

Optimal Bundle Pricing under Correlated Valuations*

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Abstract

We study optimal pricing issues for a monopolist selling two indivisible goods to a continuum of consumers with correlated private valuations over the goods, where the (positive or negative) correlation is modeled using copulas in the Fréchet family. We derive explicit optimal pricing schemes and comparative statics results for various environments in our setting. The optimal pricing schemes can take several forms, including pure bundling, partial mixed bundling, and mixed bundling, depending jointly on the degrees of asymmetry and correlation of the consumers' valuations. The explicit optimal pricing schemes also enable us to investigate whether and how the monopolist's profit can be further improved via random assignments.

Keywords: Bundling, Correlated Valuations, Monopoly Pricing, Price Discrimination, Random Mechanism.

JEL Classification: D11, D42, D82, L12.

1 Introduction

Product bundle pricing typically refers to a strategy where a firm sells multiple products together as a combo package (and sometimes with component prices for individual products as well). This strategy is a prevalent business practice in various industries, such as telecommunications services, digital information products, auto sales, and insurance products. While it is an important and heavily studied topic in the literature, our understanding of product bundle pricing has been somewhat impeded by that even determining optimal bundling prices in simple settings can often be analytically intractable. This paper provides a tractable multiproduct model of correlated valuations and derives closed-form solutions on optimal bundle pricing schemes in this framework.¹

We consider a monopolist selling two indivisible products to a continuum of consumers with unit demand and correlated private valuations of the products. We model the buyers' valuations via bivariate copulas in the Fréchet family with uniform marginal distributions on asymmetric interval supports, which can be regarded as "limiting" probability distributions given the marginal

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¹In line with the bundling literature, we use correlated valuations and stochastically dependent valuations interchangeably (dependence here is a non-linear correlation based on copulas). The precise meaning of correlated valuations is presented in Section 2.

distributions and the degree of correlation.² This enables us to fix marginal distributions and consider joint valuation distributions with varying degrees of correlation. In particular, the consumers’ valuations of the products can be independent, positively correlated, or negatively correlated.

The monopolist chooses a *deterministic* price-posting mechanism, or a pricing scheme, which can take the form of separate sales (only component prices being offered), pure bundling (posting a single bundle price), partial mixed bundling (posting a bundle price and one component price), and mixed bundling where a bundle price and both component prices are offered with the bundle price not exceeding the sum of the component prices, a natural scheme when the monopolist cannot monitor purchasing behavior of the buyers.³ Instead of treating pure bundling and partial mixed bundling as special cases of mixed bundling, we distinguish them explicitly in our study. Our main purpose is to analytically characterize the optimal pricing schemes and to analyze the efficacy and comparative statics of these pricing schemes in our setting.

We first explicitly derive the optimal pricing schemes under correlated valuations, which can be pure bundling, partial mixed bundling, or mixed bundling. The specific form of the optimal pricing schemes depends jointly on the magnitude of asymmetry in the supports and the degree of correlation of the valuation distribution. Under both positive and negative correlation, mixed bundling is optimal when the support asymmetry and the degree of correlation are both small. Partial mixed bundling however becomes the optimal pricing scheme if the support asymmetry is large (positive correlation) or if either the support asymmetry or the degree of correlation is large (negative correlation). Valuation correlation plays a salient role in driving partial mixed bundling to be optimal with negative correlation — with a higher degree of negative correlation, the monopolist stops selling the (dominant) product with possibly larger valuations separately, forcing consumers interested mainly in the dominant good to purchase the entire bundle so as to achieve a better price discrimination outcome.⁴ Finally, pure bundling arises as an optimal pricing scheme *only* in the case of symmetric distributions when the consumers valuations are sufficiently *negatively* correlated. The rationale is that pure bundling, treating both goods indiscriminately, is less effective when the goods are ex ante different to the consumers.

On comparative statics, we find that the negative (resp., positive) correlation case is a more (resp., less) favorable setting for the monopolist, in that the optimal expected profit *strictly increases* in the degree of negative correlation and *strictly decreases* in the degree of positive correlation. Such comparative statics results are widely accepted, but have not been analytically derived in the literature. The driving force of this result is that with negatively correlated valuations, the monopolist can always use *both* the bundle price and component prices effectively to sort consumers into different groups, while the component prices play rather passive roles when consumers’ valuations are positively correlated. Our analysis here provides a novel perspective on how valuation correlation affects the efficacy of price-posting mechanisms, particularly so for the positive correlation case, which is much less studied in the literature compared to negative correlation.⁵

²This copula has appeared in Chen and Riordan (2013). A copula is a joint probability distribution with given marginals (see Nelson (2006) for an introduction). Section 2 provides detailed justifications for our copula choice.

³This is the standard “no monitoring sales” assumption in the literature, starting from Adams and Yellen (1976).

⁴An (arguable) example of partial mixed bundling is the pricing scheme for base products and accessories. Consumers with an old base product (i.e., razors, guitar or laptops) typically can *only* buy a new accessory (i.e., blades, guitar strings or batteries) or a new base product with a new accessory, but cannot buy a new base product without an accessory. We consider such consumers (with an old base product) because in our setting both products are valuable for consumers and can be consumed separately and an accessory is only valuable to a consumer with a base product (though the complementarity between a base product and an accessory is absent in our setting).

⁵We have also derived optimal pure bundling pricing schemes where the monopolist sells the products at a single bundle price and there is certain similarity between the optimal pricing scheme and the optimal pure bundling scheme in terms of the comparative statics in correlation. Detailed analysis on this is available upon request.

Finally, we go beyond deterministic price-posting mechanisms and investigate whether a random mechanism enables the monopolist to achieve strictly higher profits, which is a natural and direct application of our characterization of the optimal pricing schemes. It is known in the literature that characterizing the optimal (revenue-maximizing) selling mechanisms for a multi-product monopolist is significantly more difficult than that for a one-product monopolist, and the problem is in general considered as intractable. Nevertheless, several previous studies present various examples to show that random mechanisms can dominate price-posting mechanisms in revenue when a seller sells multiple indivisible goods (see, e.g., Thanassoulis (2004), Manelli and Vincent (2006, 2007), and Hart and Reny (2015)).

Different from the previous literature, we investigate marginal deviations from our optimal (deterministic) pricing schemes toward a “nearby” random mechanism. In the random mechanism, a consumer who only purchases an individual good under the optimal pricing scheme is *instead* offered a lottery which awards the consumer with the individual good at the previous optimal component price with probability close to one and the bundle at the previous optimal bundle price with the remaining probability. The random mechanism intends to further exploit consumers who purchase an individual good under the deterministic pricing scheme by selling more products to such consumers and is in a sense *close* to the optimal deterministic pricing scheme. We find that the random mechanism strictly improves the monopolist’s profit whenever the valuations are negatively correlated and the optimal pricing scheme features partial mixed bundling. While our investigation here is restricted by the marginal-analysis approach, our analysis directly connects the random mechanism with an underlying optimal pricing scheme and our result reveals when an optimal pricing scheme is not revenue-maximizing and how the deterministic pricing scheme can be improved using lotteries, an approach that is new in the literature.

1.1 Related Literature

Various rationales and benefits of product bundling have been identified in the business literature, including cost reduction, economies of scales, and complementarities among the underlying goods. From an economics perspective, bundling has been identified as a powerful price discrimination device, starting from the classic studies of Stigler (1963) and Adams and Yellen (1976), as well as a useful tool to protect and leverage market power (e.g., Stigler (1963), Whinston (1990)).⁶

On price discrimination, Schmalensee (1984) employs a bivariate normal distribution setting to show that bundling can dominate separate sales when consumer valuations are negatively, independently, or positively correlated. Given the analytical difficulties, Schmalensee’s results are however mainly derived numerically. McAfee, McMillan and Whinston (1989) formally analyze a two-product setting with general valuation distributions and establish sufficient conditions for profitability of mixed bundling (over separate sales) using a marginal analysis. Recently, using copulas to model stochastic dependence of valuations, Chen and Riordan (2013) demonstrate profitability of bundling for settings much more general than the previous literature. In particular, the authors establish profitability of mixed bundling when product valuations are negatively dependent, independent, or have sufficiently limited positive dependence.

This paper similarly focuses on the price discriminating aspect of bundling. While the previous bundling literature has addressed specific issues in general settings, we address multiple issues in a somewhat specific setting, so as to provide a complete picture on the optimal pricing scheme instead of only evaluating whether/when separate sales is suboptimal. First, this paper is closely related to Chen and Riordan (2013). Our copula choice is motivated by and a special case of Chen

⁶Our literature review focuses on bundling decisions by a monopolist. There is also an important literature on pricing and bundling decisions among competing firms. See Zhou (2017) and references therein.

and Riordan (2013) and we follow their methodology closely in our analysis. Our main objective is however different from that in Chen and Riordan (2013) in that we derive explicitly in our setting the *globally* optimal pricing schemes and a corresponding comparative statics analysis in correlation. Given such an objective, our study is also related to Eckalbar (2010) who develops analytical results and insights for a mixed bundling problem in a two-good setting with independent and uniform marginal valuations.⁷ In comparison, our setting is more general and encompasses the entire ranges of valuation correlation and support asymmetry, which enables us to also derive some relevant comparative statics results and to analyze the efficacy of various pricing schemes under varying degrees of valuation correlation. Indeed, our main motivation is to shed some light on the optimal bundling and pricing issues when consumer valuations are correlated across products, which is largely under-explored in the current literature.

Our paper is also related to the multiproduct mechanism design literature (e.g., McAfee and McMillan (1988), Armstrong (1996), Rochet and Choné (1998), Thanassoulis (2004), Manelli and Vincent (2006, 2007), and Hart and Reny (2015)). The existing literature however lacks a general characterization so far even for the case of two goods. Our result on randomization is mainly on whether and how randomization can help improve revenue. The phenomenon where random assignments can improve revenue has appeared in Thanassoulis (2004), Manelli and Vincent (2006, 2007), and Hart and Reny (2015).⁸ Compared to these studies, our result is obtained in a specific setting via a marginal analysis approach. The novelty of our approach however lies in that we start with a completely characterized deterministic price-posting mechanism and we ask when this simple mechanism can be improved using randomization, thus establishing a novel link between price-posting mechanisms and revenue-maximizing mechanisms.

2 Framework

We consider a classic framework where a profit-maximizing monopolist sells two indivisible products indexed by $i \in \{1, 2\}$ to a continuum of buyers with measure one. The monopolist produces the products at a (normalized) zero marginal cost without capacity constraint.⁹ A representative consumer purchases at most one unit of each good and has private valuations $v = (v_1, v_2)$ with v_i being her valuation of good i . To focus on the price discrimination aspect of product bundling (rather than the complementarity aspect), we assume that a consumer with valuations v derives utility $v \cdot I$ from allocation $I = (I_1, I_2)$ where $I_i \in \{0, 1\}$ denotes whether the consumer consumes good i ($I_i = 1$) or not ($I_i = 0$). A consumer's willingness to pay on a good is hence independent of receiving the other good or not. We maintain the standard “no monitoring sales” assumption throughout, i.e., the monopolist cannot monitor consumers' purchases and thus cannot prevent them from buying the two goods separately.

We employ two well-known copulas in the Fréchet family to model dependence of the consumers' valuations of the goods. Denote the marginal cumulative distribution function of v_i as $H_i(v_i)$ over

⁷Eckalbar (2010) also considers the limiting cases of perfect (positive and negative) correlations, as well as an analysis on consumer surplus.

⁸Thanassoulis (2004) shows that with two perfectly substitutable goods, randomization in the assignment of goods typically dominates deterministic assignments. Manelli and Vincent (2006) identify conditions where price-posting mechanisms are revenue maximizing, while Manelli and Vincent (2007) characterize revenue maximizing multidimensional mechanisms for some valuation distributions. Recently, Hart and Reny (2015) among other things demonstrate (via examples) revenue-nonmonotonicity and necessity of randomization in selling multiple goods.

⁹Our results continue to hold for the case of positive yet small marginal costs. In addition, notice that normalizing the marginal costs to be zero is without loss of generality if the prices derived in the sequel are regarded as markups, i.e., prices net of marginal costs.

support $[0, a_i]$ with $a_i > 0$, and the joint distribution function of (v_1, v_2) as $F(v_1, v_2)$.¹⁰ Without loss of generality, we assume that $a_1 \geq a_2$ and call good 1 the *dominant* good in the sequel.¹¹ To model stochastic dependence of v_1 and v_2 , we specify:

$$\begin{aligned} F^N(v_1, v_2) &= C^N(H_1(v_1), H_2(v_2)) \text{ and } F^P(v_1, v_2) = C^P(H_1(v_1), H_2(v_2)), \text{ where} \\ C^N(H_1(v_1), H_2(v_2)) &= \alpha \max\{H_1(v_1) + H_2(v_2) - 1, 0\} + (1 - \alpha)H_1(v_1)H_2(v_2) \\ C^P(H_1(v_1), H_2(v_2)) &= \beta \min\{H_1(v_1), H_2(v_2)\} + (1 - \beta)H_1(v_1)H_2(v_2) \end{aligned} \quad (1)$$

Here, superscripts denote types of correlation (Positive or Negative) and parameter $\alpha \in [0, 1]$ (resp., $\beta \in [0, 1]$) captures the degree of negative dependence (resp., positive dependence). Copulas $C^N(H_1(v_1), H_2(v_2))$ and $C^P(H_1(v_1), H_2(v_2))$ are joint probability distributions of random variables $H_1(v_1)$ and $H_2(v_2)$, each being uniformly distributed over $[0, 1]$, while the joint probability distribution functions $F^N(v_1, v_2)$ and $F^P(v_1, v_2)$ are defined on the rectangle $[0, a_1] \times [0, a_2]$. For our purpose of deriving explicit optimal pricing schemes, we shall focus on the benchmark case of uniform (marginal) distributions throughout the paper.

Assumption (Uniform Margins). Valuation v_i is uniformly distributed on $[0, a_i]$ for $i \in \{1, 2\}$.

