) H - \Pr(H|L) L = (H - L) + \Pr(L|L) L - \Pr(L|H) H$. This prompts us to construct $F_H(e)$ further for $e > \underline{e}$:

$$\begin{aligned} u_i(e, \sigma_{-i}^*(\theta) | H) &= [\Pr(L|H) + \Pr(H|H) F_H(e)] V - \frac{e}{H} = 0 \\ \Rightarrow F_H(e) &= \frac{e - \Pr(L|H) HV}{\Pr(H|H) HV} \text{ for } e \in [\underline{e}, \bar{e}]. \end{aligned}$$

Solve for \bar{e} from $F_H(\bar{e}) = 1$ to obtain $\bar{e} = HV$. We further verify that $\bar{e} = HV > LV \frac{[\Pr(L|L) - \Pr(L|H)] H}{\Pr(H|H) H - \Pr(H|L) L} = \underline{e}$ and $F_H(e)$ is continuous at $e = \underline{e}$, implying that $F_H(e)$ is a legitimate probability distribution on $[0, \bar{e}]$.

To complete our proof, we verify that type L has no incentive to choose $e \in (\underline{e}, \bar{e}]$:

$$\begin{aligned} u_i(e, \sigma_{-i}^*(\theta) | L) &= ([\Pr(L|L) + \Pr(H|L) F_H(e)] V - \frac{e}{L}) \\ &= \left[\Pr(L|L) + \frac{\Pr(H|L) (e - \Pr(L|H) V H)}{\Pr(H|H) HV} \right] V - \frac{e}{L}, \\ \frac{\partial u_i(e, \sigma_{-i}^*(\theta) | L)}{\partial e} &= \frac{\Pr(H|L)}{\Pr(H|H) H} - \frac{1}{L} < 0 \Rightarrow u_i(e, \sigma_{-i}^*(\theta) | L) < u_i(\underline{e}, \sigma_{-i}^*(\theta) | L). \end{aligned}$$

¹⁰It can be verified that $F'_L(e) > F'_H(e)$ for $e \in [0, \underline{e}]$, which agrees with Figure 2.

Case (iii): $\frac{H}{L} < \frac{\Pr(H|L)}{\Pr(H|H)}$. First, notice that $F_L(\cdot)$ and $F_H(\cdot)$ are well-defined distribution functions: $F_L(0) > 0$, $F_L(e)$ is continuous at $e = \underline{e}$, and $F_L(\bar{e}) = 1$; $F_H(\underline{e}) = 0$, $F'_H(e) = \frac{(\Pr(L|H)H - \Pr(L|L)L)}{(\Pr(L|H)H - \Pr(L|L)L)\bar{e}} > 0$, and $F_H(\bar{e}) = 1$; and $\frac{\Pr(L|L)(H-L)}{\Pr(L|H)H - \Pr(L|L)L} < 1$ implies that $\underline{e} < \bar{e}$.¹¹ We next check the types' incentives by considering two subcases:

Subcase 1. For $e \in [\underline{e}, \bar{e}]$, the two types' payoffs are:

$$\begin{aligned} u_i(e, \sigma_{-i}^*(\theta) | L) &= \Pr(L|L) F_L(e) V + \Pr(H|L) F_H(e) V - \frac{e}{L} \\ &= \frac{(1 - \Pr(L|L) - \Pr(H|H))HeV}{(\Pr(H|L) - \Pr(H|H))H\bar{e}} - \frac{e}{L} = 0, \\ u_i(e, \sigma_{-i}^*(\theta) | H) &= \Pr(L|H) F_L(e) V + \Pr(H|H) F_H(e) V - \frac{e}{H} \\ &= \frac{(\Pr(H|L) - \Pr(H|H))((H-L)\bar{e} + Le)V}{(\Pr(H|L) - \Pr(H|H))H\bar{e}} - \frac{e}{H} = \frac{(H-L)V}{H}. \end{aligned}$$

Subcase 2. For $e \in [0, \underline{e}]$, the two types' payoffs are

$$\begin{aligned} u_i(e, \sigma_{-i}^*(\theta) | L) &= \Pr(L|L) F_L(e) V - \frac{e}{L} = 0, \\ u_i(e, \sigma_{-i}^*(\theta) | H) &= \Pr(L|H) F_L(e) V - \frac{e}{H} = \frac{\Pr(L|H)e}{\Pr(L|L)\bar{e}} V - \frac{e}{H} \\ &< \frac{\Pr(L|H)\underline{e}}{\Pr(L|L)\bar{e}} V - \frac{e}{H} = \frac{(H-L)V}{H}. \end{aligned}$$

where we used $\frac{\Pr(L|H)V}{\Pr(L|L)\bar{e}} > \frac{1}{H}$ to obtain the inequality for $u_i(e, \sigma_{-i}^*(\theta) | H)$.