Figure 1 provides a visual description of $C^N(H_1(v_1), H_2(v_2))$ and $C^P(H_1(v_1), H_2(v_2))$. To illustrate, $C^N(H_1(v_1), H_2(v_2))$ can be thought of resulting from drawing $(H_1(v_1), H_2(v_2))$ uniformly from the negative 45-degree line in the unit square with probability α and drawing $(H_1(v_1), H_2(v_2))$ uniformly from the rest of the unit square with probability $(1 - \alpha)$. Accordingly, the distribution $F^N(v_1, v_2)$ can be seen as drawing (v_1, v_2) uniformly from the rectangle $[0, a_1] \times [0, a_2]$ with probability $(1 - \alpha)$, and uniformly from the negative diagonal line connecting $(a_1, 0)$ and $(0, a_2)$ with probability α . Such a copula admits an interpretation that consumer valuations are driven jointly by idiosyncratic shocks as well as a common shock.¹²

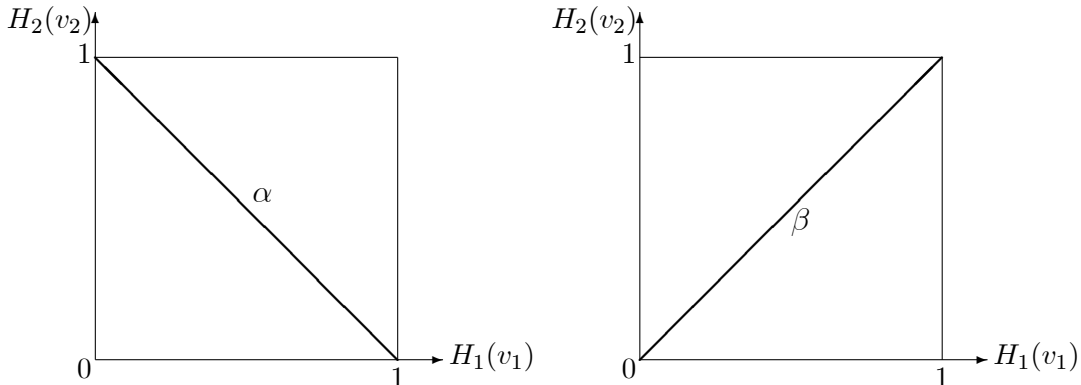


FIGURE 1. ILLUSTRATING $C^N(H_1(v_1), H_2(v_2))$ (LEFT) AND $C^P(H_1(v_1), H_2(v_2))$ (RIGHT).

There is a measure α (resp., β) consumers (uniformly) on the negative (resp., positive) diagonal line.

The motivation for our copula choice in (1) is threefold: First, given that there is insufficient understanding of optimal pricing schemes under correlated valuations, a Fréchet copula is simple

¹⁰With slight abuse of notation, we use v_1, v_2 to denote both random variables and their realized values.

¹¹Such a formulation of product asymmetry (via support asymmetry) is more general than it appears. In particular, this formulation is *equivalent* to an alternative formulation where a representative consumer with valuations $v = (v_1, v_2)$ derives utility “ $kv_1 + v_2$ ” from buying the bundle with $k \geq 1$. The interpretation is that either the consumer likes good 1 more and has an actual willingness-to-pay of kv_1 from buying a unit of good 1, or for the same physical time period, the consumer needs good 1 more often (k times to be exact) than good 2 (the price of good 1 is then the payment for k units of good 1). We thank a referee for raising this issue to us.

¹²We note that similar ideas of modelling valuation correlation have also appeared in Nalebuff (2004, Sec. III.C) on pure bundling as an entry barrier and in Armstrong and Vickers (2010, Sec. 3.5) on competitive bundling issues.

and hence a good choice in modeling correlation valuations in a two-good framework. Second, by construction, the *only* difference between joint distributions in our copulas is the degree of interdependence (parameters α and β), making it a natural environment for comparative statics analysis on such interdependence. Third, notice that our copulas are convex linear combinations of the product copula and the Fréchet-Hoeffding upper and lower bounds for joint distributions with the margins $H_1(v_1)$ and $H_2(v_2)$.¹³ The copulas are hence extreme joint distributions given the marginal distributions and correlation parameters, enabling us to interpret the derived results as the “best/worst” outcomes for the monopolist.¹⁴

Our main goal is to characterize the optimal pricing schemes. Here a pricing scheme is a deterministic price-posting mechanism where the monopolist offers *take-it-or-leave-it* prices (x, y, p) , i.e., the *component prices* of goods 1 (x) and 2 (y) and the *bundle price* (p), so as to induce the buyers to self-select into different market segments. To be specific, we focus on prices (x, y, p) satisfying $x \in [0, a_1]$, $y \in [0, a_2]$, and $p \in [\max\{x, y\}, x + y]$, where the restriction on p is a consequence of the “no monitoring sales” assumption.¹⁵

Given the representation (1) and the prices (x, y, p) , we follow Chen and Riordan (2013) to derive the demands for good 1, good 2 and the bundle respectively as $(l \in \{N, P\})$ ¹⁶

$$\begin{aligned} Q_1(x, p) &= H_2(p - x) - C^l(H_1(x), H_2(p - x)), \\ Q_2(y, p) &= H_1(p - y) - C^l(H_1(p - y), H_2(y)), \\ Q_{12}(x, y, p) &= \int_{H_1(p-y)}^{H_1(x)} [1 - C_1^l(z, H_2(p - H_1^{-1}(z)))] dz + 1 - H_1(x) - Q_1(x, p). \end{aligned} \tag{2}$$

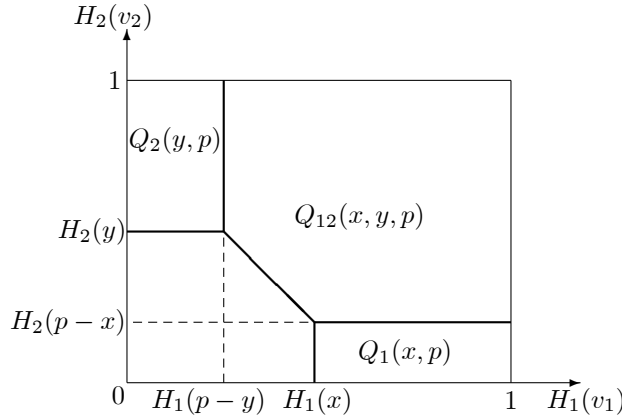


FIGURE 2. AN ILLUSTRATION OF THE DEMANDS IN (2).

Figure 2 illustrates the demands heuristically for prices (x, y, p) . Roughly each demand corresponds to the measure of buyers in the corresponding market segment — For example, $C^l(H_1(x), H_2(p-x))$ denotes the measure with $H_1(v_1) < H_1(x)$ and $H_2(v_2) < H_2(p-x)$ in the unit square. Notice

¹³In our setting, the product copula is $H_1(v_1)H_2(v_2)$, and the Fréchet-Hoeffding upper and lower bounds are respectively $\min\{H_1(v_1), H_2(v_2)\}$ and $\max\{H_1(v_1) + H_2(v_2) - 1, 0\}$ (Chapter 2.5 of Nelson (2006)).

¹⁴A further comment on our copula model is perhaps important: Strictly speaking, given that we are considering a specific class of distributions, it is *by no means* necessary to set up our model using copulas. However, our copula modeling approach permits a compact formulation and it also enables us to further illustrate such distributions through the lenses of the Fréchet-Hoeffding bounds as mentioned above.

¹⁵If the consumers’ purchasing behavior can be perfectly monitored, the monopolist can offer a pricing scheme featuring bundling with a premium (instead of bundling with a discount), i.e., $p > x + y$. Absent monitored sales, such a pricing scheme is however problematic. The restriction $p \geq \max\{x, y\}$ on the other hand is an incentive constraint for the bundle price to have some effect.

¹⁶Here the derivative $C_1^l(z_1, z_2) = \partial C^l(z_1, z_2)/\partial z_1$ is exactly the conditional probability $C^l(z_2|z_1)$ given $z_1 = H_1(v_1)$ being uniformly distributed over $[0, 1]$.

that we have implicitly accounted for the consumers’ optimal purchasing decisions in deriving the demand functions: Given prices (x, y, p) , a consumer with valuations (v_1, v_2) purchases the bundle if $v_1 + v_2 - p \geq \max\{0, v_1 - x, v_2 - y\}$, good 1 alone if $v_1 - x \geq \max\{0, v_2 - y, v_1 + v_2 - p\}$, and good 2 only if $v_2 - y \geq \max\{0, v_1 - x, v_1 + v_2 - p\}$.

The monopolist’s objective is to choose prices x, y and p to maximize his expected profit $\Pi(x, y, p)$ given the associated demands in (2):

$$(\mathcal{P}) \quad \begin{aligned} \max_{x,y,p} \quad & \Pi(x, y, p) = xQ_1(x, p) + yQ_2(y, p) + pQ_{12}(x, y, p) \\ \text{s.t.} \quad & x \in [0, a_1], y \in [0, a_2], p \in [\max\{x, y\}, x + y]. \end{aligned}$$

The complexity in solving the problem (\mathcal{P}) lies in multiple constraints and multiple parameters in the maximization and that the copulas are not differentiable, leading to too many cases to consider. Our approach to (\mathcal{P}) is to search instead for various combinations of prices (x, y, p) that can arise as optimal pricing schemes and then backtrack the parameter constellation for each pricing scheme. We explicitly identify four different pricing schemes: a **separate sale** strategy where the monopolist only posts a price for each product, corresponding to prices (x, y, p) with $x \in (0, a_1), y \in (0, a_2)$ and $p \geq x + y$; a **pure bundling** strategy with prices (x, y, p) satisfying $x \geq \min\{a_1, p\}, y \geq \min\{a_2, p\}$, and $p \in (0, a_1 + a_2)$, resulting in either purchasing the bundle or nothing for the consumers; a **mixed bundling** strategy if $x \in (0, a_1), y \in (0, a_2)$ and $p \in (\max\{x, y\}, x + y)$, which features “interior” component prices and a discount for purchasing the bundle; and finally, **partial mixed bundling** with prices (x, y, p) satisfying either “ $x \in (0, a_1), y \geq \min\{a_2, p\}$ and $p \in (x, x + a_2)$ ” or “ $x \geq \min\{a_1, p\}, y \in (0, a_2)$ and $p \in (y, y + a_1)$ ”, effectively resulting in that the consumers either purchase an individual product or both products, but not the other product individually.

3 Optimal Pricing Schemes

In this section, we explicitly characterize the optimal pricing schemes, analyze how the pricing schemes vary in the degree of valuation correlation, and discuss some welfare implications of the pricing schemes.

3.1 Negatively Correlated Valuations

It has long been recognized that (mixed) bundling is particularly attractive when the consumers’ valuations are negatively correlated (e.g., Stigler (1963), Adams and Yellen (1976), and Chen and Riordan (2013)). The rough intuition is that absent perfect correlation, the effects of lowering valuation heterogeneity via the bundle price and capturing consumers with high demands for single products via the component prices are jointly very powerful under negatively correlated valuations. We offer a complete characterization of the optimal pricing rule. In particular, we show that the monopolist unambiguously fares better with more negatively correlated valuations, and the exact form of the optimal pricing scheme is determined by both the support asymmetry and the degree of correlation.

To build up useful intuition, we start with the simple case of symmetric supports $a_1 = a_2 = a$. Proposition 1 characterizes the monopolist’s optimal pricing scheme in this case:¹⁷

¹⁷All proofs are relegated to an Appendix. We adopt the usual tie-breaking rule throughout that consumers buy the good(s) if they are indifferent between buying and not buying. In addition, we note here that in the extreme cases of perfectly correlation, pure bundling with bundle price a is equivalent to separate sales with component prices $\frac{a}{2}$ and $\frac{a}{2}$, both being optimal when $\beta = 1$ and $\alpha = 1$.

Proposition 1 *The monopolist’s optimal pricing scheme (x_N^*, y_N^*, p_N^*) with degree- α negative correlation and symmetric support ($a_1 = a_2 = a$) features*

- (i) *Mixed Bundling for $\alpha \in [0, \frac{1}{3})$ with $x_N^* = y_N^* = \frac{2a}{3(1-\alpha)}$, and $p_N^* = \frac{(4-\sqrt{2+6\alpha})a}{3(1-\alpha)}$.*
- (ii) *Pure Bundling for $\alpha \in [\frac{1}{3}, 1]$ with $x_N^* = y_N^* = p_N^* = a$.*

Under symmetric supports, the monopolist sells both products as a pure bundle if (and ‘only if’ as we shall see) the magnitude of correlation α is sufficiently large and the monopolist opts for mixed bundling otherwise. The rough intuition is as follows: With independent valuations, mixed bundling with interior prices is optimal (Eckalbar (2010)). As α increases, a larger mass of consumers’ valuations “clusters” on the negative 45-degree line, giving rise to a higher demand for both the individual goods and the bundle. This induces the monopolist to increase both the bundle price and component prices, with component prices increasing faster to push more consumers to buy the bundle. This process continues until the components prices (simultaneously) hit the choke price (a), leading to pure bundling. Hence, although mixed bundling is a more flexible pricing strategy which effectively lowers valuation heterogeneity via a bundle price (i.e., capturing consumers with high ‘ $v_1 + v_2$ ’) and serves consumers with extreme values for individual goods via component prices (i.e., attracting consumers with high ‘ v_1 ’ or ‘ v_2 ’ only), a sufficient degree of negative correlation prompts the monopolist to abandon component prices entirely so as to increase profit by “forcing” consumers to purchase the bundle.

While it has been known that pure bundling is optimal with perfect negative correlation (Eckalbar (2010)) and that the symmetric case is the most favorable to pure bundling (Schmalensee (1984)), Proposition 1 illustrates that pure bundling can be (uniquely) optimal for symmetric and sufficiently negatively correlated valuations.¹⁸

Now consider the general case with asymmetric supports, i.e., $a_1 \geq a_2$. Intuitively, asymmetry in valuations creates at least two complications for the seller. *First*, with negative correlation, a bundle price loses some of its grip in lowering valuation heterogeneity under asymmetric supports in that a bundle price can no longer capture all consumers with valuations clustered on the negative diagonal line as effectively and cleanly as before.¹⁹ We shall see that an immediate consequence is that pure bundling ceases to be optimal whenever $a_1 > a_2$, indicating that pure bundling being optimal is indeed a rare event. *Second*, if a_1 is sufficiently larger than a_2 , the component price for good 1 should be much higher than that for good 2 but this creates a tension for the bundle price, which should be at least as large as the component price for good 1 but cannot be too large — so that enough buyers will buy the bundle. We shall see that this tension prompts the monopolist to opt for partial mixed bundling. With such intuition in place, we now present the optimal pricing scheme for the general case ($a_1 \geq a_2$):²⁰

Proposition 2 (Optimal Pricing Scheme under Negative Correlation) *The optimal pricing scheme (x_N^*, y_N^*, p_N^*) with degree- α negative correlation has the following structure:*

Case (i). *If $\alpha < \frac{2a_2 - a_1}{3a_1}$, the optimal pricing scheme is **mixed bundling** given by*

$$x_N^* = \frac{2a_1}{3(1-\alpha)}, y_N^* = \frac{2a_2}{3(1-\alpha)}, p_N^* = \frac{2(a_1 + a_2) - \sqrt{a_1 a_2 (2 + 6\alpha)}}{3(1-\alpha)}.$$

¹⁸ Admittedly (as also pointed out by a referee), our result that pure bundling can be uniquely optimal is very much a consequence of our specific setting, where an α -measure of consumers have valuations clustering *exactly* on the negative diagonal line.

¹⁹ Graphically, the bundle price is represented by a negative 45-degree line in $[0, a_1] \times [0, a_2]$, which is steeper than the negative diagonal line of $[0, a_1] \times [0, a_2]$ with measure- α consumers when $a_1 > a_2$.

²⁰ In presenting the optimal pricing scheme, we sometimes choose to present the solutions (x_N^*, y_N^*, p_N^*) implicitly, as the explicit solutions are inconveniently long. We show in the proof that the solutions exist and are uniquely pinned down except in the extreme cases of $\alpha = 1$ and $\beta = 1$.

If $\alpha \geq \frac{2a_2 - a_1}{3a_1}$, then the optimal pricing scheme is **pure bundling** if $a_1 = a_2$, and is **partial mixed bundling** as long as $a_1 > a_2$ where the monopolist stops selling good 1 by itself. Specifically, there is a unique threshold $\alpha(a_1) \in [0, 1)$ with $\alpha'(a_1) > 0$ such that

Case (ii). If $\alpha \geq \alpha(a_1)$, the optimal partial mixed bundling scheme is uniquely determined by

$$\begin{aligned} x_N^* &= p_N^*, \\ \frac{y_N^*}{a_2} + \frac{p_N^* - y_N^*}{a_1} &= 1, \\ 3a_2(1 - \alpha)\left(1 + \frac{a_1 - a_2}{2a_1}\right)\left(\frac{y_N^*}{a_2}\right)^2 - [3(1 - \alpha)a_2 + 2a_1]\left(\frac{y_N^*}{a_2}\right) + a_1 + a_2 &= 0. \end{aligned}$$

Case (iii). If $\alpha < \alpha(a_1)$, the optimal partial mixed bundling scheme is uniquely determined by

$$\begin{aligned} x_N^* &= p_N^*, \\ p_N^* &= -\frac{3(1 - \alpha)(a_1 - a_2)}{4a_2[a_1 - (1 - \alpha)a_2]}\left(y_N^*\right)^2 + \frac{(1 - \alpha)(a_1 - a_2)}{a_1 - (1 - \alpha)a_2}y_N^* + \frac{a_1}{2}, \\ \alpha\left(1 - \frac{2y_N^*}{a_2}\right) + (1 - \alpha)\frac{p_N^* - y_N^*}{a_1}\left(2 - \frac{3y_N^*}{a_2}\right) &= 0. \end{aligned}$$

Proposition 2 indicates that the monopolist's optimal pricing scheme, which can take various forms, is determined jointly by the correlation parameter (α) and the magnitudes of the supports a_1 and a_2 . Figure 3 graphically illustrates the optimal pricing schemes in Proposition 2.²¹

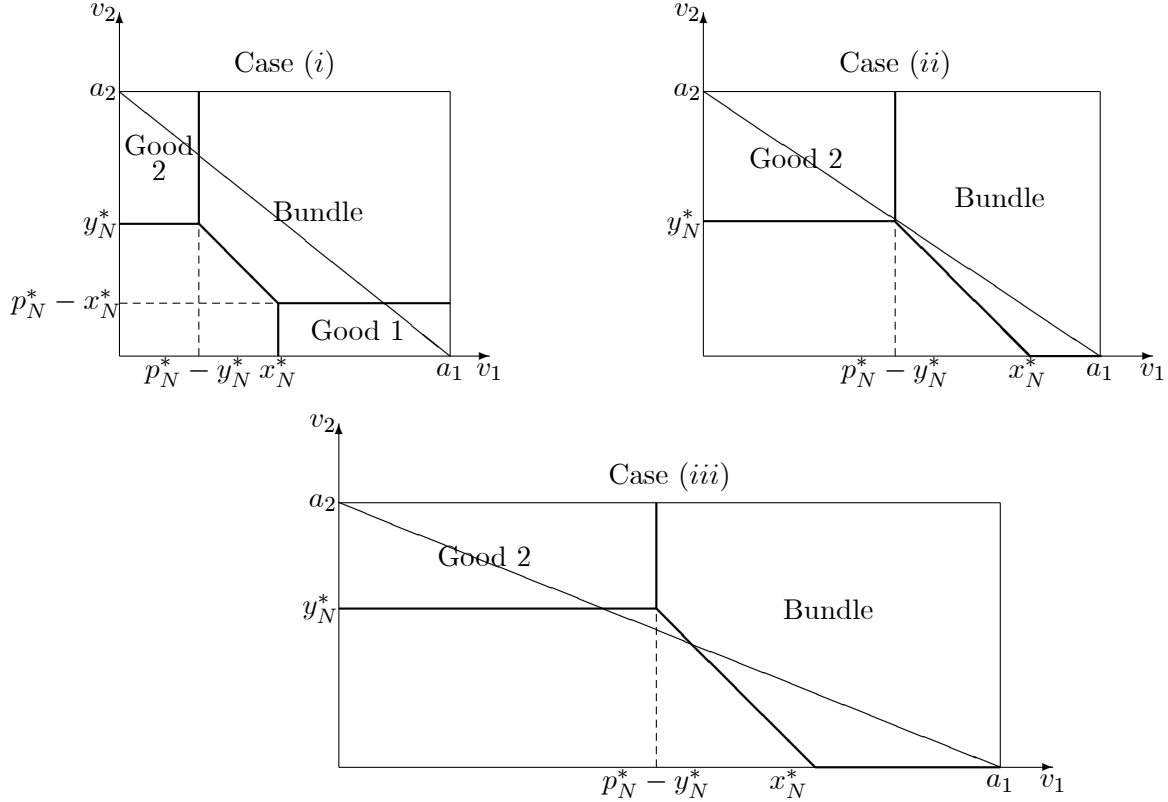


FIGURE 3. AN ILLUSTRATION OF THE OPTIMAL PRICING SCHEMES IN PROPOSITION 2.

Consumers with valuations in Area “Good 1” buy good 1 only (and similarly for Area “Good 2 and Area “Bundle”). The negative diagonal line has measure α consumers.

²¹Both case (ii) and case (iii) feature partial mixed bundling. The difference is that in case (ii) the threshold point $(p_N^* - y_N^*, y_N^*)$ lies exactly on the negative diagonal line and all consumers with valuations on the negative diagonal line buy the bundle, while in case (iii), the point $(p_N^* - y_N^*, y_N^*)$ lies above the negative diagonal line and some consumers with valuations on the negative diagonal line are excluded from consumption. We can however categorize the cases alternatively into mixed bundling ($\alpha < \frac{2a_2 - a_1}{3a_1}$), pure bundling ($\alpha \geq \frac{2a_2 - a_1}{3a_1}$ and $a_1 = a_2$), and partial mixed bundling ($\alpha \geq \frac{2a_2 - a_1}{3a_1}$ and $a_1 > a_2$).

Proposition 1 corresponds to case (i) of Proposition 2 when $a_1 = a_2$. As noted earlier, pure bundling is never optimal when $a_1 > a_2$, not surprising since pure bundling, designed to treat both goods indiscriminately, is a crude pricing strategy, particularly so when the goods are ex ante different. In addition, Proposition 2 is consistent with the result in Eckalbar (2010) on the optimal pricing scheme under independence, as shown in Corollary 1 (where part (a) is related to case (i) and part (b) is related to case (iii) of Proposition 2):

Corollary 1 (Optimal Pricing Scheme under Independence) *The monopolist's optimal pricing scheme with independent valuations ($\alpha = 0$) features*

- (a) *Mixed bundling for $a_1 \in [a_2, 2a_2]$ with $x_N^* = \frac{2a_1}{3}$, $y_N^* = \frac{2a_2}{3}$, and $p_N^* = \frac{2(a_1+a_2-\sqrt{2a_1a_2})}{3}$.*
- (b) *Partial mixed bundling for $a_1 > 2a_2$ with $x_N^* = p_N^* = \frac{a_1}{2} + \frac{a_2}{3}$ and $y_N^* = \frac{2a_2}{3}$.*

An implication of Proposition 2 is that mixed bundling is optimal when both support asymmetry and correlation are small while partial mixed bundling is optimal otherwise. Consider first the effect of asymmetric supports. Other things being equal, a larger a_1 implies higher demands for both good 1 and the bundle, resulting in higher component and bundle prices (x and p). To induce consumers to buy the bundle, x increases at a higher rate than p as a_1 increases. This however creates a tension with the constraint ' $x \leq p$ ', indicating that for some level of a_1 , the constraint ' $x \leq p$ ' is binding, i.e., good 1 is no longer sold separately. Next consider the effect of correlation. A higher degree of correlation typically makes it more likely for the monopolist to adopt a partial mixed bundling. Other things equal, a higher α implies a larger measure of consumers with valuations on the negative diagonal line, and hence attracting such consumers becomes more important. Roughly, with a higher α , forcing consumers with high v_1 to purchase the bundle is more profitable since such consumers are less likely to walk away and buy nothing. This prompts the monopolist to stop selling good 1 separately.

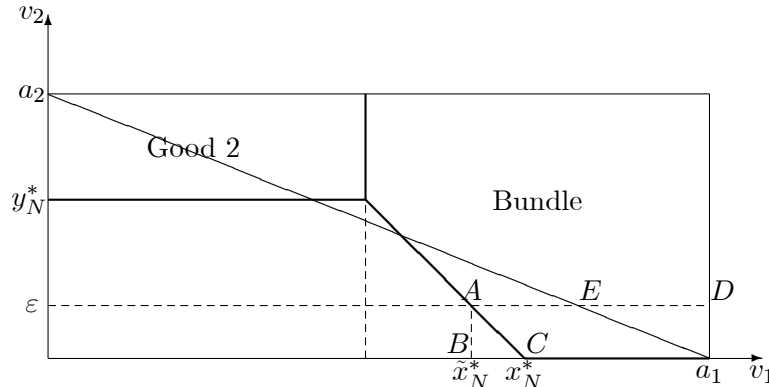


FIGURE 4. A MARGINAL ANALYSIS OF PROPOSITION 2 WITH MARGINAL VARIATION $\tilde{x}_N^* = p_N^* - \varepsilon$.

We use a marginal analysis as in McAfee et al. (1989) to illustrate the rationale of partial mixed bundling from a different angle. Consider a partial mixed bundling scheme with prices x_N^* and p_N^* (cases (ii) and (iii) in Figure 3). If we decrease x_N^* to $\tilde{x}_N^* = p_N^* - \varepsilon$ (keeping p_N^* and y_N^* the same as before), two opposing effects arise: There is extra revenue from some (new) buyers who now buy good 1 (i.e., consumers in ABC in Figure 4). At the same time, there is a net loss from some (old) buyers switching from buying the bundle to buying only good 1 (i.e., consumers in ACa_1D in Figure 4).²² With sufficiently large a_1 or α , the revenue increase from new consumers is dominated

²²To be specific, the extra revenue from the new consumers in ABC can be calculated explicitly as " $\varepsilon^2 (1 - \alpha) (p_N^* - \varepsilon) / 2a_1a_2$," while the revenue loss from switching consumers in ACa_1D (including those on Ea_1) is " $\varepsilon^2 (1 - \alpha) (2a_1 - 2x_N^* + \varepsilon) / 2a_1a_2 + \varepsilon^2 \alpha / a_2$." The revenue loss and revenue gain are exactly equal when $\alpha = \frac{2a_2 - a_1}{3a_1}$, resulting the exact threshold between mixed bundling and partial mixed bundling in Proposition 2.

by the revenue loss from switching consumers, rendering mixed bundling to be suboptimal.

We now briefly describe the role of the two thresholds in Proposition 2. The effective regions where the three optimal pricing prevail are illustrated in the (a_1, α) -parameter space in Figure 5. The threshold ‘ $\alpha = \frac{2a_2 - a_1}{3a_1}$ ’ separating case (i) and case (ii) of Proposition 2 is driven by the constraint ‘ $x \leq p$ ’ in the maximization problem (\mathcal{P}) . This threshold determines whether the optimal pricing scheme is mixed bundling or partial mixed bundling. The threshold ‘ $\alpha = \alpha(a_1)$ ’ differentiating case (ii) and case (iii) of Proposition 2 however is of less economic significance and results from a technical aspect of the Fréchet copula. In our setting, the probability distribution $F^N(v_1, v_2)$ of consumer valuations is not differentiable along the negative diagonal line, which has implications on the location of the optimal solution point $(p_N^* - y_N^*, y_N^*)$ in the rectangle, resulting in the threshold “ $\alpha = \alpha(a_1)$.”

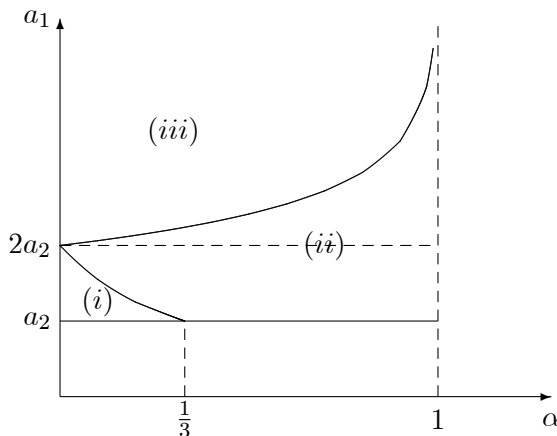


FIGURE 5. THE PARAMETER SPACE FOR THE THREE CASES OF PROPOSITION 2.

Proposition 2 enables us to obtain the following comparative statics for the general case $a_1 \geq a_2$:

Proposition 3 (Comparative Statics under Negative Correlation) *In problem (\mathcal{P}) with negative correlation, the monopolist’s optimal expected profit **strictly increases** in α .*

Hence, ceteris paribus the monopolist is strictly better off if the consumer valuations are more negative correlated. This is a well-expected result and the underlying rationale comes from the fact that other things fixed, the demands for both the individual good(s) and the bundle increase as the consumer valuations are more negatively correlated: While the monopolist faces an intricate task of adjusting the prices in response to a higher degree of correlation, the component prices and the bundle price can *all* be effectively adjusted as α changes (see Propositions 2).²³ As a result, both the component price(s) and the bundle price play an active role toward a better price discrimination outcome, making negative correlation a favorable setting for our multi-product monopolist. As we shall see, this is in drastic contrast to the positive correlation case where component prices play a rather passive role when consumers’ valuations become more positively correlated.

3.2 Positively Correlated Valuations

Compared to the negative correlation case, the bundling literature offers less insight on how positive correlation affects the profitability of bundling. Recently, Chen and Riordan (2013) establish in

²³This can also be confirmed by a direct comparative statics analysis of the optimal prices with respect to α . Our working paper contains such comparative statics results (for the optimal prices) for both the negative correlation case and the positive correlation case. These results are available upon request.

a general environment that bundling is strictly more profitable than separate sales if the degree of positive dependence of consumer valuations is not too large. As in Section 3.1, our objective here is to provide a complete characterization of the optimal pricing rule and a set of comparative statics analysis for the positive correlation case. Our main result is that short of perfect correlation, the optimal pricing rule always takes the form of mixed or partial mixed bundling. In addition, everything else being equal the monopolist's expected profit is always strictly decreasing in the degree of positive correlation of consumers' valuations.

We again begin our analysis with the simpler case of symmetric supports $a_1 = a_2 = a$.