We conclude that the strategy profile in (iii) is an equilibrium. ■

Equilibrium Strategies for Figure 2. Let $H = V = 2$, $L = 1$, and $\alpha = \frac{1}{2}$, hence $\rho \in (-1, 1)$. The equilibrium strategies in Proposition 1 can be calculated as:

(i) For $-\frac{1}{3} \leq \rho \leq \frac{1}{3}$:

$$F_H(e) = \frac{e - 1 - \rho}{2 + 2\rho}, e \in [1 + \rho, 3 + 3\rho]; F_L(e) = \frac{e}{1 + \rho}, e \in [0, 1 + \rho].$$

(ii) For $\rho > \frac{1}{3}$:

$$F_L(e) = \frac{e(3\rho + 1)}{8\rho}, e \in \left[0, \frac{8\rho}{3\rho + 1}\right]; F_H(e) = \begin{cases} \frac{e(3\rho - 1)}{8\rho}, & e \in \left[0, \frac{8\rho}{3\rho + 1}\right] \\ \frac{2\rho + e - 2}{2\rho + 2}, & e \in \left[\frac{8\rho}{3\rho + 1}, 4\right] \end{cases}.$$

(iii) For $\rho < -\frac{1}{3}$:

$$F_H(e) = \frac{2\rho - e + 3\rho e + 2}{8\rho}, e \in \left[\frac{2 + 2\rho}{1 - 3\rho}, 2\right]; F_L(e) = \begin{cases} \frac{e}{\rho + 1}, & e \in \left[0, \frac{2 + 2\rho}{1 - 3\rho}\right] \\ \frac{2\rho + e + 3\rho e - 2}{8\rho}, & e \in \left[\frac{2 + 2\rho}{1 - 3\rho}, 2\right] \end{cases}.$$

The three panels of Figure 2 represent the above strategies graphically. ■

Proof of Proposition 2. By Proposition 1 (i), a type H 's equilibrium payoff is

$$\mathbb{E}u_i(\sigma_1^*, \sigma_2^* | H) = V - \frac{\bar{e}}{H} = (1 - \alpha)(1 - \rho)V - \frac{(1 - \alpha)(1 - \rho)LV}{H},$$

¹¹We have used the following inequalities in our arguments: (1) $\frac{\Pr(H|L)}{\Pr(H|H)} > \frac{H}{L} > 1$ implies that $\Pr(L|H) > \Pr(L|L)$, (2) Given that $\Pr(H|L) + \Pr(L|L) = \Pr(L|H) + \Pr(H|H) = 1$, $\Pr(H|L)L > \Pr(H|H)H$ is equivalent to $(1 - \Pr(L|L))L > (1 - \Pr(L|H))H$, which implies that $\Pr(L|H)H - \Pr(L|L)L > H - L > 0$.

while the expected total effort in equilibrium is

$$\begin{aligned}
\mathbb{E}\mathcal{E} &= 2 \Pr(H, H) \mathbb{E}(\sigma_i^*(H)) + 2 \Pr(H, L) (\mathbb{E}(\sigma_i^*(H)) + \mathbb{E}(\sigma_i^*(L))) + 2 \Pr(L, L) \mathbb{E}(\sigma_i^*(L)) \\
&= 2(\alpha^2 + \rho\alpha(1-\alpha)) \frac{\bar{e} + \underline{e}}{2} + 2(1-\rho)\alpha(1-\alpha) \left(\frac{\bar{e} + \underline{e}}{2} + \frac{\underline{e}}{2} \right) + 2 \left((1-\alpha)^2 + \rho\alpha(1-\alpha) \right) \frac{\underline{e}}{2} \\
&= \underline{e} + \alpha\bar{e}.
\end{aligned}$$

Finally, direct computation implies that

$$\frac{\partial \mathbb{E}u_i(\sigma_1^*, \sigma_2^* | H)}{\partial \rho} = -\frac{(H - H\alpha + L\alpha)V}{H} < 0, \text{ and } \frac{\partial \mathbb{E}\mathcal{E}}{\partial \rho} = \alpha((1-\alpha)H + (1+\alpha)L)V > 0,$$

completing the proof. ■