Proposition 4 *The monopolist's optimal pricing scheme (x_P^*, y_P^*, p_P^*) with degree- β positive correlation and symmetric support features*

- (i) *Mixed Bundling for $\beta \in [0, 1)$: $x_P^* = y_P^* = \frac{2a}{3}$, $p_P^* = \frac{a(4-3\beta-\sqrt{2-2\beta+\beta^2})}{3(1-\beta)}$ with $\lim_{\beta \rightarrow 1} p_P^* = a$.*
- (ii) *Pure Bundling for $\beta = 1$: $x_P^* = y_P^* = p_P^* = a$.*

Proposition 4 shows that the optimal pricing scheme under symmetric supports is mixed bundling for all $\beta \in [0, 1)$ and hence mixed bundling strictly dominates separate sales (and pure bundling) for the entire range of positive dependence short of perfect correlation. An important difference between Proposition 4 and Proposition 1 is that the component prices x_P^* and y_P^* under positive correlation, unlike x_N^* and y_N^* , are completely independent of the correlation parameter β . While the forms of the component prices are specific to our setting, the main driving force of such difference comes from how differently consumer valuations are distributed in the positive and negative correlation cases. Such a phenomenon also arises in the general asymmetric support case and we shall come back to this point later.

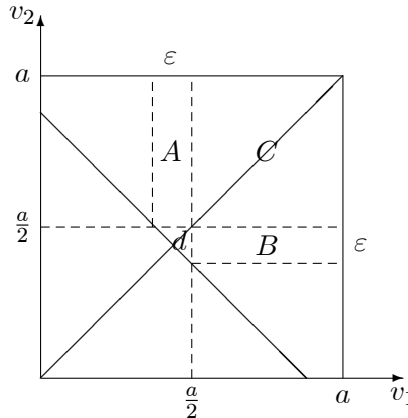


FIGURE 6. A MARGINAL ANALYSIS OF BUNDLING VS. SEPARATE SALES FOR PROPOSITION 4.

We now use a marginal analysis to illustrate Proposition 4. Starting from the optimal prices for separate sales, identically $(\frac{a}{2}, \frac{a}{2})$ for all $\beta \in [0, 1]$, consider a small discount of ε for purchasing the bundle. There are three marginal effects from this discount: revenue gains from consumers switching from buying one product to both products in areas “A” and “B”, revenue losses from consumers who already purchase the bundle in area “C”, and revenue gain from consumers switching from buying nothing to both products in area “d”. The net (first-order) effect on revenue can be found to be strictly positive for all $\beta \in [0, 1)$.²⁴ Notice that the above analysis also conveys the

²⁴Specifically, the revenue gains from areas “A” and “B” is $\frac{\varepsilon(\frac{a}{2}-\varepsilon)(1-\beta)}{a}$, the revenue loss from area “C” is $\frac{(1-\beta)\varepsilon}{4} + \frac{\beta\varepsilon}{2}$, and the revenue gain from area “d” can be calculated as $\frac{\varepsilon^2(1-\beta)(a-\varepsilon)}{2a^2} + \frac{\varepsilon\beta(a-\varepsilon)}{2a}$. Altogether, the net (first-order) effect on revenue is $\frac{1-\beta}{4}$.

intuition that for more general valuation distributions, separate sales may dominate bundled sales for highly positively correlated valuations since revenue losses from offering a bundle discount can be significant, as illustrated in Chen and Riordan (2013).

We now characterize the optimal pricing scheme for the general case with $a_1 \geq a_2$:

Proposition 5 (Optimal Pricing Scheme under Positive Correlation) *The monopolist's optimal pricing scheme (x_P^*, y_P^*, p_P^*) with degree- β positive correlation has the following structure:*

Case (i) The optimal pricing strategy is **mixed bundling** when $a_1 \in \left[a_2, \frac{a_2}{\tau(\beta)} \right]$:

$$x_P^* = \frac{2a_1}{3}, y_P^* = \frac{2a_2}{3},$$

$$p_P^* = \frac{2(1-\beta)(a_1+a_2)^2 + 2\beta a_1 a_2 - \sqrt{2a_1 a_2 [(1-\beta)(a_1^2 + a_2^2) + 2a_1 a_2 (\beta^2 - \beta + 1)]}}{3(1-\beta)(a_1 + a_2)}$$

with $\lim_{\beta \rightarrow 1} p_P^* = \frac{a_1 + a_2}{2}$.

Case (ii) The optimal pricing strategy is **partial mixed bundling** when $a_1 \in \left(\frac{a_2}{\tau(\beta)}, +\infty \right)$:

$$x_P^* = p_P^* = \frac{(a_1 + a_2)[2a_2(1-\beta) + 3a_1]}{6[a_1 + (1-\beta)a_2]}, y_P^* = \frac{2a_2}{3}, \text{ and } \lim_{\beta \rightarrow 1} p_P^* = \frac{a_1}{2}.$$

The threshold $\tau(\beta) = \frac{\sqrt{4\beta^2 - 4\beta + 9} - (1+2\beta)}{4(1-\beta)} \in \left[\frac{1}{3}, \frac{1}{2} \right]$ is strictly decreasing and strictly convex.

Proposition 5 indicates that the monopolist's optimal pricing scheme is either mixed bundling (case (i)) or partial mixed bundling (case (ii)). Figure 7 illustrates case (ii) and the (a_1, β) -parameter space of Proposition 5.

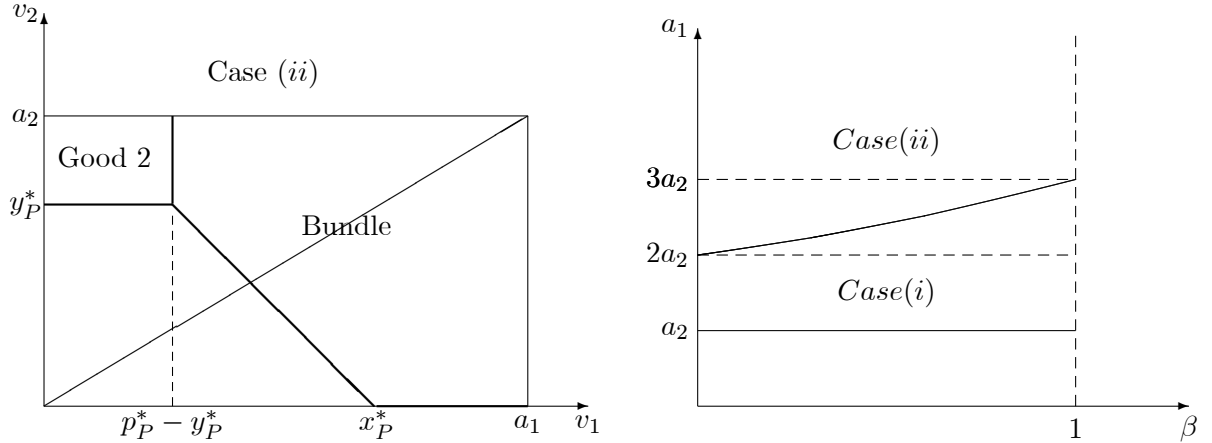


FIGURE 7. AN ILLUSTRATION OF PROPOSITION 5 AND THE CORRESPONDING PARAMETER SPACE.

The transition of the optimal pricing scheme from mixed bundling to partial mixed bundling is again shaped jointly by the magnitudes of correlation and support asymmetry. First, fixing β , a large support asymmetry prompts the monopolist to stop selling good 1 separately. The intuition is similar to that in the negative correlation case: As a_1 increases, ceteris paribus, the monopolist charges a higher price for good 1 to extract more surplus from consumers with large v_1 . This ultimately conflicts with the constraint ' $x \leq p$ ', leading to partial mixed bundling. The condition ' $a_1 \geq a_2/\tau(\beta)$ ' pins down exactly when this constraint is binding. Notice however that compared

to negative correlation, positive correlation is less likely to induce partial mixed bundling. In particular, with a small asymmetry between a_1 and a_2 , mixed bundling remains to be optimal for **all** $\beta \in [0, 1)$, while partial mixed bundling is optimal for negative correlation whenever the correlation α is large enough (Figure 5 vs. the right panel of Figure 7). The rationale of the difference is that under positive correlation, consumer valuations cluster uniformly along the positive diagonal line connecting $(0, 0)$ and (a_1, a_2) . Fixing a small support asymmetry, a larger β implies a higher demand for the bundle and lower demands for the individual goods, leading to strictly higher bundle price p and (weakly) lower component prices x and y . Such price changes effectively slacken the constraint ' $x \leq p$ ', rendering mixed bundling to be optimal for all $\beta \in [0, 1)$.

Proposition 5 leads to the following comparative statics of the monopolist's optimal profit:

Proposition 6 (Comparative Statics under Positive Correlation) *In problem (\mathcal{P}) with positive correlation, the monopolist's optimal expected profit **strictly decreases** in β .*

Proposition 6 agrees with a common belief in the literature that positive valuation correlation is a less favorable setting for a multi-product monopolist. Proposition 6 is in contrast to the case of negative correlation. In particular, component prices are much less effective here compared to the negative correlation case: As consumer valuations cluster more along the positive diagonal line, the component prices are independent of β and hence play a completely passive role (see Proposition 5). With negative correlation, however, a larger α implies more consumers with diametrically different valuations for the two goods, and component prices are always effective tools to induce the buyers to self-select into different market segments. Although the comparative statics result in Proposition 6 is specific for our setting, we believe such results should hold more generally: In a general setting, a higher degree of positive correlation implies more congregated consumer valuations around or near the positive diagonal line, resulting in fewer consumers with high valuations only for an individual good. Therefore, component prices would similarly become less important or effective in generating profits. Our analysis in Proposition 6 hence provides a novel perspective on the efficacy of pricing schemes when consumer valuations are positively correlated.

3.3 Welfare

One important issue missing in our analysis so far is a discussion of consumer surplus and total welfare under the optimal pricing schemes. The problem here is that it is technically difficult to explicitly calculate such welfare measures in our setting given that the form of the optimal pricing scheme changes as the parameters (the valuation correlation and asymmetric supports) change. To make progress, we conduct a set of Monte Carlo simulations to investigate how the (expected) consumer surplus and the (expected) total welfare vary with correlation and product asymmetry.²⁵

The following figures (Figure 9 and Figure 10) present our simulated consumer surplus and total welfare as functions of α and β respectively. Each set of curves consists of 5 cases with different

²⁵The Monte Carlo simulations are implemented using R-3.2.2. Take Figure 9 as an example. For each vector (α, a_1, a_2) with discretized $\alpha \in (0, 1)$, we first determine the optimal pricing scheme (x_N^*, y_N^*, p_N^*) according to Proposition 2. We then randomly generate 10^4 valuations (v_1, v_2) according to the copula in (1). The average consumer surplus and average total welfare are then calculated based on (x_N^*, y_N^*, p_N^*) . Finally, we use a simple smoothing algorithm – moving average with 20 points (to prevent overfitting) – to smooth the curves.

lengths of a_1 where $a_1 \in \{1, 1.5, 2, 2.5, 3\}$ and a_2 is fixed to be 1.²⁶

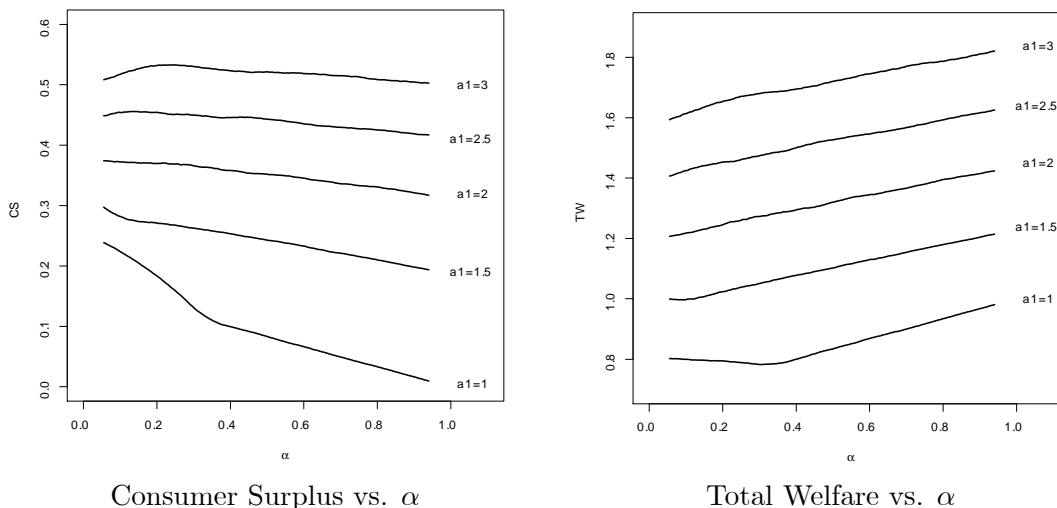


FIGURE 9. SIMULATED WELFARE FOR NEGATIVE CORRELATION WITH $a_2 = 1$.

In the negative correlation case in Figure 9, for the most part, the consumer surplus decreases while total welfare increases in α . These phenomena are mainly due to that the consumers' valuations are more predictable as α increases, enabling the monopolist to more finely price discriminate the consumers. Importantly, observe that such better price discrimination outcomes (as α increases) are achieved by adjusting the prices (x_N^*, y_N^*, p_N^*) more effectively, rather than by *excluding more consumers from consumption*, since the total welfare typically increases in α . Notice that an important caveat here is that the valuation distribution also changes as α increases and hence the monopolist can still serve relatively more consumers with higher prices. The same thing however is not true for positive correlation.

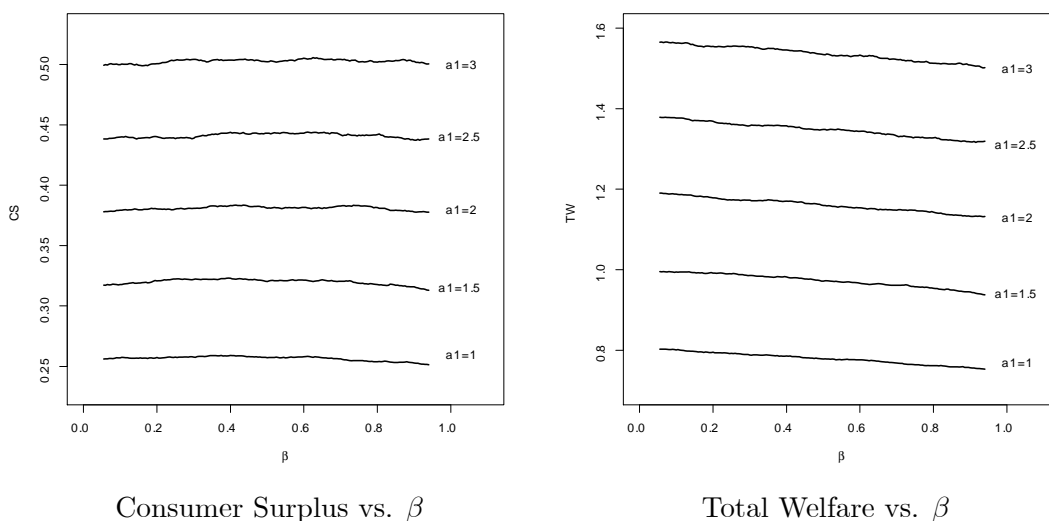


FIGURE 10. SIMULATED WELFARE FOR POSITIVE CORRELATION WITH $a_2 = 1$.

²⁶To provide a clean presentation, we have avoided labeling the cases in Propositions 2 and 5 in the figures (but the thresholds on α and β separating the cases in the simulation are indeed consistent with Propositions 2 and 5).

For the positive correlation case (Figure 10), the total welfare strictly decreases as β increases, as a result of higher bundle prices (from larger β 's) excluding more consumers from consumption. Unlike negative correlation, an increase in the bundle price here always excludes some consumers, given the β measure of consumer valuations on the positive diagonal line. For the consumer surplus, while it is difficult to exactly disentangle its overall “trend” in β (other than its variations being small), a comparison between the left panels of Figure 9 and Figure 10 reveals that more predictable consumer valuations do not result in a better price discrimination outcome for the monopolist. This is consistent with our previous findings in that component prices are less effective in the positive correlation case compared with the negative correlation case.

Finally, in both Figure 9 and Figure 10, both the consumer surplus and total welfare strictly increase in the degree of product asymmetry between the goods. This result is a direct consequence of that the marginal distribution of v_1 improves in the sense of first order stochastic dominance when a_1 increases (with a_2 being fixed).

4 Selling More to the Consumers via Randomization

The deterministic optimal pricing scheme discussed in Section 3 is a simple and commonly used mechanism. However, it is known in the literature that deterministic price-posting mechanisms may not be revenue-maximizing for a multiproduct monopolist selling to a single buyer, while for a single-product monopolist, posting an optimal price is revenue maximizing among all feasible mechanisms (e.g., Myerson (1981) and Samuelson (1984)). Indeed, several previous studies discussed in Section 1.1 show in various settings that for a multi-product monopolist, randomization in the assignment of goods can be more profitable than deterministic price-posting mechanisms.

We now present an interesting and direct application of our previous characterization of the pricing schemes. Our objective is not to characterize the revenue-maximizing mechanism in our setting. Rather, we ask the question of whether it is possible to improve profitability on the characterized optimal pricing schemes using a random assignment rule. Our starting point is the common slogan “selling more to your customers to increase your company’s profitability.” To this end, we consider a random mechanism where the monopolist forces its previous single-good buyers (in the price-posting mechanism) to purchase the bundle by offering a lottery. Similar to a partial mixed bundling which forces buyers with high valuations on good 1 to buy the bundle, such a random mechanism entails a trade-off between additional benefits and additional costs. We employ a marginal analysis to show that whenever the valuation distribution features negative correlation ($\alpha > 0$) and the monopolist’s optimal pricing scheme is partial mixed bundling, the monopolist can **strictly** improve its expected profit via such randomization.

We first describe the random assignment which builds directly on the optimal pricing scheme characterized in Section 3, denoted here as (x^*, y^*, p^*) .²⁷ Now instead of posting deterministic prices (x^*, y^*, p^*) , the monopolist offers a random mechanism, denoted as a price-allocation bundle (x^*, p^*, q) , where a consumer can choose one of three options: (A) purchase good 1 only at price x^* (this option is not available when (x^*, y^*, p^*) takes the form of partial mixed bundling), (B) purchase the bundle of both goods at price p^* , and (C) enter a lottery where with probability $(1 - q)$, the consumer purchases good 2 only at price y^* , and with probability q , the consumer purchases the bundle at price p^* .²⁸

²⁷Notice that our framework in Section 2 is equivalent to one where the monopolist sells the two indivisible goods to a *single buyer* who has additive and correlated valuations over the goods. In addition, as our purpose is to demonstrate revenue improvement using some types of random mechanisms, we omit the description of a full-fledged mechanism design problem for the monopolist.

²⁸We can consider a more general random mechanism by replacing option (A) with a similar lottery as option (C).

We can equivalently state option (C) as “purchase a lottery at price $(1 - q)y^* + qp^*$ to get the bundle with probability q and good 2 only with probability $(1 - q)$.” Indeed, the random mechanism is more appropriately understood as deterministic pricing with random allocation rather than random pricing.²⁹ Notice that as standard in mechanism design, we require certain commitment from the monopolist to carry out the transaction as stated. The lottery option, intermediate between buying good 2 and buying the bundle, can be alternatively interpreted as a result of limited quantity of good 2, as a slightly damaged or restrictive version of the bundle, or as a slightly enhanced version of good 2.

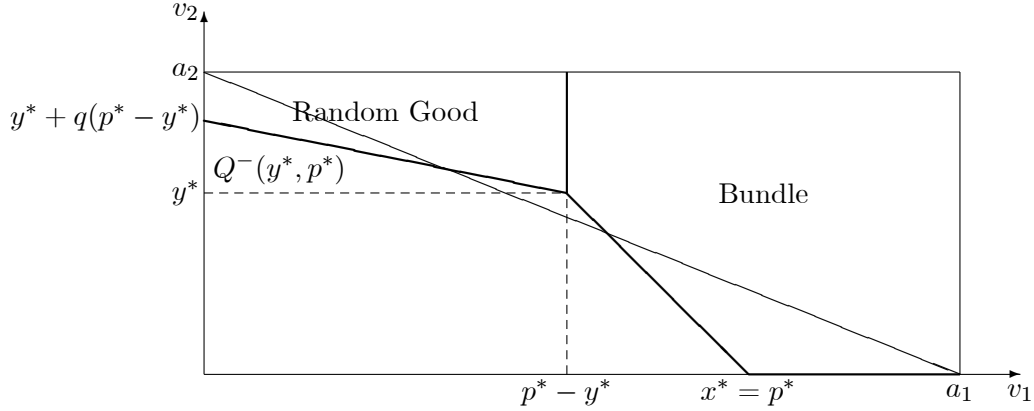


FIGURE 8. IMPROVING THE OPTIMAL PRICING SCHEME VIA RANDOMIZATION

Consumers with valuations in Area “Random Good” buy good 2 (resp., bundle) with probability $(1 - q)$ (resp., q), while consumers with valuations in Area “ $Q^-(y^*, p^*)$ ” stop buying anything.

Since the original optimal pricing scheme (x^*, y^*, p^*) is obtained by setting $q = 0$ in the random mechanism (x^*, p^*, q) , the random mechanism is ‘close’ to the pricing scheme when q is small. Nevertheless, the random mechanism induces different consumption behavior from that of the pricing scheme: While the decisions of consumers who purchase the bundle (or good 1 if available) are the same as before, the random mechanism excludes some previous good-2 buyers with low valuations on good 2, and at the same time forces the remaining previous good-2 buyers to purchase the bundle with positive probability. More precisely, for previous good-2 buyers with valuations (v_1, v_2) satisfying ‘ $q(v_1 + v_2 - p^*) + (1 - q)(v_2 - y^*) < 0$ ’, or ‘ $v_2 < y^* + q(p^* - y^* - v_1)$ ’, they stop buying anything under the random mechanism. For the remaining previous good-2 buyers, they now purchase good 2 with probability $(1 - q)$ and the bundle with probability q . Accordingly, compared with the pricing scheme, the total demand reduction $(Q^-(y^*, p^*))$ and the total demand increase $(Q^+(y^*, p^*))$ in the random mechanism are (see Figure 8):

$$\begin{aligned} Q^-(y^*, p^*) &= \int_0^{H_1(p^* - y^*)} [C_1^l(z, H_2(y^* + q(p^* - y^* - H_1^{-1}(z)))) - C_1^l(z, H_2(y^*))] dz, \\ Q^+(y^*, p^*) &= q [Q_2(y^*, p^*) - Q^-(y^*, p^*)]. \end{aligned} \quad (3)$$

The demand changes in (3) can be derived similarly as the demands in (2). Notice that while (3) offers a compact representation in copulas, the explicit demand changes can be calculated directly as measures of buyers in the relevant market segments (the complication here only arises in evaluating the measures of consumers whose valuations locate exactly on the negative diagonal line in $Q^-(y^*, p^*)$). Finally, the net change in expected profit for the monopolist from the changes

This however does not have any effect on our results in Proposition 7.

²⁹A case in point for the lottery option here is perhaps web shopping on Priceline where consumers can get killer travel deals by taking some risks of not knowing exactly which good they will get in the end.

in demand can be expressed as:

$$\Delta\Pi(q) = (p^* - y^*)Q^+(y^*, p^*) - y^*Q^-(y^*, p^*). \quad (4)$$

Our objective is to investigate whether the monopolist can improve its expected profit on the pricing scheme (x^*, y^*, p^*) by choosing a sufficiently small but positive probability q . The next proposition shows that the optimal partial mixed bundling pricing scheme identified in Proposition 2 (cases (ii) and (iii)) may *not* be revenue maximizing. Moreover, the random mechanism improves the monopolist's profit *only* in the negative correlation case.³⁰

Proposition 7 (Profit Improvement via Randomization) *Consider the random mechanism $(x^*, p^*; q)$ based on the optimal pricing scheme (x^*, y^*, p^*) in Section 3. For sufficiently small $q > 0$, the random mechanism $(x^*, p^*; q)$ generates strictly higher expected profit than the deterministic pricing scheme (x^*, y^*, p^*) if and only if consumers' valuations are negative correlated and the pricing scheme features partial mixed bundling (i.e., $a_1 > a_2$, $\alpha > 0$ and $\alpha \geq \frac{2a_2 - a_1}{3a_1}$).*

The intuition of Proposition 7 is as follows: By Proposition 2, the monopolist adopts partial mixed bundling in the presence of a large support asymmetry or a high degree of negative correlation, leaving only good 2 to be sold separately. A reasonable conjecture is that *forcing* some of the previous good-2 buyers to purchase the bundle improves profit so long as p^* is sufficiently greater than y^* , since the extra profit from an additional bundle buyer (induced by the random assignment) is $p^* - y^*$, while the lost revenue from excluding a previous good-2 buyer is y^* . Introducing the random component can improve profit only if the extra revenue exceeds the lost revenue.³¹ However, the mere requirement of p^* sufficiently exceeding y^* is not sufficient: Proposition 7 shows that for the randomization to improve profit, the magnitude of $Q^+(y^*, p^*)$ also has to be large compared to $Q^-(y^*, p^*)$, which only happens when the consumers' valuations are negatively correlated ($\alpha > 0$). As a result, profit improvement via randomization is possible only when the optimal pricing scheme is partial mixed bundling when either support asymmetry is large (so that p^* is relatively larger than y^*) or correlation is sufficiently high (large α). From a technical point of view, when the optimal pricing scheme is constrained to be partial mixed bundling, the monopolist is restricted from choosing component prices flexibly to some degree. This is especially true in the negative correlation case: While the optimal pricing scheme can also be partially mixed in the cases of independence and positive correlation, the optimal price of good 2 is fixed and not affected by whether the optimal pricing scheme is constrained to be partially mixed or not.³²

While we have not obtained the exact optimal selling mechanism, Proposition 7 demonstrates that random mechanisms can dominate deterministic mechanisms in revenue under negative correlation. In particular, Proposition 7 provides directions through which the monopolist can further increase its revenue on top of the deterministic pricing scheme. This is achieved along the lines of selling more products to the consumers and in some sense, our random mechanism also resembles the practice of offering both a standard version and a damaged version of a good by a seller.

³⁰Proposition 7 does not contradict (nor is it implied by) the result in Manelli and Vincent (2006) where a buyer's valuations over multiple indivisible goods are **independently** distributed. Moreover, it is consistent with the suggestion in Manelli and Vincent (2006) that "negative covariance of valuations poses problems" (Sec. 8. Conclusion).

³¹It can indeed be verified that in Proposition 2 $p_N^* - y_N^* < y_N^*$ in case (i) and $p_N^* - y_N^* > y_N^*$ in cases (ii) and (iii).

³²To be specific, the optimal component price of good 2 always has $y^* = \frac{2}{3}a_2$ regardless of whether the optimal pricing scheme is mixed or partially mixed in Corollary 1 and Proposition 5.

5 Conclusion

Product bundling is a prevalent business strategy that is important for both academic research and industry. We have proposed a useful multiproduct framework where consumers' valuations are ex ante asymmetric and statistically correlated across the products. We have analytically derived the optimal pricing schemes, together with a set of comparative statics results in the framework. We have also demonstrated how random mechanisms constructed along the lines of selling more products to the existing consumers can improve the monopolist's profit over that from the optimal deterministic price-posting mechanism. We have found that valuation correlation introduces several issues for a multiproduct monopolist. In particular, valuation correlation can crucially affect the form of the optimal pricing scheme and can raise the possibility that a random selling mechanism can outperform a deterministic price-posting mechanism.

Our multiproduct setting is undoubtedly specific, as it should be given our objective and the mathematical challenge of such problems in general. This however enables us to obtain explicit analytical solutions and allows us to delineate the efficacy of various pricing schemes under varying degrees of correlation and asymmetry more precisely. Overall, our results can provide useful intuition on optimal bundle pricing schemes for general multiproduct environments with correlated valuations.

Appendix³³

Proof of Proposition 2. Given that problem (\mathcal{P}) involves multiple constraints and parameters and the copula $C^N(H_1(x), H_2(p-x))$ (resp., $C^N(H_1(p-y), H_2(y))$) is not differentiable on $H_1(x) + H_2(p-x) = 1$ (resp., $H_1(p-y) + H_2(y) = 1$), we search for combinations of (x, y, p) that can be optimal, and then derive the associated parameter constellations for optimality. We consider four specific cases: **N1.** $H_1(x) + H_2(p-x) > 1$ and $H_1(p-y) + H_2(y) > 1$, **N2.** $H_1(x) + H_2(p-x) \leq 1$ and $H_1(p-y) + H_2(y) \leq 1$, **N3.** $H_1(x) + H_2(p-x) \leq 1$ and $H_1(p-y) + H_2(y) > 1$ and **N4.** $H_1(x) + H_2(p-x) > 1$ and $H_1(p-y) + H_2(y) \leq 1$. Importantly, dividing the problem into these cases also enables us to express $C^N(H_1(x), H_2(p-x))$ and $C^N(H_1(p-y), H_2(y))$ explicitly without the (inconvenient) “max” operator in (1). Roughly, **Case N1** corresponds to the situation where p is relatively large compared to x and y , i.e., both the key points $(x, p-x)$ and $(p-y, y)$ lie above the negative diagonal line. Similarly, **Case N2** roughly corresponds to the scenario where p is relatively small. And p is relatively close to x and is much larger than y in **Case N3**. Finally, since $a_1 \geq a_2$ and $p \leq x + y$, **Case N4** is impossible and hence is omitted.

In our proof (and also in the proof for Proposition 5), we will repeatedly use the following important argument: In all the maximization problems, the first-order derivatives are quadratic and convex in the corresponding choice variables. Our analysis can hence be focused on the corresponding first-order conditions. Accordingly, we typically select the smaller root of a first-order condition since the other root corresponds to a (local) minimizer.

Case N1: $H_1(x) + H_2(p-x) > 1$ and $H_1(p-y) + H_2(y) > 1$.

We will show that no price scheme (x, y, p) in **Case N1** can be optimal.

Given that $C^N(H_1(x), H_2(p-x)) = \alpha(H_1(x) + H_2(p-x) - 1) + (1 - \alpha)H_1(x)H_2(p-x)$ in this

³³Since Proposition 1 and Proposition 4 are respectively special cases of Proposition 2 and Proposition 5, we only present the proofs for Proposition 2 and Proposition 5 here.

case, the demands in (2) can be calculated as:

$$\begin{aligned} Q_1(x, p) &= (1 - \frac{x}{a_1})[\alpha + (1 - \alpha)\frac{p-x}{a_2}] \\ Q_2(y, p) &= (1 - \frac{y}{a_2})[\alpha + (1 - \alpha)\frac{p-y}{a_1}] \\ Q_{12}(x, y, p) &= (1 - \alpha)[\frac{x+y-p}{a_1} - \frac{(x+y-p)(p-x+y)}{2a_1a_2} + (1 - \frac{x}{a_1})(1 - \frac{p-x}{a_2})] \end{aligned}$$

The corresponding first-order conditions are (for example, $\frac{\partial \Pi(x, y, p)}{\partial x} = x \frac{\partial Q_1}{\partial x} + Q_1(x, p) + p \frac{\partial Q_{12}}{\partial x}$)

$$\frac{\partial \Pi(x, y, p)}{\partial x} = \alpha a_2(a_1 - 2x) + (1 - \alpha)(p - x)(2a_1 - 3x) = 0 \quad (\text{A1})$$

$$\frac{\partial \Pi(x, y, p)}{\partial y} = \alpha a_1(a_2 - 2y) + (1 - \alpha)(p - y)(2a_2 - 3y) = 0 \quad (\text{A2})$$

$$\frac{\partial \Pi(x, y, p)}{\partial p} = 4(p - x)(a_1 - x) + 2(p - y)(2a_2 - y) + (x + y - p)(3p - x + y) = 2a_1a_2 \quad (\text{A3})$$

Suppose there are optimal prices (x, y, p) . First, the two conditions defining **Case N1** immediately imply that $p > \max\{x, y\}$. Let $x_N(p, \alpha)$ and $y_N(p, \alpha)$ be the implicit solutions given p and α from (A1) – (A2). Now (A1) and (A2) imply that $x_N(p, \alpha) \in [\frac{a_1}{2}, \frac{2a_1}{3}]$ and $y_N(p, \alpha) \in [\frac{a_2}{2}, \frac{2a_2}{3}]$ for all p and $\alpha \in [0, 1]$. Now consider the two conditions defining **Case N1**:

$$\begin{aligned} H(x_N(p, \alpha)) + H(p - x_N(p, \alpha)) &> 1 \implies p > a_2 + \frac{a_1 - a_2}{a_1} x_N(p, \alpha) \\ H(p - y_N(p, \alpha)) + H(y_N(p, \alpha)) &> 1 \implies p < a_1 - \frac{a_1 - a_2}{a_2} y_N(p, \alpha) \end{aligned}$$

Hence, an optimal price p exists if

$$(a_1 - a_2) > (a_1 - a_2) \left(\frac{y_N(p, \alpha)}{a_2} + \frac{x_N(p, \alpha)}{a_1} \right) \quad (\text{A4})$$

However, given that $x_N(p, \alpha) \geq \frac{a_1}{2}$ and $y_N(p, \alpha) \geq \frac{a_2}{2}$, (A4) can never hold, a contradiction. We hence conclude that no price scheme (x, y, p) in **Case N1** can be optimal.

Case N2: $H_1(x) + H_2(p - x) \leq 1$ and $H_1(p - y) + H_2(y) \leq 1$.

We will show that **Case N2** characterizes cases (i) and (ii) of Proposition 2.

We first calculate the demands in this case as follows:

$$\begin{aligned} Q_1(x, p) &= \frac{p-x}{a_2} \left[1 - (1 - \alpha)\frac{x}{a_1} \right] \\ Q_2(y, p) &= \frac{p-y}{a_2} \left[1 - (1 - \alpha)\frac{y}{a_2} \right] \\ Q_{12}(x, y, p) &= \frac{x+y-p}{a_1} - (1 - \alpha)\frac{(x+y-p)(p-x+y)}{2a_1a_2} - \frac{p-x}{a_2} \left[1 - (1 - \alpha)\frac{x}{a_1} \right] - \frac{x}{a_1} + 1 \end{aligned}$$

The corresponding first-order conditions are

$$\frac{\partial \Pi(x, y, p)}{\partial x} = \frac{p-x}{a_2} \left[2 - 3(1 - \alpha)\frac{x}{a_1} \right] = 0 \quad (\text{A5})$$

$$\frac{\partial \Pi(x, y, p)}{\partial y} = \frac{p-y}{a_1} \left[2 - 3(1 - \alpha)\frac{y}{a_2} \right] = 0 \quad (\text{A6})$$

$$\frac{\partial \Pi(x, y, p)}{\partial p} = \frac{2(p-x)}{a_2} \left[\frac{(1-\alpha)x}{a_1} - 1 \right] + \frac{p-y}{a_1} \left[\frac{(1-\alpha)y}{a_2} - 2 \right] - \frac{(1-\alpha)(x+y-p)(3p-x+y)}{2a_1a_2} + 1 = 0 \quad (\text{A7})$$

With (A5)-(A7), any optimal bundle price should satisfy $p_N^* > \frac{2a_2}{3(1-\alpha)}$: If not, we have $x = y = p$ at the optimum by (A5) and (A6). (A7) then implies that $\frac{\partial \Pi}{\partial p} > 0$, or the monopolist should increase p , a contradiction.

Consider first the case of $\alpha < \frac{2a_2 - a_1}{3a_1}$. Simultaneously solving (A5)-(A7), we have x_N^* , y_N^* and p_N^* given in case (i) of Proposition 2, all corresponding to the smaller roots of (A5)-(A7) respectively. Further, one can immediately verify that the conditions $x_N^* < p_N^*$, $y_N^* < p_N^*$, $p_N^* < x_N^* + y_N^*$, $H_1(x_N^*) + H_2(p_N^* - x_N^*) < 1$ and $H_1(p_N^* - y_N^*) + H_2(y_N^*) < 1$ are all satisfied here.

Next consider the case of $\alpha \geq \frac{2a_2 - a_1}{3a_1}$. Notice that now $x_N^* \geq p_N^*$ if x_N^* and p_N^* are given case (i) of Proposition 2. Hence, the constraint $x \leq p$ must be binding when $\alpha \geq \frac{2a_2 - a_1}{3a_1}$. Moreover, notice that we must also have $H_1(p - y) + H_2(y) = 1$ at the optimum.³⁴ Suppose that $H_1(p - y) + H_2(y) < 1$. Then $y = \frac{2a_2}{3(1 - \alpha)}$ due to $p_N^* > \frac{2a_2}{3(1 - \alpha)}$ and (A6). With such y and $H_1(p - y) + H_2(y) = \frac{p - y}{a_1} + \frac{y}{a_2} < 1$, we have $p < \frac{(1 - 3\alpha)a_1}{3(1 - \alpha)} + \frac{2a_2}{3(1 - \alpha)}$. Substituting $y = \frac{2a_2}{3(1 - \alpha)}$ and $x = p$ into (A7) yields

$$\frac{\partial \Pi}{\partial p} = \frac{1}{3a_1} \left[\frac{3a_1(1 - \alpha) + 2a_2}{1 - \alpha} - 6p \right] > \frac{1}{3a_1(1 - \alpha)} [(1 + 3\alpha)a_1 - 2a_2] \geq 0$$

where the last inequality is due to $\alpha \geq \frac{2a_2 - a_1}{3a_1}$. This contradicts that such p is optimal.

Given $H_1(p - y) + H_2(y) = 1$ and $x = p$, the Kuhn-Tucker theorem implies that³⁵

$$\begin{aligned} \frac{\partial \Pi}{\partial y} - \lambda \left(\frac{\partial H_1}{\partial y} + \frac{\partial H_2}{\partial y} \right) &= \frac{p - y}{a_1} \left[2 - 3(1 - \alpha) \frac{y}{a_2} \right] - \lambda \left(\frac{1}{a_2} - \frac{1}{a_1} \right) = 0, \\ \frac{\partial \Pi}{\partial p} - \lambda \frac{\partial H_1}{\partial p} &= -\frac{2(p - y)}{a_1} \left[1 - (1 - \alpha) \frac{y}{a_2} \right] - (1 - \alpha) \frac{4py - y^2}{2a_1a_2} + 1 - \frac{\lambda}{a_1} = 0, \end{aligned}$$

where $\lambda > 0$ is the Lagrange multiplier for the constraint $H_1(p - y) + H_2(y) \leq 1$. For $a_1 > a_2$, we can derive the third equation in case (ii) of Proposition 2 by eliminating λ and p from the Kuhn-Tucker conditions using $H_1(p - y) + H_2(y) = 1$, i.e.,

$$3a_2(1 - \alpha) \left(1 + \frac{a_1 - a_2}{2a_1} \right) \left(\frac{y_N^*}{a_2} \right)^2 - [3(1 - \alpha)a_2 + 2a_1] \left(\frac{y_N^*}{a_2} \right) + a_1 + a_2 = 0. \quad (\text{A8})$$

Define the LHS of (A8) as $\kappa(y, \alpha)$, which is quadratic and strictly convex in y for $\alpha < 1$. An application of the Intermediate Value theorem to $\kappa(y, \alpha)$ shows the existence and uniqueness (by strict convexity) of $y_N^* \in (0, a_2)$ given $\kappa(0, \alpha) > 0$ and $\kappa(a_2, \alpha) < 0$ when $\alpha \geq \frac{2a_2 - a_1}{3a_1}$. Moreover, applying $a_1 = a_2 = a$ to case (ii) of Proposition 2 immediately leads to part (ii) of Proposition 1.³⁶

Case N3: $H_1(x) + H_2(p - x) \leq 1$ and $H_1(p - y) + H_2(y) > 1$.

We will show that **Case N3** characterizes case (iii) of Proposition 2.

Note first that the demand $Q_1(x, p)$ (resp., $Q_2(y, p)$) here is the same as that in **Case N2** (resp., **Case N1**). For the bundle demand, we have

$$Q_{12}(x, y, p) = 1 - \frac{(1 - \alpha)(p - y)}{a_1} - \frac{(1 - \alpha)(x + y - p)(p + y - x)}{2a_1a_2} - \frac{\alpha(p - a_2)}{a_1 - a_2} - \frac{p - x}{a_2} \left[1 - (1 - \alpha) \frac{x}{a_1} \right].$$

The first-order conditions are ((A5) in **Case N2** and (A2) in **Case N1** are reproduced below)

$$\frac{\partial \Pi(x, y, p)}{\partial x} = \frac{p - x}{a_2} \left[2 - 3(1 - \alpha) \frac{x}{a_1} \right] = 0 \quad (\text{A5})$$

$$\frac{\partial \Pi(x, y, p)}{\partial y} = \alpha \left(1 - \frac{2y}{a_2} \right) + (1 - \alpha) \frac{p - y}{a_1} \left(2 - \frac{3y}{a_2} \right) = 0 \quad (\text{A2})$$

$$\frac{\partial \Pi(x, y, p)}{\partial p} = (1 - \alpha) \frac{p - y}{a_1} \left(\frac{y}{a_2} - 2 \right) + \frac{2(p - x)}{a_2} \left[\frac{(1 - \alpha)x}{a_1} - 1 \right] - \frac{\alpha(2p - a_2)}{a_1 - a_2} - \frac{(1 - \alpha)(x + y - p)(3p - x + y)}{2a_1a_2} + 1 = 0. \quad (\text{A9})$$

³⁴This also means that the solution point $(p_N^* - y_N^*, y_N^*)$ lies on the negative diagonal line.

³⁵It can be verified that the Hessian matrix of the corresponding Lagrangian is negative definite if $y \leq \frac{2a_2}{3(1 - \alpha)}$, which holds here since if $y > \frac{2a_2}{3(1 - \alpha)}$, we have (1) no (p, y) can satisfy $H_1(p - y) + H_2(y) = 1$ and (2) $\frac{\partial \Pi}{\partial y} < 0$.

³⁶The (unique) threshold $\alpha(a_1)$ will be determined in **Case N3** below.

We first show that there is partial mixed bundling ($x = p$) at the optimum. Suppose not, or $p > x$. The condition $H_1(p - y) + H_2(y) > 1$ implies that y is also interior, i.e., $y < p$. We hence have $x = \frac{2a_1}{3(1-\alpha)}$ by (A5) and $y \in [\frac{a_2}{2}, \frac{2a_2}{3}]$ by (A2). Using the conditions in **Case N3**, we have

$$\begin{aligned} H_1(x) + H_2(p - x) &\leq 1 : p \leq a_2 + \frac{a_1 - a_2}{a_1}x = a_2 + \frac{a_1 - a_2}{3(1-\alpha)} \\ H_1(p - y) + H_2(y) &> 1 : p > a_1 - \frac{a_1 - a_2}{a_2}y \geq \frac{a_1 + a_2}{2} \\ \Rightarrow a_2 + \frac{a_1 - a_2}{3(1-\alpha)} &> \frac{a_1 + a_2}{2} \text{ or } \alpha > \frac{1}{3}. \end{aligned}$$

However, this implies that $x = \frac{2a_1}{3(1-\alpha)} > a_1$, which is never optimal. A contradiction. Notice that given we have partial mixed bundling in this case, the condition $\alpha \geq \frac{2a_2 - a_1}{3a_1}$ is also satisfied here.

We next show that given $x = p$, there are unique solutions y_N^* and p_N^* , written here respectively as $y(\alpha)$ and $p(\alpha)$, by simultaneously solving (A2) and (A9).³⁷ This is immediate for the simple cases of $\alpha = 0$ and $\alpha = 1$, where $y(0) = \frac{2a_2}{3}$, $p(0) = \frac{a_1}{2} + \frac{a_2}{3}$, $y(1) = \frac{a_2}{2}$ and $p(1) = \frac{a_1}{2}$. For the case of $\alpha \in (0, 1)$, we derive two bundle prices $p_1(y, \alpha)$ and $p_2(y, \alpha)$ from (A9) and (A2) respectively as

$$\begin{aligned} p_1(y, \alpha) &= -\frac{3(1-\alpha)(a_1 - a_2)}{4a_2[a_1 - (1-\alpha)a_2]}y^2 + \frac{(1-\alpha)(a_1 - a_2)}{a_1 - (1-\alpha)a_2}y + \frac{a_1}{2}, \\ p_2(y, \alpha) &= y + \frac{\alpha}{1-\alpha} \frac{a_1(2y - a_2)}{2a_2 - 3y}. \end{aligned}$$

Define a function $\delta(y, \alpha) = p_1(y, \alpha) - p_2(y, \alpha)$. Applying the Intermediate Value theorem to $\delta(y, \alpha)$ on $y \in [\frac{a_2}{2}, \frac{2a_2}{3}]$ and noting that $\delta(y, \alpha)$ strictly decreases in y for $y > \frac{a_2}{2}$, we obtain the existence and uniqueness of solution $y(\alpha)$ solving the equations that define $\delta(y, \alpha)$. The existence and uniqueness of $y(\alpha)$ also establishes the existence and uniqueness of solution $p(\alpha)$.

Finally, we are left to show that there is a unique threshold that separates **Case N2** and **Case N3**, i.e., there is a unique $\alpha(a_1) \in [0, 1)$ with $\alpha'(a_1) > 0$ such that $H_1(p(\alpha) - y(\alpha)) + H_2(y(\alpha)) > 1$ for all $\alpha < \alpha(a_1)$ and $H_1(p(\alpha) - y(\alpha)) + H_2(y(\alpha)) \leq 1$ for all $\alpha \geq \alpha(a_1)$. And $\alpha(a_1)$ exists whenever we have $2a_2 \geq a_1$. First, suppose we are in **Case N3** and define a function

$$w(\alpha, a_1) = H_1(p(\alpha) - y(\alpha)) + H_2(y(\alpha)) - 1 = \frac{p(\alpha) - y(\alpha)}{a_1} + \frac{y(\alpha)}{a_2} - 1.$$

First, it can be verified that $p'(\alpha) = \frac{\partial p_N^*}{\partial \alpha} = \frac{\partial x_N^*}{\partial \alpha} < 0$ and $y'(\alpha) = \frac{\partial y_N^*}{\partial \alpha} < 0$, which imply that w is strictly decreasing in α . Further, using our previous results on $y(0)$, $p(0)$, $y(1)$ and $p(1)$, we have

$$w(0, a_1) = \frac{1}{6} - \frac{a_2}{3a_1} \geq (<)0 \Leftrightarrow a_1 \geq (<)2a_2 \text{ and } w(1, a_1) = -\frac{a_2}{2a_1} < 0.$$

The Intermediate Value theorem implies that there exists a unique $\alpha(a_1) \in [0, 1)$ such that $w(\alpha(a_1), a_1) = 0$, and this is true if and only if $a_1 \geq 2a_2$ (and we can verify that $\alpha(2a_2) = 0$).

To show $\alpha'(a_1) > 0$, recall that $\alpha(a_1)$ is implicitly defined by $w(\alpha, a_1) = 0$. Treating α as a function of a_1 and totally differentiating $w(\alpha, a_1) = 0$ w.r.t. a_1 , we have

$$\frac{\partial w}{\partial \alpha} \alpha'(a_1) + \frac{\partial w}{\partial a_1} = 0.$$

Given that $\frac{\partial w}{\partial \alpha} < 0$, we shall have $\alpha'(a_1) > 0$ if $\frac{\partial w}{\partial a_1} > 0$. First, direct calculation implies that

$$\frac{\partial p_1(y, \alpha)}{\partial a_1} = \left(1 - \frac{3y}{4a_2}\right) \frac{\alpha a_2 y (a_1 - a_2)}{[a_1 - (1-\alpha)a_2]^2} + \frac{1}{2} > 0, \text{ and } \frac{\partial p_2(y, \alpha)}{\partial a_1} = \frac{\alpha(2y - a_2)}{(1-\alpha)(2a_2 - 3y)} \leq 0,$$

³⁷One can again verify that the associated Hessian matrix is negative definite if $y \leq \frac{2a_2}{3}$, which is implied by (A2).

which hold by $y \in [\frac{a_2}{2}, \frac{2a_2}{3}]$. We hence have $\frac{\partial \delta}{\partial a_1} > 0$. Together with $\frac{\partial \delta(y, \alpha)}{\partial y} < 0$, we have $\frac{\partial y(\alpha)}{\partial a_1} > 0$. Further, using the first-order condition (A2) directly, we have

$$\frac{\partial \left[\frac{p(\alpha) - y(\alpha)}{a_1} \right]}{\partial a_1} = \frac{\alpha a_2}{(1 - \alpha)[2a_2 - 3y(\alpha)]^2} \frac{\partial y(\alpha)}{\partial a_1} > 0.$$

By the definition of $w(\alpha, a_1)$, we have $\frac{\partial w}{\partial a_1} > 0$, proving $\alpha'(a_1) > 0$.³⁸

Finally, notice that our above discussion also implies that if either $a_1 < 2a_2$ (so that $w(0, a_1) < 0$ and $H_1(p(\alpha) - y(\alpha)) + H_2(y(\alpha)) < 1$ for all $\alpha \geq 0$) or $\alpha \geq \alpha(a_1)$ holds, it is then impossible for the monopolist's optimal pricing scheme to be in **Case N3** or case (iii). This implies that if $\alpha < \alpha(a_1)$ (and hence $a_1 > 2a_2$ which automatically implies that $\alpha \geq \frac{2a_2 - a_1}{3a_1}$), the optimal pricing scheme will be in **Case N3** representing case (iii) of Proposition 2, while if $\alpha \geq \alpha(a_1)$ and $\alpha \geq \frac{2a_2 - a_1}{3a_1}$, the optimal pricing scheme will be in **Case N2**, featuring case (ii) of Proposition 2. ■

Proof of Proposition 3. We apply the Envelope Theorem to show $\frac{\partial \Pi(x_N^*, y_N^*, p_N^*)}{\partial \alpha} > 0$.

Case (i): $\alpha < \frac{2a_2 - a_1}{3a_1}$.

$$\begin{aligned} \frac{\partial \Pi(x_N^*, y_N^*, p_N^*)}{\partial \alpha} &= \left\{ \begin{aligned} &x_N^* H_1(x_N^*) H_2(p_N^* - x_N^*) + y_N^* H_1(p_N^* - y_N^*) H_2(y_N^*) \\ &+ p_N^* \left[\int_{H_1(p_N^* - y_N^*)}^{H_1(x_N^*)} H_2(p_N^* - H_1^{-1}(z)) dz - H_1(p_N^* - y_N^*) H_2(y_N^*) \right] \end{aligned} \right\} \\ &= \frac{(x_N^*)^2 (p_N^* - x_N^*)}{a_1 a_2} + \frac{(y_N^*)^2 (p_N^* - y_N^*)}{a_1 a_2} + \frac{p_N^* [(x_N^*)^2 + (y_N^*)^2 - (p_N^*)^2]}{2a_1 a_2} \\ &> 0, \end{aligned}$$

where the last inequality is due to $(x_N^*)^2 + (y_N^*)^2 > (p_N^*)^2$, which can be verified directly.

Case (ii): $\alpha \geq \alpha(a_1)$ and $\alpha \geq \frac{2a_2 - a_1}{3a_1}$.

$$\begin{aligned} \frac{\partial \Pi(x_N^*, y_N^*, p_N^*)}{\partial \alpha} &= \left\{ \begin{aligned} &y_N^* (1 - H_2(y_N^*)) (1 - H_1(p_N^* - y_N^*)) \\ &+ p_N^* \left[\int_{H_1(p_N^* - y_N^*)}^{H_1(x_N^*)} H_2(p_N^* - H_1^{-1}(z)) dz + H_1(p_N^* - y_N^*) - H_1 \left(\frac{a_1(p_N^* - a_2)}{a_1 - a_2} \right) \right] \end{aligned} \right\} \\ &= y_N^* H_2(y_N^*) (1 - H_2(y_N^*)) + p_N^* \int_{H_1(p_N^* - y_N^*)}^{H_1(x_N^*)} H_2(p_N^* - H_1^{-1}(z)) dz > 0, \end{aligned}$$

where the second equality follows from $H_2(y_N^*) + H_1(p_N^* - y_N^*) = 1$.

Case (iii): $\alpha < \alpha(a_1)$.

$$\begin{aligned} \frac{\partial \Pi(x_N^*, y_N^*, p_N^*)}{\partial \alpha} &= \left\{ \begin{aligned} &y_N^* (1 - H_2(y_N^*)) (1 - H_1(p_N^* - y_N^*)) \\ &+ p_N^* \left[\int_{H_1(p_N^* - y_N^*)}^{H_1(x_N^*)} H_2(p_N^* - H_1^{-1}(z)) dz + H_1(p_N^* - y_N^*) - H_1 \left(\frac{a_1(p_N^* - a_2)}{a_1 - a_2} \right) \right] \end{aligned} \right\} \\ &= p_N^* \left[\frac{y_N^*}{p_N^*} \left(1 - \frac{y_N^*}{a_2} \left(1 - \frac{p_N^* - y_N^*}{a_1} \right) \right) + \frac{(y_N^*)^2}{2a_1 a_2} + \frac{p_N^* - y_N^*}{a_1} - \frac{p_N^* - a_1}{a_1 - a_2} \right] \quad (\text{by } x_N^* = p_N^*) \\ &\equiv p_N^* \eta(y_N^*, p_N^*). \end{aligned}$$

Next, notice that

$$\begin{aligned} \frac{\partial \eta(y_N^*, p_N^*)}{\partial y_N^*} &= \frac{-3(y_N^*)^2 - 2(a_1 - a_2)y_N^* + 3p_N^* y_N^* + a_2(a_1 - 2p_N^*)}{a_1 a_2 p_N^*} \\ &< \frac{3y_N^*(p_N^* - y_N^*) - a_1 a_2}{a_1 a_2 p_N^*} \leq \frac{3y_N^* \left(\frac{a_1}{2} - \frac{a_2}{6} \right) - a_1 a_2}{a_1 a_2 p_N^*} \leq \frac{-a_2}{3a_1 p_N^*} < 0, \end{aligned}$$

³⁸The result that $\alpha'(a_1) > 0$ gives the shape of the curve that separates case (ii) and case (iii) in Figure 5, though notice that the above mentioned curve in Figure 5 is the inverse of $\alpha(a_1)$.

where the first inequality is due to $H_2(y_N^*) + H_1(p_N^* - y_N^*) = \frac{y_N^*}{a_2} + \frac{p_N^* - y_N^*}{a_1} > 1$, while the second and third follow from $p_N^* \leq \frac{a_1}{2} + \frac{a_2}{3}$ and $\frac{2a_2}{3} \geq y_N^* \geq \frac{a_2}{2}$ as $\alpha < \alpha(a_1)$. Similarly, we have

$$\frac{\partial \eta(y_N^*, p_N^*)}{\partial p_N^*} = \frac{(y_N^* - a_2)[y_N^*(a_1 - a_2)(a_2 + y_N^*) + a_2(p_N^*)^2]}{a_1 a_2 p_N^* (a_1 - a_2)} - \frac{y_N^*}{a_1(a_1 - a_2)} < 0.$$

Finally, one can check that

$$\eta(y_N^*|_{\alpha=0}, p_N^*|_{\alpha=0}) = \eta\left(\frac{2a_2}{3}, \frac{a_1}{2} + \frac{a_2}{3}\right) = \frac{3a_1 a_2 + 2a_2(a_1 - a_2)}{18a_1(a_1 - a_2)} > 0,$$

implying that $\eta(y_N^*, p_N^*) > 0$ for all $\alpha < \alpha(a_1)$.³⁹ ■

Proof of Proposition 5. Since copulas $C^P(H_1(x), H_2(p-x))$ and $C^P(H_1(p-y), H_2(y))$ are not differentiable when $H_1(x) = H_2(p-x)$ and $H_1(p-y) = H_2(y)$ respectively, we similarly discuss four cases: **P1.** $H_2(y) \geq H_1(p-y)$ and $H_1(x) < H_2(p-x)$, **P2.** $H_2(y) \geq H_1(p-y)$ and $H_1(x) \geq H_2(p-x)$, **P3.** $H_2(y) < H_1(p-y)$ and $H_1(x) \geq H_2(p-x)$, and **P4.** $H_2(y) < H_1(p-y)$ and $H_1(x) < H_2(p-x)$. **Case P4** here is impossible and can be dropped as $p \leq x + y$. Notice further that **Case P1** and **Case P3** can be treated symmetrically, and we hence only discuss **Case P1** and **Case P2** below. As in the proof of Proposition 2, dividing the problem into these cases enables us to write out $C^P(H_1(x), H_2(p-x))$ and $C^P(H_1(p-y), H_2(y))$ explicitly without the (inconvenient) “min” operator in (1).

Case P1: $H_2(y) \geq H_1(p-y)$ and $H_1(x) < H_2(p-x)$.

The demand functions can be explicitly expressed as

$$\begin{aligned} Q_1(x, p) &= \frac{p-x}{a_2} \left[1 - (1-\beta) \frac{x}{a_1} \right] - \beta \frac{x}{a_1}, \\ Q_2(y, p) &= (1-\beta) \frac{p-y}{a_1} \left(1 - \frac{y}{a_2} \right), \\ Q_{12}(x, y, p) &= (1-\beta) \left[\frac{x+y-p}{a_1} - \frac{(x+y-p)(p-x+y)}{2a_1 a_2} \right] + \left(1 - \frac{p-x}{a_2} \right) \left[1 - (1-\beta) \frac{x}{a_1} \right]. \end{aligned}$$

The first-order conditions with regard to x and y can be derived as

$$\frac{\partial \Pi(x, y, p)}{\partial x} = \frac{p-x}{a_2} \left[2 - 3(1-\beta) \frac{x}{a_1} \right] - 2\beta \frac{x}{a_1} = 0 \quad (\text{A10})$$

$$\frac{\partial \Pi(x, y, p)}{\partial y} = (1-\beta) \frac{p-y}{a_1} (2 - \frac{3y}{a_2}) = 0 \quad (\text{A11})$$

We will show that no optimal price scheme exists in **Case P1**. Suppose not and (x, y, p) are optimal prices here. First, notice that

$$H_1(x) < H_2(p-x) \text{ and } H_2(y) \geq H_1(p-y) \implies \frac{(a_1 + a_2)}{a_1} x < p \leq \frac{(a_1 + a_2)}{a_2} y.$$

In addition, (A11) implies that $y = \frac{2a_2}{3}$ at the optimum. Consider an alternative first-order condition $\frac{\partial \Pi'}{\partial x} = (1-\beta) \frac{p-x}{a_2} [2 - \frac{3x}{a_1}] = 0$. It can be verified that $\frac{\partial \Pi}{\partial x} - \frac{\partial \Pi'}{\partial x} = -2 \frac{\beta}{a_1 a_2} (x a_1 - p a_1 + x a_2) > 0$ given $a_1 p > (a_1 + a_2) x$. Since both $\frac{\partial \Pi}{\partial x}$ and $\frac{\partial \Pi'}{\partial x}$ are convex in x , (A10) and $\frac{\partial \Pi}{\partial x} > \frac{\partial \Pi'}{\partial x}$ jointly imply that $x > \frac{2a_1}{3}$ at the optimum. However, the result that $x > \frac{2a_1}{3}$ and $y = \frac{2a_2}{3}$ obviously contradicts the above implication from the two conditions defining **Case P1**.

³⁹Strictly speaking, this $(\eta(y_N^*, p_N^*) > 0)$ also requires $\frac{\partial y_N^*}{\partial \alpha} < 0$ and $\frac{\partial p_N^*}{\partial \alpha} < 0$. The latter can be verified directly here. This has also been obtained in our working paper.

Case P2: $H_2(y) \geq H_1(p-y)$ and $H_1(x) \geq H_2(p-x)$.

The demand functions are as follows:

$$Q_1(x, p) = (1 - \beta) \frac{p-x}{a_2} \left(1 - \frac{x}{a_1}\right),$$

$$Q_2(y, p) = (1 - \beta) \frac{p-y}{a_1} \left(1 - \frac{y}{a_2}\right)$$

$$Q_{12}(x, y, p) = (1 - \beta) \left[\frac{x+y-p}{a_1} - \frac{(x+y-p)(p-x+y)}{2a_1a_2} \right] + \beta \left(\frac{x}{a_1} - \frac{p}{a_1+a_2} \right) + (1 - \frac{x}{a_1}) \left[1 - (1 - \beta) \frac{p-x}{a_2} \right]$$

The corresponding first-order conditions can then be derived as

$$\frac{\partial \Pi(x, y, p)}{\partial x} = (1 - \beta) \frac{p-x}{a_2} \left(2 - \frac{3x}{a_1}\right) = 0 \quad (\text{A12})$$

$$\frac{\partial \Pi(x, y, p)}{\partial y} = (1 - \beta) \frac{p-y}{a_1} \left(2 - \frac{3y}{a_2}\right) = 0 \quad (\text{A13})$$

$$\frac{\partial \Pi(x, y, p)}{\partial p} = 1 - (1 - \beta) \left[\frac{2(1 - \frac{x}{a_1})(p-x)}{a_2} + \frac{(p-y)(2 - \frac{y}{a_2})}{a_1} + \frac{(x+y-p)(3p-x+y)}{2a_1a_2} \right] - \beta \frac{2p}{a_1+a_2} = 0 \quad (\text{A14})$$

We first show that no $p \leq \frac{2a_2}{3}$ can be optimal. For any $p \leq \frac{2a_2}{3}$, (A12) and (A13) imply respectively that $x = p$ and $y = p$ at the optimum. Plugging these prices into (A14), we have

$$\frac{\partial \Pi}{\partial p} = 1 - \frac{2\beta p}{a_1+a_2} - \frac{3(1-\beta)p^2}{2a_1a_2} \geq 1 - \frac{4\beta a_2}{3(a_1+a_2)} - \frac{2(1-\beta)a_2}{3a_1} \geq \frac{1}{3},$$

where the two inequalities follow from $p \leq \frac{2a_2}{3}$ and $a_1 \geq a_2$ respectively. Hence, the monopolist's profit strictly increases in p for all $p \leq \frac{2a_2}{3}$.

Define a threshold $\tau(\beta) = \frac{\sqrt{4\beta^2 - 4\beta + 9} - (1+2\beta)}{4(1-\beta)}$. It can be verified that $\tau(0) = \frac{1}{2}$, $\lim_{\beta \rightarrow 1} \tau(\beta) = \frac{1}{3}$ and $\tau'(\beta) < 0$. We now solve for the optimal prices (x_P^*, y_P^*, p_P^*) . First consider mixed bundling where $p_P^* \geq \frac{2a_1}{3}$. Such p_P^* implies that $x_P^* = \frac{2a_1}{3}$ and $y_P^* = \frac{2a_2}{3}$, both being the smaller solutions to (A12) and (A13) respectively. Next, plug $x_P^* = \frac{2a_1}{3}$ and $y_P^* = \frac{2a_2}{3}$ into (A14) to obtain

$$\frac{\partial \Pi}{\partial p} = \frac{3(1-\beta)}{2a_1a_2} p^2 - 2 \left(\frac{(1-\beta)(a_1+a_2)}{a_1a_2} + \frac{\beta}{a_1+a_2} \right) p + \frac{2(1-\beta)(a_1^2+a_2^2)}{3a_1a_2} + 1 = 0. \quad (\text{A15})$$

One can verify that p_P^* given in case (i) of Proposition 5 is the smaller root of the quadratic equation (A15). In addition, we have that in this case, $a_1 \leq \frac{a_2}{\tau(\beta)}$ implies $p_P^* \geq \frac{2a_1}{3}$. We next verify that the prices (x_P^*, y_P^*, p_P^*) are consistent with $H_2(y_P^*) \geq H_1(p_P^* - y_P^*)$ and $H_1(x_P^*) \geq H_2(p_P^* - x_P^*)$, both identically reducing to $p_P^* \leq \frac{2(a_1+a_1)}{3}$ given $x_P^* = \frac{2a_1}{3}$ and $y_P^* = \frac{2a_2}{3}$. By (A15), $\frac{\partial \Pi}{\partial p}$ is convex and quadratic in p . It can be verified that in (A15), $\frac{\partial \Pi}{\partial p} = -\frac{1}{3}$ when $p = \frac{2(a_1+a_2)}{3}$, which immediately implies that $p_P^* < \frac{2(a_1+a_1)}{3}$. Finally, the constraint $p_P^* < \frac{2(a_1+a_2)}{3} = x_P^* + y_P^*$ also holds here.

Now consider partial mixed bundling, which arises when $a_1 > \frac{a_2}{\tau(\beta)}$ (and hence $p_P^* < \frac{2a_1}{3}$). Such p_P^* implies that $x_P^* = p_P^*$ and $y_P^* = \frac{2a_2}{3}$ by (A12) and (A13). Plug x_P^* and y_P^* into (A14) to obtain

$$\frac{\partial \Pi}{\partial p} = 1 - \left[\frac{2(1-\beta)}{a_1} + \frac{2\beta}{a_1+a_2} \right] p + \frac{2a_2(1-\beta)}{3a_1} = 0. \quad (\text{A16})$$

One can verify that $p_P^* = \frac{(a_1+a_2)[2a_2(1-\beta)+3a_1]}{6[a_1+(1-\beta)a_2]}$ in case (ii) of Proposition 5 solves (A16). Direct calculations also imply that $p_P^* > \frac{2a_2}{3}$ is equivalent to $3a_1^2 + a_1a_2 > 2a_2^2 + 2\beta a_2(a_1 - a_2)$, which holds here as $a_1 \geq a_2$ and $\beta \leq 1$. Finally, $x_P^* = p_P^*$ and $p_P^* > \frac{2a_2}{3}$ imply that $H_1(x_P^*) \geq H_2(p_P^* - x_P^*)$, $H_2(y_P^*) \geq H_1(p_P^* - y_P^*)$ and $p_P^* < x_P^* + y_P^*$ are all satisfied. ■

Proof of Proposition 6. We again apply the Envelope theorem which implies

$$\frac{\partial \Pi(x_P^*, y_P^*, p_P^*)}{\partial \beta} = \frac{(p_P^* - x_P^*)^2}{a_2} \left(1 - \frac{x_P^*}{a_1}\right) - \frac{y_P^*(p_P^* - y_P^*)}{a_1} \left(1 - \frac{y_P^*}{a_2}\right) + p_P^* \left[\frac{(x_P^* + y_P^* - p_P^*)(p_P^* - x_P^* + y_P^*)}{2a_1 a_2} - \frac{p_P^*}{a_1 + a_2} + \frac{p_P^* - y_P^*}{a_1} \right] \quad (\text{A17})$$

When $\tau(\beta) \leq \frac{a_2}{a_1}$, plugging $x_P^* = \frac{2a_1}{3}$ and $y_P^* = \frac{2a_2}{3}$ into (A17) to obtain

$$\frac{\partial \Pi}{\partial \beta} \Big|_{\left(\frac{2a_1}{3}, \frac{2a_2}{3}, p_P^*\right)} \equiv \tilde{\Pi}(p_P^*) = \left[\frac{(p_P^* - \frac{2a_1}{3})^2}{3a_2} - \frac{2a_2(p_P^* - \frac{2a_2}{3})}{9a_1} \right] + p_P^* \left[\frac{(\frac{2(a_1+a_2)}{3} - p_P^*)(p_P^* - \frac{2(a_1-a_2)}{3})}{2a_1 a_2} - \frac{p_P^*}{a_1 + a_2} + \frac{p_P^* - \frac{2a_2}{3}}{a_1} \right]$$

where $\tilde{\Pi}(p_P^*)$ is defined to simplify notation and $p_P^* \in \left[\frac{2a_1}{3}, \frac{2(a_1+a_2)}{3}\right)$ is given in case (i) of Proposition 5. We need to show that $\tilde{\Pi}(p_P^*) < 0$. Since p_P^* is inconveniently long, we prove this indirectly. First, one can verify that $\tilde{\Pi}\left(\frac{2(a_1+a_2)}{3}\right) = 0$. Hence to show $\tilde{\Pi}(p_P^*) < 0$, it suffices to show that function $\tilde{\Pi}(p)$ is increasing on $\left[\frac{2a_1}{3}, \frac{2(a_1+a_2)}{3}\right)$. Differentiate $\tilde{\Pi}(p)$ to obtain

$$\tilde{\Pi}'(p) = -\frac{3}{2a_1 a_2} p^2 + 2 \left(\frac{a_1 + a_2}{a_1 a_2} - \frac{1}{a_1 + a_2} \right) p - \frac{2(a_1^2 + a_2^2)}{3a_1 a_2}.$$

And $\tilde{\Pi}'(p)$ is quadratic and strictly concave in p . Since $\tilde{\Pi}'\left(p = \frac{2(a_1+a_2)}{3}\right) = 0$ and $\tilde{\Pi}'\left(p = \frac{2a_1}{3}\right) = \frac{2a_2(a_1-a_2)}{3a_1(a_1+a_2)} \geq 0$, we immediately obtain that $\tilde{\Pi}'(p) \geq 0$ or $\tilde{\Pi}(p)$ is strictly increasing on $\left[\frac{2a_1}{3}, \frac{2(a_1+a_2)}{3}\right)$ except possibly on $p = \frac{2a_1}{3}$. Therefore, $\tilde{\Pi}(p_P^*) < 0$.

When $\tau(\beta) > \frac{a_2}{a_1}$, substituting $x_P^* = p_P^*$ and $y_P^* = \frac{2a_2}{3}$ into (A17), we have

$$\frac{\partial \Pi}{\partial \beta} \Big|_{\left(p_P^*, \frac{2a_2}{3}, p_P^*\right)} \equiv \bar{\Pi}(p_P^*) = \frac{4a_2^2}{27a_1} - \frac{2a_2}{3a_1} p_P^* + \frac{a_2}{a_1(a_1 + a_2)} (p_P^*)^2$$

where again $\bar{\Pi}(p_P^*)$ is defined to simplify notation and $p_P^* \in \left[\frac{2a_2}{3}, \frac{2a_1}{3}\right)$ is given in case (ii) of Proposition 5. Consider function $\bar{\Pi}(p)$ where $p \in \left[\frac{2a_2}{3}, \frac{2a_1}{3}\right)$. We first calculate that $\bar{\Pi}\left(p = \frac{2a_2}{3}\right) = \bar{\Pi}\left(p = \frac{2a_1}{3}\right) = \frac{4a_2^2(a_2 - 2a_1)}{27a_1(a_1 + a_2)} < 0$. Since $\bar{\Pi}(p)$ is quadratic and strictly convex in p , we conclude that $\bar{\Pi}(p) < 0$ for all $p \in \left[\frac{2a_2}{3}, \frac{2a_1}{3}\right)$. Hence $\bar{\Pi}(p_P^*) < 0$. ■

Proof of Proposition 7. We prove the “if” part by demonstrating that $\Delta \Pi'(0) > 0$ under the conditions in Proposition 7. Under such conditions, we are in either case (ii) or case (iii) of Proposition 2 with the restriction that $\alpha > 0$ and $a_1 > a_2$ (also see Figure 5).

First consider case (iii) of Proposition 2. Denoting (y^*, p^*) in (3) and (4) as (y_N^*, p_N^*) in case (iii) of Proposition 2, we have that

$$\Delta \Pi(q) = \left\{ \begin{array}{l} (1 - \alpha) \left[\frac{q(p_N^* - y_N^*)^2 (2(a_2 - y_N^*) - q(p_N^* - y_N^*))}{2a_1 a_2} - \frac{q y_N^* (p_N^* - y_N^*)^2}{2a_1 a_2} \right] + \\ \alpha \left[\frac{q(p_N^* - y_N^*) ((a_2 - y_N^*) - q(p_N^* - y_N^*))}{a_2 - a_1 q} - y_N^* \left(1 - \frac{y_N^*}{a_2} - \frac{(a_2 - y_N^*) - q(p_N^* - y_N^*)}{a_2 - a_1 q} \right) \right] \end{array} \right\}$$

Differentiating and using the first-order condition for y in (A2) to derive (recall that $\alpha > 0$)

$$\Delta \Pi'(0) = \alpha a_1 \left[\frac{p_N^* - y_N^*}{2a_1} \left(1 - \frac{2y_N^*}{a_2}\right) + \frac{y_N^*}{a_2} \left(1 - \frac{y_N^*}{a_2}\right) \right] \geq \frac{\alpha a_1}{18} > 0, \quad (\text{A18})$$

where the first inequality is from $\frac{y_N^*}{a_2} \leq \frac{2}{3}$ and $\frac{p_N^* - y_N^*}{a_1} \leq 1$ in **Case N3** in the proof of Proposition 2.

Now consider case (ii) of Proposition 2. Again let (y^*, p^*) in (3) and (4) be the (y_N^*, p_N^*) given in case (ii) of Proposition 2. We can similarly calculate that

$$\Delta\Pi'(0) = \frac{(p_N^* - y_N^*)^2}{a_1} \left[1 - \frac{3(1-\alpha)y_N^*}{2a_2} \right] > 0,$$

where the inequality follows from $p_N^* > y_N^*$ (given that $a_1 > a_2$) and $y_N^* < \frac{2a_2}{3(1-\alpha)}$ (implied by the Kuhn-Tucker condition in **Case N2** in the proof of Proposition 2).

For the “only if” part, we will show that we either have $\Delta\Pi(q) < 0$ or $\Delta\Pi'(0) = 0$ and $\Delta\Pi'(q) < 0$ for (small) $q > 0$ when the conditions in Proposition 7 are **not** satisfied (i.e., either $\alpha = 0$, or $\alpha < \frac{2a_2 - a_1}{3a_1}$, or $a_1 = a_2$, or $0 < \beta < 1$).

Suppose we are in case (iii) of Proposition 2 and $\alpha = 0$. Our result in (A18) above shows that $\Delta\Pi'(0) = 0$. And differentiating $\Delta\Pi'(q)$ further leads to

$$\Delta\Pi''(0)|_{\alpha=0} = -\frac{(p_N^* - y_N^*)^3}{a_1 a_2} < 0,$$

establishing that $\Delta\Pi'(q) < 0$ for sufficiently small but positive q .

Suppose we are in case (i) of Proposition 2 with $0 \leq \alpha < \frac{2a_2 - a_1}{3a_1}$ and $a_1 \geq a_2$. We have

$$\Delta\Pi(q) = \frac{q(p_N^* - y_N^*)^2}{a_1} \left[1 - \frac{3(1-\alpha)y_N^*}{2a_2} - \frac{q(1-\alpha)(p_N^* - y_N^*)}{2a_2} \right] < 0, \text{ for all } q > 0,$$

where the inequality is from $y_N^* = \frac{2a_2}{3(1-\alpha)}$ in case (i) of Proposition 2.

Suppose we have that $a_1 = a_2$ and pure bundling is optimal (i.e., $\alpha \geq \frac{2a_2 - a_1}{3a_1} = \frac{2}{3}$). The random mechanism is suboptimal since there is no buyer buying only good 2.

Finally, we can similarly show that $\Delta\Pi'(q) < 0$ for all $\beta < 1$ under positive correlation. ■

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